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Simple and Compact Expressions for Neutrino Oscillation Probabilities in Matter

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Abstract: We reformulate perturbation theory for neutrino oscillations in matter with an expansion parameter related to the ratio of the solar to the atmospheric Δm^2 scales. Unlike previous works, we use a renormalized basis in which certain first-order effects are taken into account in the zeroth-order Hamiltonian. Using this perturbation theory we derive extremely compact expressions for the neutrino oscillation probabilities in matter. We find, for example, that the ν_e disappearance probability at this order is of a simple two flavor form with an appropriately identified mixing angle and Δm^2. Despite exceptional simplicity in their forms they accommodate all order effects of θ_{13} and the matter potential.

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1 Introduction

Neutrino oscillation based on the standard three-flavor scheme provides the best possible theoretical framework available to date to describe most of the experimental results obtained in the atmospheric, solar, reactor, and the accelerator neutrino experiments. Although numerically calculated neutrino oscillation probabilities suffice to analyze experimental data, understanding of the framework, in particular the one in matter [1], has not yet reached a sufficient level, in our opinion. Under the assumption of uniform matter density distribution, the exact expressions of the eigenvalues, the mixing angles and oscillation probabilities in matter have been obtained [2–4]. Yet, the results for these quantities are generally too complicated to facilitate understanding of the structure of the three flavor neutrino oscillations in matter primarily due to the complexities of solving the cubic eigenvalue characteristic equation. For a recent comprehensive treatment of neutrino oscillation in the matter, see ref. [5].

Analytic approaches to the neutrino oscillation phenomenon, so far, are mostly based on variety of perturbative frameworks. If the matter effect is small one can treat it as a small perturbation [6]. In the environments in which the matter effect is comparable to the vacuum mixing effect, the only available small expansion parameter known to us is the ratio of the solar-scale $\Delta m^2_\odot$ to the atmospheric-scale $\Delta m^2_\oplus$, $\Delta m^2_\odot/\Delta m^2_\oplus \simeq 0.03$. $\sin\theta_{13}$ has been often used as an expansion parameter (there are enormous number of references, see e.g., [7]), but it is now known that its value is not so small, $\sin\theta_{13} \simeq 0.15$, which is of the order of $\sqrt{\Delta m^2_\odot/\Delta m^2_\oplus}$. Moreover, expansion around $\sin\theta_{13} = 0$ misses the physics of the resonance which exists at an energy around $E \sim 10$ GeV for earth densities. Therefore, it appears that the suitable perturbative framework is the one with the unique expansion parameter $\Delta m^2_\odot/\Delta m^2_\oplus$. This framework was indeed examined in the past, to our knowledge in refs. [7–10].

In this paper, we present a new framework of perturbative treatment of neutrino oscillation in matter. We follow the reasoning stated above which led to identification of the unique expansion parameter $\epsilon \approx \Delta m^2_\odot/\Delta m^2_\oplus$. But, unlike the preceding works, we use a "renormalized basis" as the basis of perturbation. That is, we absorb certain terms of order ϵ to our "zeroth-order" Hamiltonian around which we perturb. Or, in other word, we take the zeroth-order eigenvalues in matter such that it matches the exact eigenvalues to order ϵ. We will show that use of the renormalized basis makes the structure of the perturbation theory transparent, and allows us to obtain simple, elegant and compact expressions for the oscillation probabilities. For example, ν_e survival probability takes the form to order ϵ as

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\phi \ \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \tag{1.1}$$

where ϕ is θ_{13} in matter, and $\lambda_\pm$ denote the eigenvalues of the sates which participate the 1-3 level crossing. Despite its extremely simple form, $P(\nu_e \to \nu_e)$ in (1.1) takes into account all order effects of θ_{13} and the matter potential. Since we will only consider terms up to order ϵ, in this paper, our results here are not applicable to the region near the solar

MSW resonance [1, 11]. Our perturbative framework will be called as the "renormalized helio-perturbation theory" in the rest of this paper.

The section plan of this paper is somewhat unusual: in the next section 2 we describe the principle results of this work including the oscillation probabilities for all channels in matter. This section does not describe the derivations but provides a self contained summary of the results of this paper. Following this section, see section 3, we present a systematic exposition of our perturbative framework and how the results of the section 2 are derived. In the appendices A, B, and C we present, respectively, calculational details of the S matrix, the results of oscillation probabilities to order ϵ in the standardized form, and useful relationships to verify the equivalence of various expressions.

2 Results of our perturbative expansion for all oscillation probabilities

In this section, we describe the main results of this paper without derivations and with only minimal discussion. In later sections we provide the derivation and more in depth discussions. We start with the approximate eigenvalues of the Hamiltonian, the approximate neutrino mixing matrix and then give the oscillation probabilities for all channels to first order in the expansion parameter, ϵ, see eq. (2.3) for the precise definition.

2.1 Mass squared eigenvalues in matter

In vacuum the three eigenvalues of the full Hamiltonian which governs the neutrino oscillation is given in the form $m_i^2/2E$, where m_i is the mass of i-th mass eigenstate of neutrinos, $i = 1, 2, 3$. Similarly, in matter we write the three eigenvalues as

$$\frac{\lambda_i}{2E},$$

where the state label runs over $i = -, 0, +$ for the approximate Hamiltonian of three flavor mixing system. To treat the normal and the inverted mass orderings (NO and IO respectively) in a unified way, we define the eigenvalues as follows[1]

$$\begin{aligned}
\lambda_- &= \frac{1}{2}\left[\left(\Delta m^2_{\rm ren} + a\right) - {\rm sign}(\Delta m^2_{\rm ren})\sqrt{\left(\Delta m^2_{\rm ren} - a\right)^2 + 4 s^2_{13} a \Delta m^2_{\rm ren}}\right] + \epsilon \Delta m^2_{\rm ren} s^2_{12}, \\
\lambda_0 &= c^2_{12}\ \epsilon\ \Delta m^2_{\rm ren}, \qquad\qquad (2.1) \\
\lambda_+ &= \frac{1}{2}\left[\left(\Delta m^2_{\rm ren} + a\right) + {\rm sign}(\Delta m^2_{\rm ren})\sqrt{\left(\Delta m^2_{\rm ren} - a\right)^2 + 4 s^2_{13} a \Delta m^2_{\rm ren}}\right] + \epsilon \Delta m^2_{\rm ren} s^2_{12}.
\end{aligned}$$

[1] We note that the eigenvalues in (2.1) above appear in ref. [5]. See section 3.1 for the derivation and a comment on the treatment in [5].

In eq. (2.1), the renomalized $\Delta m^2 \equiv \Delta m^2_{\rm ren}$, the expansion parameter ϵ, and the Wolfenstein matter potential, a, [1] are defined as follows:[2]

$$\Delta m^2_{\rm ren} \equiv \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}, \tag{2.2}$$

$$\epsilon \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{\rm ren}}, \tag{2.3}$$

$$\text{and} \qquad a = 2\sqrt{2}G_F N_e E \approx 1.52\times 10^{-4}\left(\frac{Y_e\rho}{\text{g.cm}^{-3}}\right)\left(\frac{E}{\text{GeV}}\right)\text{eV}^2. \tag{2.4}$$

This choice of $\Delta m^2_{\rm ren}$ is crucial to the compact formulas for the oscillation probabilities that will be given in this paper. Note also that the sign of $\Delta m^2_{\rm ren}$ signals the mass ordering, both $\Delta m^2_{\rm ren}$ and ϵ are positive (negative) for NO (IO). However, for both orderings $\epsilon\Delta m^2_{\rm ren} = \Delta m^2_{21} > 0$, as required by nature. Notice that λ_0 is the same for the both mass orderings, and when we switch from NO to IO we also switch the sign in front of the square root in eq. (2.1). The nicest feature of the sign choice is that the oscillation probability has a unified expression and the solar resonance is in ν_--ν_0 level crossing for the both mass orderings.

$\Delta m^2_{\rm ren}$ is equal to the effective atmospheric Δm^2 measured in a electron (anti-) neutrino disappearance experiment in vacuum, $\Delta m^2_{ee} \equiv c^2_{12}\Delta m^2_{31} + s^2_{12}\Delta m^2_{32}$ [12]. This quantity is identical to Δm^2_{ee} recently measured by the reactor θ_{13} experiment [13] up to effects of $\mathcal{O}(\Delta m^2_{21}/\Delta m^2_{31})^2$. Whether the coincidence between Δm^2_{ee} and $\Delta m^2_{\rm ren}$ reflects a deep aspect of neutrino oscillation or not will be judged depending upon what happens at second order in ϵ [3] . This point as well as the relevance of the other effective $\Delta m^2_{\mu\mu}$ [12], ν_μ equivalent of Δm^2_{ee}, will be discussed in depth in a forthcoming communication.

2.2 The mixing angle θ_{13} and mixing matrix in matter

We use the angle ϕ to represent the mixing angle θ_{13} in matter. With the definitions of the eigenvalues (2.1), the following mass-ordering independent expressions for cosine and sine 2ϕ (see section 3.2) are given by

$$\cos 2\phi = \frac{\Delta m^2_{\rm ren}\cos 2\theta_{13} - a}{\lambda_+ - \lambda_-},$$
$$\sin 2\phi = \frac{\Delta m^2_{\rm ren}\sin 2\theta_{13}}{\lambda_+ - \lambda_-}. \tag{2.5}$$

It is easy to show that ϕ goes from $0 \to \pi/2$ as a goes from $-\infty$ to $+\infty$ for the NO and as a goes from $+\infty$ to $-\infty$ for the IO. In vacuum ($a = 0$), $\phi = \theta_{13}$ and $\phi = \pi/4$ at the atmospheric resonance, when $a = \Delta m^2_{\rm ren}\cos 2\theta_{13}$, for both mass orderings.

[2] The following notation is used throughout: $\Delta m^2_{ij} \equiv m^2_i - m^2_j$, $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$ where θ_{ij} are the standard neutrino mixing angles and G_F is the Fermi constant, N_e is the number density of electrons, E is the energy of the neutrino, Y_e the electron fraction and ρ is the density of matter.

[3] The authors respectfully disagree with each other on this point.

The mixing matrix in matter, V, relates the flavor eigenstates, ν_e, ν_μ, ν_τ, to the matter mass eigenstates, ν_-, ν_0, ν_+ as follows (see section 3.4):

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = V \begin{pmatrix} \nu_- \\ \nu_0 \\ \nu_+ \end{pmatrix} \tag{2.6}$$

where the matrix V is unitary. It is convenient to split V into a zeroth order term, $V^{(0)}$, and a first order term, $V^{(1)}$ in our ϵ expansion,

$$V \equiv V^{(0)} + \epsilon V^{(1)}, \tag{2.7}$$

where the zeroth order matrix is given by

$$V^{(0)} = \begin{bmatrix} c_\phi & 0 & s_\phi \\ -s_\phi s_{23} e^{i\delta} & c_{23} & c_\phi s_{23} e^{i\delta} \\ -s_\phi c_{23} & -s_{23} e^{-i\delta} & c_\phi c_{23} \end{bmatrix}. \tag{2.8}$$

whereas the first order correction is given by

$$V^{(1)} = c_{12} s_{12}\, \Delta m^2_{\rm ren} \left\{ \left(\frac{c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} \right) \begin{bmatrix} 0 & -c_\phi & 0 \\ c_{23} & s_\phi s_{23} e^{i\delta} & 0 \\ -s_{23} e^{-i\delta} & s_\phi c_{23} & 0 \end{bmatrix} \right.$$
$$\left. + \left(\frac{s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} \right) \begin{bmatrix} 0 & -s_\phi & 0 \\ 0 & -c_\phi s_{23} e^{i\delta} & c_{23} \\ 0 & -c_\phi c_{23} & -s_{23} e^{-i\delta} \end{bmatrix} \right\}. \tag{2.9}$$

As an outcome of the consistent perturbative treatment the total V matrix given by (2.7) with (2.8) and (2.9) must be unitary to order ϵ. In fact it is, since the following two conditions are satisfied

$$V^{(0)}(V^{(0)})^\dagger = 1 \quad \text{and} \quad V^{(0)}(V^{(1)})^\dagger + V^{(1)}(V^{(0)})^\dagger = 0. \tag{2.10}$$

Of course, none of what follows is self consistent without unitary here.

