

# A MODEL RING WITH EXACTLY SOLVABLE NONLINEAR MOTION

T. Zolkin\*, The University of Chicago, Department of Physics, Chicago, IL 60637–1434, USA  
 Y. Kharkov, I. Morozov†, BINP SB RAS, Novosibirsk 630090, Russia  
 S. Nagaitsev‡, FNAL, Batavia, IL 60510–5011, USA

## Abstract

Recently, a concept of nonlinear accelerator lattices with two analytic invariants has been proposed [1]. Based on further studies [3], the Integrable Optics Test Accelerator (IOTA) was designed and is being constructed at FNAL. Despite the clarity and transparency of the proposed idea, the detailed analysis of the beam motion remains quite complicated and should be understood better even for the case when no perturbations are taken into account. In this paper we will review one of the three proposed realizations of integrable optics, where the variables separation is possible in polar coordinates. This system allows for an exact analytical solution expressed in terms of elliptic integrals and Jacobi elliptic functions [2]. It gives the possibility to check numerical algorithms used for tracking and to perform more rigorous analysis of the motion in comparison with the "crude" analysis of the topology of the phase space. In addition we will discuss some difficulties associated with numerical simulations of such a comparatively complex dynamical system and will take a look at the possible perturbations for a model machine.

## INTRODUCTION

Below we will consider the design of the accelerator lattice which includes nonlinear lens, while the transverse motion remains integrable. Several such systems possessing a second invariant of motion, which is quadratic in momentum, have been proposed in [1]. One of the possible realization is based on the application of a point-like magnetic quadrupole inserted into the center of a vacuum pipe. In this case, the motion allows the variables separation in polar coordinates and its analytical solution was described in [2]. Here, we will focus on the implementation of this idea and will design a model accelerator ring. For more details on dynamics in such a magnetic field one should consult the list of references.

## ACCELERATOR LATTICE DESIGN

The particle dynamics in the magnetic field of point-like magnetic quadrupole is unusual in that there is no equilibrium orbit of motion; trapped particles are only in dynamical equilibrium, i.e. in constant motion in at least one degree of freedom. This feature is related to the presence of singularity at the origin. On the other hand, it means that under the action of friction force, let say caused by the

synchrotron radiation, particles will fall towards the singularity and eventually will be lost on the inner aperture of the vacuum pipe,  $\rho_{in}$ . Thus, below we will consider the design of an accelerator ring for protons, so far as the damping of oscillations due to radiation effects is negligible for them.

## Axially Symmetric Focusing

As it was described in [1], the concept of nonlinear integrable optics under consideration requires an axially symmetric focusing in the accelerator ring. A super-period of such a lattice can be realized as  $\frac{E}{2}O\frac{E}{2}$  lattice, with a drift space of length  $L$ , where the nonlinear lens is located, and an optics insert (also so-called T-insert), which is equivalent to a thin axially symmetric lens with the focal length equal to  $1/k$  (see Fig. 1).

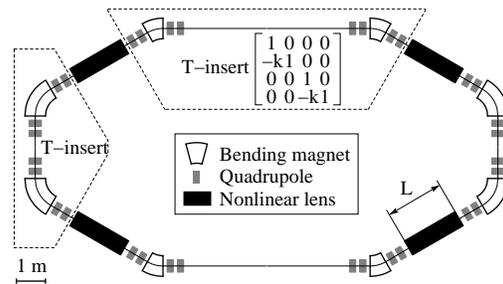


Figure 1: Schematic layout of the IOTA ring.

To study the transverse motion of the monochromatic beam the principal realization of the optics insert is not essential, and for all further simulations we will use a parameters designed for IOTA ring [3]. The dependence of the minimum and maximum of beta function in the drift space as well as betatron frequency as a function of the T-insert strength are show in Fig. 2. Main parameters of the ring are listed in Table 1.

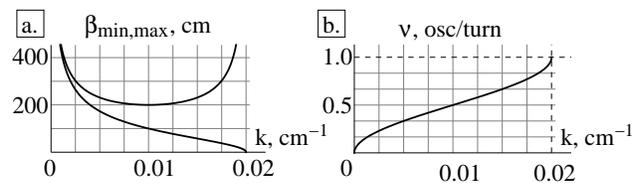


Figure 2: Minimum and maximum of beta-function (a.) and betatron frequency (b.) as a function of axially symmetric lens strength parameter,  $k$ .

\* zolkin@uchicago.edu

† imorozov@fnal.gov

‡ nsergei@fnal.gov

Table 1: IOTA ring parameters used in simulations and optimized for compatibility with nonlinear lens under consideration

Linear Lattice Parameters		
# of super-periods		4
# of nonlinear lenses		2
Circumference, $\Pi$	(m)	38.7
Bending dipole field, $B$	(T)	0.7
Drift space length, $L$	(cm)	200
T-insert strength parameter, $k$	( $\text{cm}^{-1}$ )	$\in (0; 0.02)$
Beam at the Injection		
Beam kinetic energy, $\mathcal{E}_{\text{kin}}$	(MeV)	1.91
Beam momentum, $P_{\text{eq}}$	(MeV/c)	60
Normalized emittance, $\epsilon_{\text{norm}}^{\perp}$	(cm rad)	$2 \times 10^{-5}$

### Nonlinear Lens

The design of nonlinear lenses, corresponding to integrable potentials proposed at [1], brings in two major inevitable perturbations. They are associated with the special longitudinal dependence of the field, and, the physical realization of poles of the lens. The first condition is automatically satisfied due to invariance of the potential under the transformation to normalized coordinates and time, except the effect from the fringe fields. While the second one can not be realized without introducing the perturbation, since poles can not be placed at the same point in the space.

Point-like magnetic quadrupole can be approximated with four wires of diameter  $D$ , which are spaced very closely to each other at the vertices of a square,  $\vec{R}_{\alpha}$ , and numbered counter-clockwise from the corner located at  $\vec{R}_1 = (b, 0)$  (see Fig. 3). ‘‘Dot’’ and ‘‘cross’’ symbols in the figure show the current direction: coming out and into the page (along the beam direction) respectively. In order to keep particles which are performing the libration ( $0 < W < A$ ), the vacuum chamber can be made of two nested half-cylinders of radii  $\rho_{\text{in,out}}$ , forming a c-shape.