With the matter eigenvalues, λ's , definite by eq. (2.1) and the matter mixing matrix, V, given by eq. (2.7), simple and compact expressions can be easily derived for the oscillation probabilities in matter for all channels, to leading order in ϵ, as will be shown in the next section.

2.3 Compact formulas for the oscillation probabilities in matter

In this section we present the shortest path to the oscillation probabilities by using the eigenvalues, $\lambda_{\pm,0}$, and mixing matrix, V, given in the previous section to order ϵ. Other methods are described in later sections which give identical results. Note however, that the agreement between these different methods is highly nontrivial, and its meaning will be explained in section 3.5.

2.3.1 $\nu_e \to \nu_e$ disappearance channel

The derivation of the ν_e survival oscillation probability, $P(\nu_e \to \nu_e)$, in our renormalized helio-perturbation theory is extremely simple. Starting from the general expression

$$\begin{aligned} P(\nu_e \to \nu_e) = 1 &- 4|V_{e+}|^2|V_{e-}|^2 \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\ &- 4|V_{e+}|^2|V_{e0}|^2 \sin^2 \frac{(\lambda_+ - \lambda_0)L}{4E} \\ &- 4|V_{e0}|^2|V_{e-}|^2 \sin^2 \frac{(\lambda_0 - \lambda_-)L}{4E} \end{aligned}$$

where, L, is the baseline. Now $|V_{e0}|^2 = \mathcal{O}(\epsilon^2)$, so we obtain to order ϵ

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\phi \ \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \tag{2.11}$$

with

$$\sin^2 2\phi = \left(\frac{\Delta m^2_{\rm ren}}{\lambda_+ - \lambda_-} \right)^2 \sin^2 2\theta_{13}$$

and $\lambda_\pm$ are given in eq. (2.1).

Notice that the formula in eq. (2.11) takes into account the matter effect as well as the effect of s_{13} to *all* orders. Nonetheless, it keeps an exceptional simplicity, an effective two-flavor form with only the unique eigenvalue difference $\lambda_+ - \lambda_-$. The feature stems from the fact that there is no ν_e component at zeroth order in ϵ in the "0" state in matter. It is expressed in the zero in the V_{e0} element of the zeroth-order V matrix as in (2.8), see also section 3.4.

2.3.2 $\nu_e \to \nu_\mu$ appearance channel

Now, we discuss the appearance channel $\nu_e \to \nu_\mu$. We describe here the simplest way to derive the formulas for the oscillation probabilities starting from the V matrix by using unitarity. The oscillation probability $P(\nu_e \to \nu_\mu)$ can be computed as

$$P(\nu_e \to \nu_\mu) = \left| V_{\mu-} V^*_{e-} e^{-i\frac{\lambda_- L}{2E}} + V_{\mu 0} V^*_{e0} e^{-i\frac{\lambda_0 L}{2E}} + V_{\mu+} V^*_{e+} e^{-i\frac{\lambda_+ L}{2E}} \right|^2 \tag{2.12}$$

We use unitarity relation $V_{\mu-}V^*_{e-} + V_{\mu 0}V^*_{e0} + V_{\mu+}V^*_{e+} = 0$ to eliminate the $V_{\mu-}V^*_{e-}$ term in (2.12). Then, we obtain

$$\begin{aligned} P(\nu_e \to \nu_\mu) &= 4|V_{\mu+}V^*_{e+} \sin\Delta_{+-} e^{-i\Delta_{+0}} - V_{\mu 0}V^*_{e0} \sin\Delta_{-0}|^2 \\ &= 4|V_{\mu+}|^2|V_{e+}|^2 \sin^2\Delta_{+-} \\ &\quad -8\mathcal{R}(V_{\mu+}V^*_{e+}V^*_{\mu 0}V_{e0}) \sin\Delta_{+-} \sin\Delta_{-0} \cos\Delta_{+0} \\ &\quad -8\mathcal{I}(V_{\mu+}V^*_{e+}V^*_{\mu 0}V_{e0}) \sin\Delta_{+-} \sin\Delta_{-0} \sin\Delta_{+0} \\ &\quad +4|V_{\mu 0}|^2|V_{e0}|^2 \sin^2\Delta_{-0} \end{aligned} \tag{2.13}$$

where the common shorthand notation for the kinematic phase $\Delta_{ij} = (\lambda_i - \lambda_j)L/4E$ is used. Again, since $|V_{e0}|^2 = \mathcal{O}(\epsilon^2)$, we have to order ϵ

$$
\begin{aligned}
& P(\nu_e \to \nu_\mu) \\
&= \left[s_{23}^2 \sin^2 2\theta_{13} + 4\epsilon J_r \cos\delta \left\{ \frac{(\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a)}{(\lambda_+ - \lambda_0)} \right\} \right] \left(\frac{\Delta m^2_{\rm ren}}{\lambda_+ - \lambda_-} \right)^2 \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\
&+ 8\epsilon J_r \frac{(\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \cos \left(\delta - \frac{(\lambda_+ - \lambda_0)L}{4E} \right)
\end{aligned}
\tag{2.14}
$$

here J_r, the reduced Jarlskog factor, is

$$
J_r \equiv c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13}. \tag{2.15}
$$

This expression for the $\nu_e \to \nu_\mu$ appearance channel probability is quite compact, despite that it contains all-order contributions of s_{13} and a. In particular, it keeps the similar structure as the one derived by the Cervera *et al.* formula [7], which retains terms of order ϵ^2 but expanded by s_{13} only up to second order.

This method of computing $P(\nu_e \to \nu_\mu)$ in the above offers the shortest path to the expression of the oscillation probability which is manifestly free from the apparent singularity as $\lambda_- \to \lambda_0$ because $1/(\lambda_- - \lambda_0)$ always appears adjacent to $\sin \frac{(\lambda_- - \lambda_0)L}{4E}$. We will refer this method as the "shortcut method" in the rest of this paper.

2.3.3 $\nu_e \to \nu_\tau$ appearance channel

The oscillation probability for $\nu_e \to \nu_\tau$ channel can be obtained in the shortcut method used in the previous subsection, or by using the unitarity relation $P(\nu_e \to \nu_\tau) = 1 - P(\nu_e \to \nu_e) - P(\nu_e \to \nu_\mu)$. Another method to obtain $P(\nu_e \to \nu_\tau)$ is to make the transformation $c_{23} \to -s_{23}$ and $s_{23} \to c_{23}$ in $P(\nu_e \to \nu_\mu)$ [10]. The result obtained in either one of these ways can be written as

$$
\begin{aligned}
& P(\nu_e \to \nu_\tau) \\
&= \left[c_{23}^2 \sin^2 2\theta_{13} - 4\epsilon J_r \cos\delta \left\{ \frac{(\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a)}{(\lambda_+ - \lambda_0)} \right\} \right] \left(\frac{\Delta m^2_{\rm ren}}{\lambda_+ - \lambda_-} \right)^2 \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\
&- 8\epsilon J_r \frac{(\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \cos \left(\delta - \frac{(\lambda_+ - \lambda_0)L}{4E} \right)
\end{aligned}
\tag{2.16}
$$

Given the expression of $P(\nu_e \to \nu_\mu)$ in (2.14), it is very easy to understand the result (2.16). To satisfy unitarity $P(\nu_e \to \nu_\tau)$ and $P(\nu_e \to \nu_\mu)$ must add up to $1 - P(\nu_e \to \nu_e)$, which means that the explicit order ϵ terms must cancel out and the coefficient of $\sin^2 2\theta_{13}$ term must add up to unity.

2.3.4 $\nu_\mu \to \nu_\mu$ disappearance channel

By using the shortcut method, or by rewriting the expression in appendix B using the formulas in appendix C, one obtains $P(\nu_\mu \to \nu_\mu)$ as:

$$
\begin{aligned}
&P(\nu_\mu \to \nu_\mu) \\
&= 1 - \left[s_{23}^4 \sin^2 2\phi + 8\epsilon J_r \cos\delta \; s_{23}^2 \; \frac{(\Delta m^2_{\rm ren})^2 \left\{ (\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a) \right\}}{(\lambda_+ - \lambda_-)^2 (\lambda_+ - \lambda_0)} \right] \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\
&- \left[\sin^2 2\theta_{23} c_\phi^2 - 4\epsilon \left(J_r \cos\delta / c_{13}^2 \right) \; \cos 2\theta_{23} \; \frac{\Delta m^2_{\rm ren} \left\{ (\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} + a) \right\}}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)} \right] \sin^2 \frac{(\lambda_+ - \lambda_0)L}{4E} \\
&- \left[\sin^2 2\theta_{23} s_\phi^2 - 4\epsilon \left(J_r \cos\delta / c_{13}^2 \right) \; \cos 2\theta_{23} \; \frac{\Delta m^2_{\rm ren} \left\{ (\lambda_+ - \lambda_-) + (\Delta m^2_{\rm ren} + a) \right\}}{(\lambda_+ - \lambda_-)(\lambda_- - \lambda_0)} \right] \sin^2 \frac{(\lambda_- - \lambda_0)L}{4E} \\
&- 16\epsilon J_r \cos\delta \; s_{23}^2 \; \frac{(\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \cos \frac{(\lambda_+ - \lambda_0)L}{4E}.
\end{aligned}
\tag{2.17}
$$

The last term in (2.17) can be casted into the canonical $\sin^2 \Delta_{ji}$ terms by using the identity

$$
\sin^2 \Delta_{+-} + \sin^2 \Delta_{-0} - \sin^2 \Delta_{+0} = -2 \sin \Delta_{+-} \sin \Delta_{-0} \cos \Delta_{+0}, \tag{2.18}
$$

which leads to the form in (B.5). However, if this is done then some of the terms are singular when $\lambda_- = \lambda_0$, yet the total expression is equivalent to eq. (2.17) and is finite.

2.3.5 $\nu_\mu \to \nu_\tau$ appearance channel

Similarly, one can easily derive the expression of $P(\nu_\mu \to \nu_\tau)$ by the shortcut method with use of the identity (2.18), or by using unitarity, or by rewriting the expression in appendix B. The result can be written as

$$
\begin{aligned}
&P(\nu_\mu \to \nu_\tau) \\
&= - \left[c_{23}^2 s_{23}^2 \sin^2 2\phi + 4\epsilon J_r \cos\delta \cos 2\theta_{23} \frac{(\Delta m^2_{\rm ren})^2 \left\{ (\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a) \right\}}{(\lambda_+ - \lambda_-)^2 (\lambda_+ - \lambda_0)} \right] \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\
&+ \left[\sin^2 2\theta_{23} c_\phi^2 - 4\epsilon \left(J_r \cos\delta / c_{13}^2 \right) \; \cos 2\theta_{23} \; \frac{\Delta m^2_{\rm ren} \left\{ (\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} + a) \right\}}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)} \right] \sin^2 \frac{(\lambda_+ - \lambda_0)L}{4E} \\
&+ \left[\sin^2 2\theta_{23} s_\phi^2 - 4\epsilon \left(J_r \cos\delta / c_{13}^2 \right) \; \cos 2\theta_{23} \; \frac{\Delta m^2_{\rm ren} \left\{ (\lambda_+ - \lambda_-) + (\Delta m^2_{\rm ren} + a) \right\}}{(\lambda_+ - \lambda_-)(\lambda_- - \lambda_0)} \right] \sin^2 \frac{(\lambda_- - \lambda_0)L}{4E} \\
&- 8\epsilon J_r \frac{(\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \\
&\qquad \times \left[\cos 2\theta_{23} \; \cos\delta \cos \frac{(\lambda_+ - \lambda_0)L}{4E} - \sin\delta \sin \frac{(\lambda_+ - \lambda_0)L}{4E} \right].
\end{aligned}
\tag{2.19}
$$

2.3.6 $\nu_\tau \to \nu_\tau$ disappearance channel

The simplest way to obtain $P(\nu_\tau \to \nu_\tau)$ is to make the transformation $c_{23} \to -s_{23}$ and $s_{23} \to c_{23}$ in $P(\nu_\mu \to \nu_\mu)$. Or, one may use the shortcut method, or the unitarity relation $P(\nu_\tau \to \nu_\tau) = 1 - P(\nu_\tau \to \nu_e) - P(\nu_\tau \to \nu_\mu)$. The results are all the same, and given by

$$
\begin{aligned}
& P(\nu_\tau \to \nu_\tau) \\
&= 1 - \left[c_{23}^4 \sin^2 2\phi - 8\epsilon J_r \cos\delta c_{23}^2 \frac{(\Delta m^2_{\rm ren})^2 \left\{ (\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a) \right\}}{(\lambda_+ - \lambda_-)^2 (\lambda_+ - \lambda_0)} \right] \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\
&- \left[\sin^2 2\theta_{23} c_\phi^2 - 4\epsilon \left(J_r \cos\delta / c_{13}^2 \right) \ \cos 2\theta_{23} \ \frac{\Delta m^2_{\rm ren} \left\{ (\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} + a) \right\}}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)} \right] \sin^2 \frac{(\lambda_+ - \lambda_0)L}{4E} \\
&- \left[\sin^2 2\theta_{23} s_\phi^2 - 4\epsilon \left(J_r \cos\delta / c_{13}^2 \right) \ \cos 2\theta_{23} \ \frac{\Delta m^2_{\rm ren} \left\{ (\lambda_+ - \lambda_-) + (\Delta m^2_{\rm ren} + a) \right\}}{(\lambda_+ - \lambda_-)(\lambda_- - \lambda_0)} \right] \sin^2 \frac{(\lambda_- - \lambda_0)L}{4E} \\
&+ 16\epsilon J_r \cos\delta \ c_{23}^2 \ \frac{(\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \cos \frac{(\lambda_+ - \lambda_0)L}{4E}.
\end{aligned}
\tag{2.20}
$$

2.4 Comments and range of applicability

So far we have given the expressions of oscillation probabilities in all the oscillation channels. The simplicity of our expressions for the oscillation probabilities may be understood by the readers, if they compare our expressions to the ones in the existing literatures to the same order in the expansion parameter, for example, $P(\nu_e \to \nu_\mu)$ in (2.14) to the equations (36a)−(36f) in [9], or $P(\nu_\mu \to \nu_\tau)$ in (2.19) to eqs. (10) and (11) in [10].

Two relevant comments on the properties of the oscillation probabilities are in order:

- In vacuum, $a = 0$, the above oscillation probabilities reproduce the standard results, to first order in ϵ. The form is somewhat unusual but we have checked that the expressions are identical.

- If one looks at eqs. (2.14), (2.16), and (2.19), the terms proportional to $\sin\delta$ are all equal to

  $$8 \left\{ J_r \sin\delta \frac{\epsilon \ (\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \right\} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \sin \frac{(\lambda_+ - \lambda_0)L}{4E}$$

  up to an overall sign. These is because the quantity in $\{\cdots\}$ is just the Jarlskog factor in matter due to the Naumov-Harrison-Scott identity, [16, 17].

To discuss the range of applicability of our expressions, it is useful to first consider the vacuum expressions to first order in the expansion parameter ϵ. For all channels, the expansion of the vacuum oscillation probabilities to first order in ϵ does not include terms proportional to $\sin^2 \Delta_{21}$ which starts at second order in ϵ,

$$\sin^2 \Delta_{21} = \epsilon^2 \Delta^2_{ren} + \mathcal{O}(\epsilon^4) \simeq \epsilon^2 \Delta^2_{31} + \mathcal{O}(\epsilon^4). \tag{2.21}$$

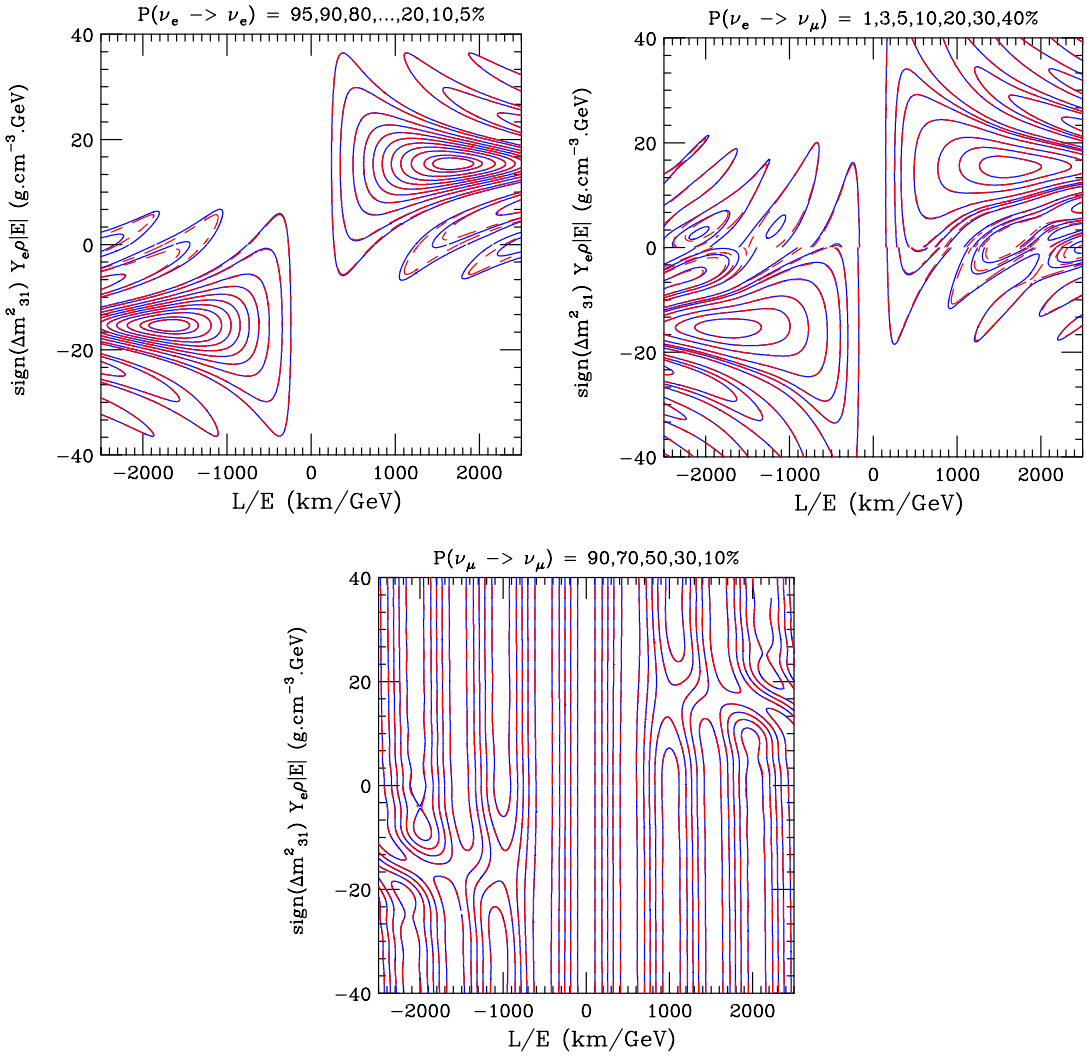

Figure 1. The iso-probability contours for the exact (solid blue) and approximate (dashed red) oscillation probabilities for upper left, $\nu_e \to \nu_e$, upper right, $\nu_e \to \nu_\mu$ and lower, $\nu_\mu \to \nu_\mu$. The upper (lower) half plane is for normal ordering (inverted ordering), whereas positive (negative) L/E is for neutrinos (antineutrinos). (See footnote 4.) The order of the contours given in the title is determined from the line L/E=0. The discontinued as one crosses $Y_e\rho|E| = 0$ is because we are switching mass orderings at this point. In most of parameter space the approximate and exact contours sit on top of one another so the lines appear to alternate blue-red dashed. Note that, for L/E >1000 km/GeV and $|Y_e\rho E| < 5$ g.cm^{-3}.GeV, the difference between the exact and approximate contours becomes noticeable at least for $\nu_e \to \nu_e$ and $\nu_e \to \nu_\mu$.

where $\Delta_{ji} \equiv \Delta m^2_{ji} L/4E$. When, $\Delta_{31} = \pi/2$, that is at the first atmospheric oscillation maximum, $\epsilon^2\Delta^2_{31} \approx 0.002$ which is small for the channels $\nu_e \to \nu_x$ where $x = e, \mu$, or τ since $1 - P(\nu_e \to \nu_e)$, $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ are all of order $\sin^2 2\theta_{13} \approx 0.1$. However, at the second atmospheric oscillation maximum, $\Delta^2_{31} = 3\pi/2$ and $\epsilon^2\Delta^2_{31} \approx 0.02$, which is significant compared to the $\sin^2 2\theta_{13}$ term. So in vacuum our first order expansion is only a good approximation for $\Delta^2_{31} \lesssim \pi$ or $L/E \lesssim 1000$ km/GeV for these ν_e channels. For the other channels, $\nu_\mu \to \nu_\mu$, $\nu_\tau \to \nu_\tau$ and $\nu_\mu \to \nu_\tau$, our first order expansion is a good

approximation to somewhat beyond L/E = 1000 km/GeV because the leading terms are not suppressed by the smallness of $\sin^2 2\theta_{13}$.

Then, what about the validity in matter? In section 4 we will argue that our perturbative description is valid outside the solar resonance. Notice that the region without validity (no guarantee for approximation being good) is rather wide and includes the vacuum because the solar resonance width $|\Delta a| = \sqrt{3}(\sin 2\theta_{12}/\cos^2\theta_{13})\Delta m^2_{21}$ is larger than the solar resonance position $a = (\cos 2\theta_{12}/\cos^2\theta_{13})\Delta m^2_{21}$. We expect then that our helio-perturbation theory works for the matter potential a larger than a few tenth of $|\Delta m^2_{\rm ren}|$.

To give the reader a sense of the precision of our approximation we have plotted in Fig. 1, the contours of equal probability for the exact and the approximate solutions for the channels $\nu_e \to \nu_\mu$, $\nu_e \to \nu_e$ and $\nu_\mu \to \nu_\mu$.[4] As expected, for large values of the matter potential, $|a| > \frac{1}{3}|\Delta m^2_{\rm ren}|$ we find we have no restrictions on L/E, to have a good approximation to the exact numerical solutions. Whereas for small values of the matter potential, $|a| < \frac{1}{3}|\Delta m^2_{\rm ren}|$ we still need the restriction $L/E \lesssim 1000$ km/GeV.[5]

We note that most of the settings for the ongoing and the proposed experiments, except possibly for the one which utilizes the second oscillation maximum, fall into the region $L/E \lesssim 1000$ km/GeV. To improve the accuracy to larger values of L/E, especially for values of $|a| < \frac{1}{3}|\Delta m^2_{\rm ren}|$, second order perturbation theory in ϵ is needed, which will be the subject of a future publication.

3 Formulating the renormalized helio-perturbation theory

In this section, we formulate the helio-to-terrestrial ratio perturbation theory, for short the helio-perturbation theory, which has the unique expansion parameter

$$\epsilon \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{\rm ren}}. \tag{3.1}$$

We will show that use of its renormalized version is the key to the very simple formulas of the oscillation probabilities exhibited in section 2.3 and appendix B. In fact, there are two ways of deriving the oscillation probabilities, the S matrix method and the wave function method. Here we sketch both of them, leaving technical or computational parts into Appendices A and B. The meaning of the agreement between results obtained by both the S matrix and the wave function methods will be discussed at the end of this section.

The S matrix describes neutrino flavor changes $\nu_\beta \to \nu_\alpha$ after traversing a distance L,

$$\nu_\alpha(L) = S_{\alpha\beta}\nu_\beta(0), \tag{3.2}$$

and the oscillation probability is given by

$$P(\nu_\beta \to \nu_\alpha; L) = |S_{\alpha\beta}|^2. \tag{3.3}$$

[4] One can show by taking complex conjugate of the evolution equation of anti-neutrinos of energy E, it is equivalent to solve the neutrino evolution equation with energy $-E$. Therefore, the right (left) half plane of each panel of Fig. 1 corresponds to the neutrino (anti-neutrino) channel.

[5] Of course, the boundary between these two regions should be interpreted as an approximate one. In fact, an exact boundary would sensitively depend on the definition of the difference allowed between the exact and the approximate probabilities.

When the neutrino evolution is governed by the Schrödinger equation, $i\frac{d}{dx}\nu = H\nu$, S matrix is given as

$$S = T\exp\left[-i\int_0^L dx H(x)\right] \tag{3.4}$$

where T symbol indicates the "time ordering" (in fact "space ordering" here). In the standard three-flavor neutrinos, Hamiltonian is given by

$$H = \frac{1}{2E}\left\{U\begin{bmatrix}0 & 0 & 0\\ 0 & \Delta m^2_{21} & 0\\ 0 & 0 & \Delta m^2_{31}\end{bmatrix}U^\dagger + \begin{bmatrix}a(x) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right\}, \tag{3.5}$$

where the symbols are defined in an earlier footnote. For the case of constant matter density, the right-hand side of (3.4) may be written as e^{-iHx}. We recapitulate here the earlier footnote: In (3.5) $\Delta m^2_{ji} \equiv m^2_j - m^2_i$ where m_i denotes the mass of i-th mass eigenstate neutrinos. Position dependent function $a(x) \equiv 2\sqrt{2}G_F N_e(x)E$ is a coefficient for measuring the matter effect on neutrinos propagating in medium of electron number density $N_e(x)$ [1] where G_F is the Fermi constant and E is the neutrino energy.

The neutrino flavor mixing matrix U is usually taken to be the standard form $U_{\rm PDG}$ given by Particle Data Group. We, however, prefer to work in a slightly different basis, for this paper, in which the flavor mixing matrix has a form

$$\begin{aligned} U &= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-i\delta}\end{bmatrix} U_{\rm PDG} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{i\delta}\end{bmatrix} \\ &= \begin{bmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}e^{i\delta}\\ 0 & -s_{23}e^{-i\delta} & c_{23}\end{bmatrix}\begin{bmatrix}c_{13} & 0 & s_{13}\\ 0 & 1 & 0\\ -s_{13} & 0 & c_{13}\end{bmatrix}\begin{bmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{bmatrix} \equiv U_{23}U_{13}U_{12} \end{aligned} \tag{3.6}$$

with understanding that the left phase matrix in the first line in eq.(3.6) is to be absorbed into the ν_τ neutrino wave functions. By using the convention of the mixing matrix as in (3.6), the perturbative calculation is entirely free from δ. We use the obvious notations $s_{ij} \equiv \sin\theta_{ij}$ etc. and δ is the CP violating phase.

3.1 Choosing the basis for the renormalized helio-perturbation theory

It is convenient to work with the tilde basis defined as $\tilde{\nu}_\alpha = (U^\dagger_{23})_{\alpha\beta}\nu_\beta$, in which the Hamiltonian is related to the flavor basis one as

$$\tilde{H} = U^\dagger_{23} H U_{23}, \tag{3.7}$$

where U_{23} is defined in eq.(3.6). The S matrix in the flavor basis is related to the S matrix in the tilde basis $\tilde{S}$ as

$$S(L) = U_{23}\tilde{S}(L)U^\dagger_{23}, \qquad \tilde{S}(L) = T\exp\left[-i\int_0^L dx\tilde{H}(x)\right]. \tag{3.8}$$

The simplest formulation of hello-perturbative treatment in the tilde-basis includes decomposition of $\tilde{H}$ into the zeroth and the first order terms in the expansion parameter $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ as

$$\tilde{H}(x) = \frac{\Delta m^2_{31}}{2E}\left\{\begin{bmatrix} \frac{a(x)}{\Delta m^2_{31}} + s^2_{13} & 0 & c_{13}s_{13} \\ 0 & 0 & 0 \\ c_{13}s_{13} & 0 & c^2_{13} \end{bmatrix} + \frac{\Delta m^2_{21}}{\Delta m^2_{31}}\begin{bmatrix} s^2_{12}c^2_{13} & c_{12}s_{12}c_{13} & -s^2_{12}c_{13}s_{13} \\ c_{12}s_{12}c_{13} & c^2_{12} & -c_{12}s_{12}s_{13} \\ -s^2_{12}c_{13}s_{13} & -c_{12}s_{12}s_{13} & s^2_{12}s^2_{13} \end{bmatrix}\right\} \tag{3.9}$$

To derive the compact formulas of oscillation probabilities we use slightly different basis, a renormalized basis, to formulate the perturbation theory. That is, we absorb a certain order ϵ terms into the zeroth order Hamiltonian, $\tilde{H}(x) = \tilde{H}_0(x) + \tilde{H}_1(x)$:

$$\tilde{H}_0(x) = \frac{\Delta m^2_{\rm ren}}{2E}\left\{\begin{bmatrix} \frac{a(x)}{\Delta m^2_{\rm ren}} + s^2_{13} & 0 & c_{13}s_{13} \\ 0 & 0 & 0 \\ c_{13}s_{13} & 0 & c^2_{13} \end{bmatrix} + \epsilon\begin{bmatrix} s^2_{12} & 0 & 0 \\ 0 & c^2_{12} & 0 \\ 0 & 0 & s^2_{12} \end{bmatrix}\right\} \tag{3.10}$$

$$\tilde{H}_1(x) = \epsilon c_{12}s_{12}\frac{\Delta m^2_{\rm ren}}{2E}\begin{bmatrix} 0 & c_{13} & 0 \\ c_{13} & 0 & -s_{13} \\ 0 & -s_{13} & 0 \end{bmatrix} \tag{3.11}$$

where $\Delta m^2_{\rm ren} \equiv \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}$ and $\epsilon \equiv \Delta m^2_{21}/\Delta m^2_{\rm ren}$, as defined in (2.2) and (2.3). $\tilde{H}(x)$ with (3.10) and (3.11) is identical with the tilde-Hamiltonian in (3.9). Note the simplicity of the perturbing Hamiltonian, $\tilde{H}_1$ and that the diagonalization of $\tilde{H}_0$ leads to the eigenvalues given in section 2.1, eq. (2.1).

The authors of [5] treat the order ϵ effect in the Hamiltonian as a renormalization of the matter potential, whereas we regard it as a renormalization of Δm^2_{31}.

Though our treatment can be easily generalized to cases with matter density variation as far as the adiabatic approximation holds, we derive, for ease of presentation, the formulas with constant matter density approximation in the rest of this paper.

3.2 Hat basis

We transform the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$, from the"tilde" basis to the "hat" basis, using

$$\hat{H}_0 = U^\dagger_\phi \tilde{H}_0 U_\phi, \qquad \hat{H}_1 = U^\dagger_\phi \tilde{H}_1 U_\phi \tag{3.12}$$

where the unperturbed Hamiltonian $\hat{H}_0$ is diagonal,

$$\hat{H}_0 = \frac{1}{2E}\begin{bmatrix} \lambda_- & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_+ \end{bmatrix}. \tag{3.13}$$

We take the following form of unitary matrix U_ϕ to diagonalize $\tilde{H}_0$:

$$U_\phi = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}. \tag{3.14}$$

The expressions of the zeroth order eigenvalues λ_-, λ_0, and λ_+, are given in eqn (2.1). Similarly, cosine and sine ϕ are given in eqn. (2.5).

Also the perturbing Hamiltonian, $\hat{H}_1$, retains it's simple form thanks to that the U_ϕ rotation keeps "zero" in $\tilde{H}_1$ unchanged,

$$\begin{aligned}\hat{H}_1 &= U_\phi^\dagger \tilde{H}_1 U_\phi \\ &= \epsilon c_{12} s_{12} \frac{\Delta m^2_{\text{ren}}}{2E} \begin{bmatrix} 0 & \cos(\phi - \theta_{13}) & 0 \\ \cos(\phi - \theta_{13}) & 0 & \sin(\phi - \theta_{13}) \\ 0 & \sin(\phi - \theta_{13}) & 0 \end{bmatrix}. \end{aligned} \tag{3.15}$$

In fact, $\hat{H}_1$ is identical to $\tilde{H}_1$ with θ_{13} replaced by $(\theta_{13} - \phi)$.

3.3 S Matrix and the oscillation probability

The S matrix in the flavor basis is related to the S matrix in the tilde and the hat bases as

$$S(L) = U_{23} \tilde{S}(L) U_{23}^\dagger = U_{23} U_\phi \hat{S}(L) U_\phi^\dagger U_{23}^\dagger \tag{3.16}$$

where we have used explicitly the fact that the matter density is constant:

$$\tilde{S}(L) = T\exp\left[-i\int_0^L dx \tilde{H}(x)\right] = U_\phi T \exp\left[-i\int_0^L dx \hat{H}(x)\right] U_\phi^\dagger \equiv U_\phi \hat{S}(L) U_\phi^\dagger. \tag{3.17}$$

To calculate $\hat{S}(L)$ we define $\Omega(L)$ as

$$\Omega(L) = e^{i\hat{H}_0 L} \hat{S}(L). \tag{3.18}$$

Then, $\Omega(L)$ obeys the evolution equation

$$i\frac{d}{dx}\Omega(x) = \check{H}_1 \Omega(x) \tag{3.19}$$

where

$$\check{H}_1 \equiv e^{i\hat{H}_0 x} \hat{H}_1 e^{-i\hat{H}_0 x}. \tag{3.20}$$

Then, $\Omega(x)$ can be computed perturbatively as

$$\Omega(L) = 1 + (-i)\int_0^L dx \check{H}_1(x) + \mathcal{O}(\epsilon^2). \tag{3.21}$$

Collecting the formulas the S matrix can be written as

$$S(L) = U_{23} U_\phi e^{-i\hat{H}_0 L} \Omega(L) U_\phi^\dagger U_{23}^\dagger \tag{3.22}$$

Thus, we are left with perturbative computation of $\Omega(L)$ with use of (3.20) to calculate the S matrix. With the S matrix in hand it is straightforward to compute the oscillation probabilities by using (3.3). We leave these tasks to appendices A and B.

3.4 Mass eigenstate in matter: V matrix method

In this section we calculate the V-matrix directly using our perturbation theory. If we switch off the perturbation $\hat{H}_1$, the mass eigenstates in matter, to lowest order, are given by the hat-basis wave function $\hat{\nu}_i^{(0)}$, which are the eigenstates of $\hat{H}_0$ in (3.12), and since $\hat{H}_0$ is diagonal, we have

$$\hat{\nu}_i^{(0)} = (U_{23}U_\phi)^\dagger_{i\alpha}\nu_\alpha. \tag{3.23}$$

Thus, the V matrix is given to zeroth order by $V^{(0)} = U_{23}U_\phi$ whose explicit form is given in section 4.3, and also in eq. (2.8).

In order to obtain the mass eigenstates in matter to first order in ϵ, $\nu_i = \hat{\nu}_i^{(0)} + \hat{\nu}_i^{(1)}$, let us compute the first order correction to the hat basis wave functions. Using the familiar perturbative formula for the perturbed wave functions

$$\hat{\nu}_i^{(1)} = \sum_{j\neq i} \frac{(\hat{H}_1)_{ji}}{\lambda_i - \lambda_j}\hat{\nu}_j^{(0)} \tag{3.24}$$

with $\hat{H}_1$ in (3.12), and the λ_i's are given by the eigenvalues of $\hat{H}_0$, see (2.1). Then the mass eigenstate in matter ν_i can be written to first order in ϵ as:

$$\begin{pmatrix} \nu_- \\ \nu_0 \\ \nu_+ \end{pmatrix} = \begin{pmatrix} 1 & \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 0 \\ -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 1 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} \\ 0 & \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} & 1 \end{pmatrix} \begin{pmatrix} \hat{\nu}_-^{(0)} \\ \hat{\nu}_0^{(0)} \\ \hat{\nu}_+^{(0)} \end{pmatrix}.$$

Using (3.23), this equation is of the form $\nu_i = V^\dagger\nu_\alpha$ which can be inverted to easily obtained the V-matrix given in eq. (2.7),

$$\begin{aligned} V &= U_{23}U_\phi \begin{pmatrix} 1 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 0 \\ \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 1 & \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} \\ 0 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} & 1 \end{pmatrix}, \\ &= U_{23}U_\phi \\ &+\epsilon c_{12}s_{12}\Delta m^2_{\rm ren}\left\{\frac{c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0}U_{23}U_\phi\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0}U_{23}U_\phi\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\right\}. \end{aligned} \tag{3.25}$$

This can be used to directly compute the oscillation probabilities as was performed in section (2.3) or by using the V matrix the oscillation probabilities for $\nu_\beta \to \nu_\alpha$ in matter is given (under the adiabatic approximation) as

$$\begin{aligned} P(\nu_\beta \to \nu_\alpha) &= \left| \sum_i V_{\alpha i}V^*_{\beta i}\, e^{-i\frac{\lambda_i L}{2E}} \right|^2 \\ &= \delta_{\alpha\beta} - 4\sum_{j>i}{\rm Re}[V_{\alpha i}V^*_{\beta i}V^*_{\alpha j}V_{\beta j}]\sin^2\frac{(\lambda_j-\lambda_i)L}{4E} - 2\sum_{j>i}{\rm Im}[V_{\alpha i}V^*_{\beta i}V^*_{\alpha j}V_{\beta j}]\sin\frac{(\lambda_j-\lambda_i)L}{2E} \end{aligned} \tag{3.26}$$

where $i = (-, 0, +)$. This V matrix method was used to calculate the matter effect correction in the oscillation probabilities [14].

3.5 The unique oscillation probability formulas and their meaning

We have sketched the two different methods for calculating the oscillation probabilities, the S-matrix method (section 3.3) and the V-matrix methods (section 3.4). The results obtained by the both methods agree with each other. A careful choice of the eigenvalues and the mixing angle ϕ in sections 2.1 and 2.2 allows us the unified expressions of the oscillation probabilities for both NO and IO. They are presented in the two different styles: In appendix B we present all of them in a manner loyal to the general expression (3.26), which is convenient for demonstrating unitarity. In section 2, we already presented them in a way manifestly free from the spurious singularity at $1/(\lambda_- - \lambda_0)$, and showing explicit dependence on the eigenvalues and the mixing angles.[6] The formulas useful to show the equivalence between the general expressions of the oscillation probabilities in appendix B and the ones given in section 2 will be provided in appendix C.

It turned out that this property, the agreement between the results obtained by the V and S matrix methods, is *highly nontrivial*. That is, we find that in most formulations of perturbative framework they don't agree with each other. As a concrete example, we briefly describe what happens if we formulate a perturbation theory by using the simplest zeroth-order basis, the first term in (3.9). One can calculate the V matrix elements as we did in the previous subsection, and obtain the oscillation probabilities by inserting them into (3.26). But, they do not agree with the oscillation probabilities calculated by using the S matrix method in an obvious way. Namely, the latter contains the types of terms that do not exist in the general form (3.26). A clearest example is the presence of the term proportional to $\sin \frac{(\lambda_j - \lambda_i)L}{2E}$ in $P(\nu_e \to \nu_e)$, which is absent in (3.26) because of no CP violation in $P(\nu_e \to \nu_e)$.

But, then how the oscillation probabilities calculated by the two method can agree with each other? It occurs if one can redefine the zeroth-order eigenvalues such that the extra terms in the oscillation probabilities can be absorbed into the canonical terms which exist in the general formula (3.26); This is the meaning of the agreement between the oscillation probabilities calculated by using the S matrix as well as V matrix methods. Our result demonstrates that it can happen to order ϵ. It is one of the nicest features of our renormalized helio-perturbation theory that it occurs automatically. Apparently, our framework with the normalized zeroth-order basis in (3.9) is the unique case which possesses this property, as any other choice of zeroth order eigenvalues will generate additional, non-canonical terms in the oscillation probabilities.

[6] In section 2, we made the structure of δ dependent terms transparent by exhibiting their Jarlskog factor [15] dependence explicit. Notice that the $\sin\delta$ terms must come with the Jarlskog factor J_r defined in (2.15) [16, 17]. Whereas the $\cos\delta$ terms must be accompanied with a further reduced Jarlskog factor $c_{12}s_{12}c_{23}s_{23}s_{13}$ [4, 18].

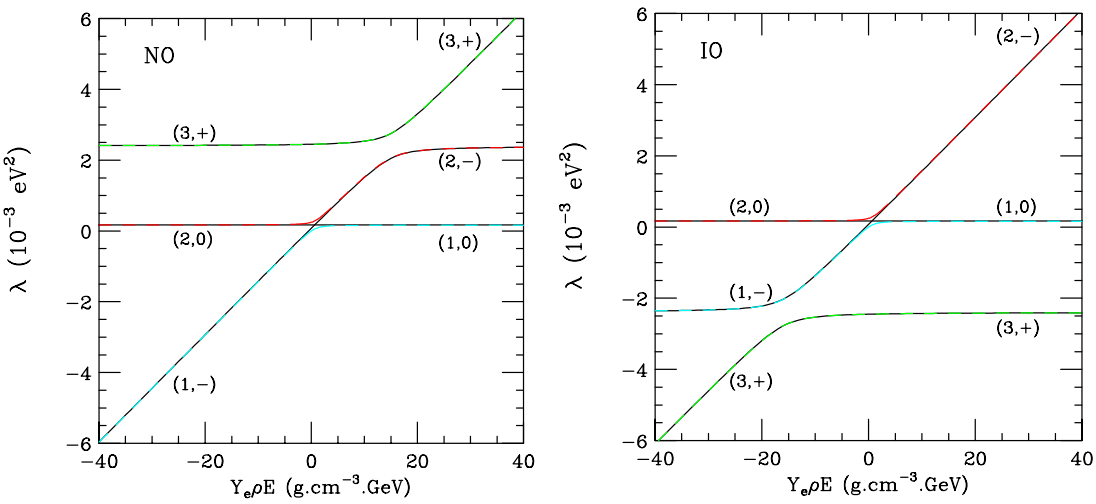

Figure 2. The flow of the three eigenvalues in the normal mass ordering, abbreviated as NO, (left panel) and in the inverted mass ordering, abbreviated as IO, (right panel). The exact eigenvalues are depicted with colored lines, green for ν_1, red for ν_2, and flight blue for ν_3. The eigenvalues calculated by using our renormalized helio-perturbation theory are drawn by the black dashed lines whose state labels are marked on the figures. The approximate eigenvalues for ν_0 and ν_- cross at the solar resonance, whereas the exact eigenvalues for ν_1 and ν_2 repelled at solar resonance. To make these features visible we used a solar Δm^2_{21} three times as large as the measured value.

4 More about the renormalized helio-perturbation theory

In this section, we critically examine the framework of the renormalized helio-perturbation theory. Despite a drawback of the current framework (which is to be described below) we argue that our perturbation theory works apart from the vicinity of the solar resonance crossing.

4.1 Exact versus zeroth-order eigenvalues in matter

The three eigenvalues of the Hamiltonian are written as $\frac{\lambda_i}{2E}$, where i runs over $1, 2, 3$ for the exact eigenvalues, and $i = -, 0, +$ for the zeroth-order eigenvalues in our perturbative framework. In Figure 2 the λ_i are plotted as a function of a, the Wolfenstein matter potential [1] for both the exact and the zeroth-order ones given by eq. (2.1). It is clear in Fig. 2 that our zeroth-order eigenvalues fail to treat the solar-Δm^2 scale level crossing correctly. As one goes through the solar resonance in the exact solution the two eigenvalues involved, the red and green, repel one another, whereas in our perturbative solution the two corresponding eigenvalues cross with each other.[7]

We first note that, despite the feature, the atmospheric and solar resonances occur at the correct values of the matter potential with our zeroth-order eigenvalues. The atmospheric resonance occurs when $\lambda_+ - \lambda_-$ is a minimum, which is at

$$a = \Delta m^2_{\rm ren} \cos 2\theta_{13}$$

[7] We note, however, that this is the common feature possessed by the similar perturbative treatments of neutrino oscillation available in the market. In this section we argue, for the first time, that despite the drawback, our perturbative framework successfully treat the flavor content of these two states at reasonably far from the solar resonance point.

and the minimum difference between λ_+ and λ_- is $\Delta m^2_{\rm ren}\sin 2\theta_{13}$ as expected. The solar resonance occurs when $\lambda_- - \lambda_0$ is a minimum; this occurs when

$$a = \epsilon \Delta m^2_{\rm ren}\cos 2\theta_{12}/\cos^2\theta_{13}$$

and the minimum difference is zero! This is not the value determined by the full Hamiltonian which is $\approx \epsilon\Delta m^2_{\rm ren}\sin 2\theta_{12}$. Therefore, while our perturbative scheme treats the atmospheric resonance correctly to order ϵ, it misses the effects of the solar resonance.

4.2 Eigenvalues in vacuum and in the asymptotic regions $a \to \pm\infty$

In this and the next subsections, we will give the arguments to indicate that our renormalized helio-perturbation theory works apart from the vicinity of the solar resonance despite the issue mentioned above.

We first show that the zeroth-order eigenvalues given in (2.1) agrees with the exact ones to order ϵ in the asymptotic regions $a \to \pm\infty$. We use the characteristic equation of the full Hamiltonian (3.5) to derive ($i = 1, 2, 3$)

$$\begin{aligned}
\sum_i \lambda_i &= \left(a + \Delta m^2_{31} + \Delta m^2_{21}\right), \\
\sum_{i,j} \lambda_i\lambda_j &= \Delta m^2_{21}\Delta m^2_{31} + a\left\{(c^2_{12} + s^2_{12}s^2_{13})\Delta m^2_{21} + c^2_{13}\Delta m^2_{31}\right\}, \\
\lambda_1\lambda_2\lambda_3 &= c^2_{12}c^2_{13}a\Delta m^2_{21}\Delta m^2_{31}.
\end{aligned} \tag{4.1}$$

Then, by using the asymptotic expansion of λ's one can obtain to leading order in the $1/a$ expansion (at $a \to +\infty$ in the NO case)

$$\begin{aligned}
\lambda_2 &= \quad c^2_{13}\Delta m^2_{31} + s^2_{12}s^2_{13}\Delta m^2_{21} &&= (c^2_{13} + s^2_{12}\epsilon)\Delta m^2_{\rm ren}, \\
\lambda_1 &= \qquad\qquad c^2_{12}\Delta m^2_{21} &&= c^2_{12}\epsilon\Delta m^2_{\rm ren}, \\
\lambda_3 &= a + s^2_{13}\Delta m^2_{31} + s^2_{12}c^2_{13}\Delta m^2_{21} &&= a + (s^2_{13} + s^2_{12}\epsilon)\Delta m^2_{\rm ren}.
\end{aligned} \tag{4.2}$$

At $a \to -\infty$, λ_+ and λ_- must be interchanged. They are identical to the ones computed with our zeroth-order eigenvalues given in (2.1):

$$\begin{pmatrix} \lambda_- \\ \lambda_0 \\ \lambda_+ \end{pmatrix}^{NO}_{a=+\infty} = \begin{pmatrix} (c^2_{13} + s^2_{12}\epsilon)\Delta m^2_{\rm ren} \\ c^2_{12}\epsilon\Delta m^2_{\rm ren} \\ a + (s^2_{13} + s^2_{12}\epsilon)\Delta m^2_{\rm ren} \end{pmatrix} \tag{4.3}$$

$$\begin{pmatrix} \lambda_- \\ \lambda_0 \\ \lambda_+ \end{pmatrix}^{NO}_{a=-\infty} = \begin{pmatrix} a + (s^2_{13} + s^2_{12}\epsilon)\Delta m^2_{\rm ren} \\ c^2_{12}\epsilon\ \Delta m^2_{\rm ren} \\ (c^2_{13} + s^2_{12}\epsilon)\Delta m^2_{\rm ren} \end{pmatrix} \tag{4.4}$$

For the IO, the analytic expressions of λ_i ($i = -, 0, +$) at $a \to \pm\infty$ is the same as those of λ_i at $a \to \mp\infty$ for the case of NO.[8]

[8] In fact, the first line in (4.1) holds exactly.

When the matter potential vanishes, $a = 0$, the mass squared eigenvalues are given for the both mass orderings by

$$\begin{pmatrix} \lambda_- \\ \lambda_0 \\ \lambda_+ \end{pmatrix}^{NO,IO}_{a=0} = \begin{pmatrix} s_{12}^2 \epsilon \Delta m^2_{\rm ren} \\ c_{12}^2 \epsilon \Delta m^2_{\rm ren} \\ \Delta m^2_{31} \end{pmatrix} \tag{4.5}$$

whose order ϵ terms are different from the exact values, $\lambda_1 = 0$, $\lambda_2 = \Delta m^2_{21}$, and $\lambda_3 = \Delta m^2_{31}$. To understand the meaning of the failure at order ϵ we need to discuss what happens to the flavor content of the matter mass eigenstates as the matter potential changes from below to above the solar resonance. This will be done in the next subsection.

Similarly, the asymptotic behaviour of the angle ϕ, i.e., θ_{13} in matter can be easily worked out. With the definition of ϕ in (2.5), it is easy to show that ϕ takes on the following values as a is varied from $-\infty$ to $+\infty$. In the case of normal mass ordering,

$$\phi_{NO} = \begin{cases} 0, & a = -\infty \\ \theta_{13}, & a = 0 \\ \frac{\pi}{4}, & a = \Delta m^2_{\rm ren} \cos 2\theta_{13} \\ \frac{\pi}{2} - \theta_{13}, & a = 2\Delta m^2_{\rm ren} \cos 2\theta_{13} \\ \frac{\pi}{2}, & a = +\infty. \end{cases} \tag{4.6}$$

while in the case of inverted mass ordering,

$$\phi_{IO} = \begin{cases} \frac{\pi}{2}, & a = -\infty \\ \frac{\pi}{2} - \theta_{13}, & a = 2\Delta m^2_{\rm ren} \cos 2\theta_{13} \\ \frac{\pi}{4}, & a = \Delta m^2_{\rm ren} \cos 2\theta_{13} \\ \theta_{13} & a = 0 \\ 0, & a = +\infty. \end{cases} \tag{4.7}$$

It reflects a natural view that physics at $a \to +\infty$ for the normal mass ordering corresponds to the one at $a \to -\infty$ for the inverted mass ordering at least to leading order in ϵ.

4.3 Neutrino mixing matrix in matter and the ν flavor content at $a \to \pm\infty$

Since the two levels cross at the solar resonance in our perturbative treatment, one may expect that our treatment fails completely beyond the solar resonance, i.e., in the region with matter density higher than the resonance. However, we will show in this subsection that the flavor contents of the three eigenstates are correctly reproduced at least in the asymptotic region. That is, the zeroth-order V matrix describes correctly the asymptotic behaviour of the exact eigenstates in matter.

Suppose we denote the exact flavor mixing matrix in matter as the (almost standard) U matrix defined in (3.6). Let us discuss the case of NO first. At $a \to -\infty$ U is nothing but $V^{(0)}$ in (2.8). Notice that s_{23} in matter is frozen to its vacuum value for $\epsilon < 0.1$, and $s_{12} \simeq 0$ at $a \to -\infty$ [3]. At $a \to +\infty$, $s_{12} \simeq 1$ and $c_{12} \simeq 0$ the matter U matrix is identical to (2.8) if we interchange ν_1 and ν_2 apart from re-phasing factor -1 for the new second mass eigenstate.

To make the meaning of this feature clearer we write down here the flavor content of the states ν_i $(i = 1, 2, 3)$ at $a \to \pm\infty$. Noticing that $\nu_i = (V^\dagger)_{i\alpha}\nu_\alpha$, the flavor composition at $a \to -\infty$ is given at zeroth order by

$$\begin{aligned}
\nu_1 &= c_\phi \nu_e - s_\phi(s_{23}e^{-i\delta}\nu_\mu + c_{23}\nu_\tau) = \nu_- \\
\nu_2 &= \{c_{23}\nu_\mu - s_{23}e^{i\delta}\nu_\tau\} = \nu_0 \\
\nu_3 &= s_\phi \nu_e + c_\phi(s_{23}e^{-i\delta}\nu_\mu + c_{23}\nu_\tau) = \nu_+
\end{aligned} \tag{4.8}$$

Whereas the composition at $a \to +\infty$ is given by

$$\begin{aligned}
\nu_1 &= -\{c_{23}\nu_\mu - s_{23}e^{i\delta}\nu_\tau\} = -\nu_0 \\
\nu_2 &= c_\phi \nu_e - s_\phi(s_{23}e^{-i\delta}\nu_\mu + c_{23}\nu_\tau) = \nu_- \\
\nu_3 &= s_\phi \nu_e + c_\phi(s_{23}e^{-i\delta}\nu_\mu + c_{23}\nu_\tau) = \nu_+
\end{aligned} \tag{4.9}$$

The flavor compositions given in (4.8) and (4.9) imply that the flavor content of the lower two mass eigenstate in matter is correctly described in our perturbative framework despite the failure of describing the solar level crossing. Note, the combinations of ν_μ and ν_τ in the $(\cdots)$ and $\{\cdots\}$ are orthogonal in the above expressions.

In the case of IO, essentially the same discussion goes through. The asymptotic behavior of θ_{12} at $a \to \pm\infty$ is the same as that in NO. The asymptotic behavior of θ_{13} at $a \to \pm\infty$ for NO is mapped into the one at $a \to \mp\infty$ for IO. But, this is already taken care of by the definition of ϕ given in (2.5). Therefore, the same U matrix in matter as the one in NO are obtained at $a \to \pm\infty$. Then, the same flavor compositions as in (4.8) and (4.9) follow for IO. Again it is consistent with those we expect from the level crossing diagram.

We add that our ν_+ state always corresponds to ν_3 state. At $a \to +\infty$ in NO and at $a \to -\infty$ in IO the electron neutrino component is all in this $\nu_+ = \nu_3$ state. At $a \to -\infty$ in NO and at $a \to +\infty$ in IO the electron neutrino component is all in ν_- (ν_1 for NO, ν_2 for IO) state. In vacuum for both NO and IO, $V^{(0)}_{e+} = s_\phi = s_{13}$, which is the correct value for the $\nu_3 = \nu_+$-state.

To summarize: Despite that the eigenvalues calculated by our helio-perturbation theory do not show the correct behavior at around the solar level crossing, the flavor composition of the states are correctly represented by our zeroth order states. Therefore, we believe that our perturbative framework correctly describe the system of three flavor mixing in matter for both NO and IO apart from the vicinity of the solar resonance.

5 Concluding remarks

In this paper, we have derived compact expressions of the neutrino oscillation probabilities in the standard three-flavor framework, which are valid to order $\epsilon \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{\rm ren}} \simeq \frac{\Delta m^2_{21}}{\Delta m^2_{31}}$. We believe that our formulas have the simplest possible forms among the available ones in the market derived by using various perturbative frameworks. Though extremely simple in their forms, they keep all-order effects of s_{13} and matter potential a, having ϵ as the only expansion parameter.

To derive the compact formulas we have developed a perturbative framework dubbed as the "renormalized helio-perturbation theory". It differs from the perturbative framework discussed before mainly on the following two points:

- We have used the renormalized basis (3.10) with an order ϵ corrected atmospheric mass squared difference, $\Delta m^2_{\rm ren}$, whose zeroth-order eigenvalues agree with the exact ones to order ϵ in the asymptotic regions of the matter potential $a \to \pm\infty$.
- We have presented two ways of computing the oscillation probabilities, the S-matrix and the V-matrix methods described, respectively, in sections 3.3 and 3.4. The results obtained by the two methods agree with each other, the fact we emphasized to be highly nontrivial.

To our knowledge they represent the unique feature among the perturbative frameworks available in the market. Despite the success of our current framework in almost all regions including the one around the first oscillation maximum, it has a clear drawback. It fails to accommodate the physics at around the solar-scale resonance. This is related to the unphysical feature that the two eigenvalues (λ_- and λ_0) cross with each other at the solar resonance. We hope that we can return to this problem in the future.

As we noticed at the end of section 2.1 our renormalized $\Delta m^2_{\rm ren}$ is identical, to order ϵ, to the effective Δm^2 measurable in an (anti-) ν_e disappearance experiment in vacuum. It is a tantalizing question whether it is just a coincidence, or is an indication of something deep.

It is quite possible that the second feature of the current framework mentioned above naturally generalizes to higher orders. That is, one can demand that the oscillation probabilities of the general form in (3.26) calculated by the V-matrix method, with the well prepared eigenvalues, be correct to certain order in ϵ. Our result in this paper is the existence proof of the concept to order ϵ. Since we know that this is true in the exact form of the oscillation probabilities (assuming adiabaticity) [3], it is likely to be correct in each order in perturbation theory. It may or may not require higher order renormalization in $\Delta m^2_{\rm ren}$.

A Calculation of S matrix elements

A.1 Computation of $\hat{S}$ matrix elements

By using $\hat{H}_1$ in (3.15) and $e^{i\hat{H}_0 x} = \mathrm{diag}\left[e^{i\lambda_- x}, e^{i\lambda_0 x}, e^{i\lambda_+ x}\right]$, one can easily compute $\check{H}_1 \equiv e^{i\hat{H}_0 x}\hat{H}_1 e^{-i\hat{H}_0 x}$. Then, using eq. (3.21), the first order term of $\Omega(x)$ can be calculated as

$$\begin{aligned}\Omega_1(x) &= (-i)\int_0^x dx' \check{H}_1(x') \\ &= -i\epsilon\Delta m^2_{\rm ren} c_{12}s_{12}\begin{bmatrix} 0 & c_{(\phi-\theta_{13})}\frac{e^{i(\lambda_- - \lambda_0)x}-1}{i(\lambda_- - \lambda_0)}, & 0 \\ c_{(\phi-\theta_{13})}\frac{e^{-i(\lambda_- - \lambda_0)x}-1}{-i(\lambda_- - \lambda_0)}, & 0 & s_{(\phi-\theta_{13})}\frac{e^{-i(\lambda_+ - \lambda_0)x}-1}{-i(\lambda_+ - \lambda_0)} \\ 0 & s_{(\phi-\theta_{13})}\frac{e^{i(\lambda_+ - \lambda_0)x}-1}{i(\lambda_+ - \lambda_0)}, & 0 \end{bmatrix}\end{aligned} \tag{A.1}$$

where we have introduced the simplified notations, $c_{(\phi-\theta_{13})} \equiv \cos(\phi-\theta_{13})$, $s_{(\phi-\theta_{13})} \equiv \sin(\phi-\theta_{13})$, etc. The simplicity in the structure of (A.1) with many zeros is the mathematical reason why the expressions of neutrino oscillation probabilities are so simple in our renormalized hello perturbation theory.

The $\hat{S}$ matrix is given by $\hat{S} = e^{-i\hat{H}_0 x}\Omega = e^{-i\hat{H}_0 x}\left[\mathbf{1} + \Omega_1(x)\right]$ where $\mathbf{1}$ denotes the unit matrix. Then, the elements of $\hat{S}$ matrix are given by:

$$
\begin{aligned}
\hat{S}_{ee} &= e^{-i\lambda_- x}, \qquad \hat{S}_{\mu\mu} = e^{-i\lambda_0 x}, \qquad \hat{S}_{\tau\tau} = e^{-i\lambda_+ x}, \\
\hat{S}_{e\tau} &= \hat{S}_{\tau e} = 0, \\
\hat{S}_{e\mu} &= -i\epsilon\Delta m^2_{\rm ren} c_{12} s_{12} c_{(\phi-\theta_{13})} \frac{(e^{-i\lambda_0 x} - e^{-i\lambda_- x})}{i(\lambda_- - \lambda_0)}, \\
\hat{S}_{\mu e} &= -i\epsilon\Delta m^2_{\rm ren} c_{12} s_{12} c_{(\phi-\theta_{13})} \frac{(e^{-i\lambda_0 x} - e^{-i\lambda_- x})}{i(\lambda_- - \lambda_0)} = \hat{S}_{e\mu}, \\
\hat{S}_{\mu\tau} &= -i\epsilon\Delta m^2_{\rm ren} c_{12} s_{12} s_{(\phi-\theta_{13})} \frac{(e^{-i\lambda_0 x} - e^{-i\lambda_+ x})}{i(\lambda_+ - \lambda_0)}, \\
\hat{S}_{\tau\mu} &= -i\epsilon\Delta m^2_{\rm ren} c_{12} s_{12} s_{(\phi-\theta_{13})} \frac{(e^{-i\lambda_0 x} - e^{-i\lambda_+ x})}{i(\lambda_+ - \lambda_0)} = \hat{S}_{\mu\tau}.
\end{aligned}
\tag{A.2}
$$

A.2 The relationships between $\hat{S}$, $\tilde{S}$, and S matrices

The relationships between $\hat{S}$, $\tilde{S}$, and S matrices are summarized as

$$
\tilde{S}(x) = U_\phi \hat{S}(x) U_\phi^\dagger \qquad S(x) = U_{23}\tilde{S}(x)U_{23}^\dagger \tag{A.3}
$$

To define the notations for their elements let us write the first equation in (A.3) explicitly:

$$
\begin{bmatrix} \tilde{S}_{ee} & \tilde{S}_{e\mu} & \tilde{S}_{e\tau} \\ \tilde{S}_{\mu e} & \tilde{S}_{\mu\mu} & \tilde{S}_{\mu\tau} \\ \tilde{S}_{\tau e} & \tilde{S}_{\tau\mu} & \tilde{S}_{\tau\tau} \end{bmatrix} = \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} \hat{S}_{ee} & \hat{S}_{e\mu} & \hat{S}_{e\tau} \\ \hat{S}_{\mu e} & \hat{S}_{\mu\mu} & \hat{S}_{\mu\tau} \\ \hat{S}_{\tau e} & \hat{S}_{\tau\mu} & \hat{S}_{\tau\tau} \end{bmatrix} \begin{bmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{bmatrix} \tag{A.4}
$$

The relationships can be written by matrix elements as

$$
\begin{aligned}
\tilde{S}_{ee} &= c_\phi^2 \hat{S}_{ee} + s_\phi^2 \hat{S}_{\tau\tau} + c_\phi s_\phi \left(\hat{S}_{e\tau} + \hat{S}_{\tau e}\right), \\
\tilde{S}_{e\mu} &= c_\phi \hat{S}_{e\mu} + s_\phi \hat{S}_{\tau\mu}, \\
\tilde{S}_{e\tau} &= c_\phi^2 \hat{S}_{e\tau} - s_\phi^2 \hat{S}_{\tau e} - c_\phi s_\phi \left(\hat{S}_{ee} - \hat{S}_{\tau\tau}\right), \\
\tilde{S}_{\mu e} &= c_\phi \hat{S}_{\mu e} + s_\phi \hat{S}_{\mu\tau} = \tilde{S}_{e\mu} \\
\tilde{S}_{\mu\mu} &= \hat{S}_{\mu\mu} \\
\tilde{S}_{\mu\tau} &= -s_\phi \hat{S}_{\mu e} + c_\phi \hat{S}_{\mu\tau}, \\
\tilde{S}_{\tau e} &= -s_\phi^2 \hat{S}_{e\tau} + c_\phi^2 \hat{S}_{\tau e} - c_\phi s_\phi \left(\hat{S}_{ee} - \hat{S}_{\tau\tau}\right) = \tilde{S}_{e\tau}, \\
\tilde{S}_{\tau\mu} &= -s_\phi \hat{S}_{e\mu} + c_\phi \hat{S}_{\tau\mu} = \tilde{S}_{\mu\tau} \\
\tilde{S}_{\tau\tau} &= s_\phi^2 \hat{S}_{ee} + c_\phi^2 \hat{S}_{\tau\tau} - c_\phi s_\phi \left(\hat{S}_{e\tau} + \hat{S}_{\tau e}\right).
\end{aligned}
\tag{A.5}
$$

Similarly, by using the similar notations, S matrix elements can be written by $\tilde{S}$ matrix elements as follows:

$$\begin{aligned}
S_{ee} &= \tilde{S}_{ee}, \\
S_{e\mu} &= c_{23}\tilde{S}_{e\mu} + s_{23}e^{-i\delta}\tilde{S}_{e\tau}, \\
S_{e\tau} &= c_{23}\tilde{S}_{e\tau} - s_{23}e^{i\delta}\tilde{S}_{e\mu}, \\
S_{\mu e} &= c_{23}\tilde{S}_{\mu e} + s_{23}e^{i\delta}\tilde{S}_{\tau e} = \tilde{S}_{e\mu}(-\delta), \\
S_{\mu\mu} &= c_{23}^2\tilde{S}_{\mu\mu} + s_{23}^2\tilde{S}_{\tau\tau} + c_{23}s_{23}(e^{-i\delta}\tilde{S}_{\mu\tau} + e^{i\delta}\tilde{S}_{\tau\mu}), \\
S_{\mu\tau} &= c_{23}^2\tilde{S}_{\mu\tau} - s_{23}^2e^{2i\delta}\tilde{S}_{\tau\mu} + c_{23}s_{23}e^{i\delta}(\tilde{S}_{\tau\tau} - \tilde{S}_{\mu\mu}), \\
S_{\tau e} &= c_{23}\tilde{S}_{\tau e} - s_{23}e^{-i\delta}\tilde{S}_{\mu e} = \tilde{S}_{e\tau}(-\delta), \\
S_{\tau\mu} &= c_{23}^2\tilde{S}_{\tau\mu} - s_{23}^2e^{-2i\delta}\tilde{S}_{\mu\tau} + c_{23}s_{23}e^{-i\delta}(\tilde{S}_{\tau\tau} - \tilde{S}_{\mu\mu}) = S_{\mu\tau}(-\delta), \\
S_{\tau\tau} &= s_{23}^2\tilde{S}_{\mu\mu} + c_{23}^2\tilde{S}_{\tau\tau} - c_{23}s_{23}(e^{-i\delta}\tilde{S}_{\mu\tau} + e^{i\delta}\tilde{S}_{\tau\mu}).
\end{aligned} \tag{A.6}$$

B Expressions of neutrino oscillation probabilities

With the expressions of S matrix elements obtained in appendix 3.3 and using (3.3) it is straightforward to calculate the neutrino oscillation probabilities. Similarly, one can insert the V matrix elements given in (2.7) into (3.26) to obtain the equivalent results. In this appendix we only give the results.

The only comment worth to give here is about the simple method for transformation $c_{23} \to -s_{23}$ and $s_{23} \to c_{23}$ to obtain $P(\nu_e \to \nu_\tau)$ from $P(\nu_e \to \nu_\mu)$, or $P(\nu_\tau \to \nu_\tau)$ from $P(\nu_\mu \to \nu_\mu)$, which is utilized in section 2. Though we work with the rephased flavor mixing matrix defined in (3.6), the transformations produces $S_{e\tau}$ from $S_{e\mu}$, and $S_{\tau\tau}$ from $S_{\mu\mu}$, up to an overall phase, see eq. (A.6).

B.1 Oscillation probabilities in ν_e−row

$P(\nu_e \to \nu_e)$ is extremely simple as

$$P(\nu_e \to \nu_e) = 1 - 4c_\phi^2 s_\phi^2 \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E}. \tag{B.1}$$

Here and in the rest of this Appendix we use x as the baseline. The reasons for the simplicity is discussed in depth in section 2.3. $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ are given by

$$\begin{aligned}
&P(\nu_e \to \nu_\mu) = 4c_\phi^2 s_\phi^2 s_{23}^2 \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} \\
&+ 4\epsilon\Delta m_{\rm ren}^2 c_{12}s_{12}c_{23}s_{23}c_\phi s_\phi \cos\delta \\
&\times \left[\left\{-c_\phi c_{(\phi-\theta_{13})}\frac{1}{(\lambda_- - \lambda_0)} + s_\phi s_{(\phi-\theta_{13})}\frac{1}{(\lambda_+ - \lambda_0)}\right\}\sin^2\frac{(\lambda_+ - \lambda_-)x}{4E}\right. \\
&\quad \left. + \left\{c_\phi c_{(\phi-\theta_{13})}\frac{1}{(\lambda_- - \lambda_0)} + s_\phi s_{(\phi-\theta_{13})}\frac{1}{(\lambda_+ - \lambda_0)}\right\}\left\{\sin^2\frac{(\lambda_+ - \lambda_0)x}{4E} - \sin^2\frac{(\lambda_- - \lambda_0)x}{4E}\right\}\right] \\
&+ 2\epsilon\Delta m_{\rm ren}^2 c_{12}s_{12}c_{23}s_{23}c_\phi s_\phi \sin\delta \left\{c_\phi c_{(\phi-\theta_{13})}\frac{1}{(\lambda_- - \lambda_0)} + s_\phi s_{(\phi-\theta_{13})}\frac{1}{(\lambda_+ - \lambda_0)}\right\} \\
&\times \left[\sin\frac{(\lambda_+ - \lambda_-)x}{2E} - \sin\frac{(\lambda_+ - \lambda_0)x}{2E} + \sin\frac{(\lambda_- - \lambda_0)x}{2E}\right]
\end{aligned} \tag{B.2}$$

$$
\begin{aligned}
&P(\nu_e \to \nu_\tau) = 4c_\phi^2 s_\phi^2 c_{23}^2 \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} \\
&- 4\epsilon \Delta m^2_{\rm ren} c_{12} s_{12} c_{23} s_{23} c_\phi s_\phi \cos\delta \\
&\times \left[\left\{ -c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} + s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} \right. \\
&\left. + \left\{ c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} + s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \left\{ \sin^2 \frac{(\lambda_+ - \lambda_0)x}{4E} - \sin^2 \frac{(\lambda_- - \lambda_0)x}{4E} \right\} \right] \\
&- 2\epsilon \Delta m^2_{\rm ren} c_{12} s_{12} c_{23} s_{23} c_\phi s_\phi \sin\delta \left\{ c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} + s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \\
&\times \left[\sin \frac{(\lambda_+ - \lambda_-)x}{2E} - \sin \frac{(\lambda_+ - \lambda_0)x}{2E} + \sin \frac{(\lambda_- - \lambda_0)x}{2E} \right]
\end{aligned} \tag{B.3}
$$

It is almost trivial to verify unitarity in the ν_e-row: $P(\nu_e \to \nu_e) + P(\nu_e \to \nu_\mu) + P(\nu_e \to \nu_\tau) = 1$.

To compare with the results of section (2.3), one needs the following identity

$$
\begin{aligned}
&\sin \frac{(\lambda_+ - \lambda_-)x}{2E} - \sin \frac{(\lambda_+ - \lambda_0)x}{2E} + \sin \frac{(\lambda_- - \lambda_0)x}{2E} \\
&\qquad = 4 \sin \frac{(\lambda_+ - \lambda_-)x}{4E} \sin \frac{(\lambda_+ - \lambda_0)x}{4E} \sin \frac{(\lambda_- - \lambda_0)x}{4E}
\end{aligned} \tag{B.4}
$$

as well as the identities given in Appendix (C).

B.2 Oscillation probabilities in $\nu_\mu-$row

$P(\nu_\mu \to \nu_e)$ is related to the T-conjugate channel probability $P(\nu_e \to \nu_\mu)$ as $P(\nu_\mu \to \nu_e : \delta) = P(\nu_e \to \nu_\mu : -\delta)$, whose latter can be obtained by replacing δ by $-\delta$ in (B.2). Therefore, we only give the expressions of $P(\nu_\mu \to \nu_\mu)$ and $P(\nu_\mu \to \nu_\tau)$:

$$
\begin{aligned}
&P(\nu_\mu \to \nu_\mu) \\
&= 1 - 4s_{23}^4 c_\phi^2 s_\phi^2 \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} - 4c_{23}^2 s_{23}^2 \left[c_\phi^2 \sin^2 \frac{(\lambda_+ - \lambda_0)x}{4E} + s_\phi^2 \sin^2 \frac{(\lambda_- - \lambda_0)x}{4E} \right] \\
&+ 8\epsilon \Delta m^2_{\rm ren} c_{12} s_{12} c_{23} s_{23} \cos\delta \\
&\times \left[s_{23}^2 c_\phi s_\phi \left\{ c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} - s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} \right. \\
&- c_\phi \left\{ s_{23}^2 s_\phi c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} + \left(c_{23}^2 - s_{23}^2 c_\phi^2\right) s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_+ - \lambda_0)x}{4E} \\
&\left. + s_\phi \left\{ \left(c_{23}^2 - s_{23}^2 s_\phi^2\right) c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} + s_{23}^2 c_\phi s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_- - \lambda_0)x}{4E} \right].
\end{aligned} \tag{B.5}
$$

$$
\begin{aligned}
&P(\nu_\mu \to \nu_\tau) \\
&= 4c_{23}^2 s_{23}^2 \left[-c_\phi^2 s_\phi^2 \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} + c_\phi^2 \sin^2 \frac{(\lambda_+ - \lambda_0)x}{4E} + s_\phi^2 \sin^2 \frac{(\lambda_- - \lambda_0)x}{4E} \right] \\
&+ 4\epsilon \Delta m_{\rm ren}^2 c_{12} s_{12} c_{23} s_{23} \left(c_{23}^2 - s_{23}^2 \right) \cos\delta \\
&\times \left[c_\phi s_\phi \left\{ c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} - s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} \right. \\
&\quad - \left\{ s_\phi c_\phi^2 c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} - c_\phi \left(1 + c_\phi^2\right) s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_+ - \lambda_0)x}{4E} \\
&\quad \left. - \left\{ s_\phi \left(1 + s_\phi^2\right) c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} - c_\phi s_\phi^2 s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_- - \lambda_0)x}{4E} \right] \\
&+ 2\epsilon \Delta m_{\rm ren}^2 c_{12} s_{12} c_{23} s_{23} c_\phi s_\phi \sin\delta \left\{ c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} + s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \\
&\times \left[\sin \frac{(\lambda_+ - \lambda_-)x}{2E} - \sin \frac{(\lambda_+ - \lambda_0)x}{2E} + \sin \frac{(\lambda_- - \lambda_0)x}{2E} \right].
\end{aligned} \tag{B.6}
$$

With the above results the unitarity in $\nu_\mu-$row can also be verified: $P(\nu_\mu \to \nu_e) + P(\nu_\mu \to \nu_\mu) + P(\nu_\mu \to \nu_\tau) = 1$.

B.3 Oscillation probabilities in $\nu_\tau-$row

$P(\nu_\tau \to \nu_e)$ and $P(\nu_\tau \to \nu_\mu)$ can be given by their T-conjugate channels: $P(\nu_\tau \to \nu_\alpha : \delta) = P(\nu_\alpha \to \nu_\tau : -\delta)$. Therefore, we only give the expressions of $P(\nu_\tau \to \nu_\tau)$ below.

$$
\begin{aligned}
&P(\nu_\tau \to \nu_\tau) \\
&= 1 - 4c_{23}^4 c_\phi^2 s_\phi^2 \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} - 4c_{23}^2 s_{23}^2 \left[c_\phi^2 \sin^2 \frac{(\lambda_+ - \lambda_0)x}{4E} + s_\phi^2 \sin^2 \frac{(\lambda_- - \lambda_0)x}{4E} \right] \\
&- 8\epsilon \Delta m_{\rm ren}^2 c_{12} s_{12} c_{23} s_{23} \cos\delta \\
&\times \left[c_{23}^2 c_\phi s_\phi \left\{ c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} - s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_+ - \lambda_-)x}{4E} \right. \\
&- c_\phi \left\{ c_{23}^2 s_\phi c_\phi c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} + \left(s_{23}^2 - c_{23}^2 c_\phi^2\right) s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_+ - \lambda_0)x}{4E} \\
&\left. + s_\phi \left\{ \left(s_{23}^2 - c_{23}^2 s_\phi^2\right) c_{(\phi-\theta_{13})} \frac{1}{(\lambda_- - \lambda_0)} + c_{23}^2 c_\phi s_\phi s_{(\phi-\theta_{13})} \frac{1}{(\lambda_+ - \lambda_0)} \right\} \sin^2 \frac{(\lambda_- - \lambda_0)x}{4E} \right].
\end{aligned} \tag{B.7}
$$

The unitarity in $\nu_\tau-$row can also be verified: $P(\nu_\tau \to \nu_e) + P(\nu_\tau \to \nu_\mu) + P(\nu_\tau \to \nu_\tau) = 1$.

C Some useful Identities

Here we list some formulas which may be useful to understand the relationship between different expressions of the oscillation probabilities:

$$
\cos 2(\phi - \theta_{13}) = \frac{\Delta m_{\rm ren}^2 - a \ \cos 2\theta_{13}}{(\lambda_+ - \lambda_-)}, \qquad \sin 2(\phi - \theta_{13}) = \frac{a \ \sin 2\theta_{13}}{(\lambda_+ - \lambda_-)}. \tag{C.1}
$$

$$
\begin{aligned}
\frac{1}{(\lambda_- - \lambda_0)} c_\phi c_{(\phi-\theta_{13})} &= \frac{c_{13}}{2(\lambda_- - \lambda_0)(\lambda_+ - \lambda_-)} \left[(\lambda_+ - \lambda_-) + (\Delta m^2_{\rm ren} - a)\right] \\
\frac{1}{(\lambda_+ - \lambda_0)} s_\phi s_{(\phi-\theta_{13})} &= \frac{c_{13}}{2(\lambda_+ - \lambda_0)(\lambda_+ - \lambda_-)} \left[(\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a)\right]
\end{aligned} \tag{C.2}
$$

$$
\begin{aligned}
\frac{1}{(\lambda_+ - \lambda_0)} c_\phi s_{(\phi-\theta_{13})} &= \frac{-s_{13}}{2(\lambda_+ - \lambda_0)(\lambda_+ - \lambda_-)} \left[(\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} + a)\right] \\
&= \frac{s_{13}}{(\lambda_+ - \lambda_0)(\lambda_+ - \lambda_-)} \left[\lambda_- - s^2_{12}\epsilon\Delta m^2_{\rm ren}\right] \\
\frac{1}{(\lambda_- - \lambda_0)} s_\phi c_{(\phi-\theta_{13})} &= \frac{s_{13}}{2(\lambda_- - \lambda_0)(\lambda_+ - \lambda_-)} \left[(\lambda_+ - \lambda_-) + (\Delta m^2_{\rm ren} + a)\right] \\
&= \frac{s_{13}}{(\lambda_- - \lambda_0)(\lambda_+ - \lambda_-)} \left[\lambda_+ - s^2_{12}\epsilon\Delta m^2_{\rm ren}\right]
\end{aligned} \tag{C.3}
$$

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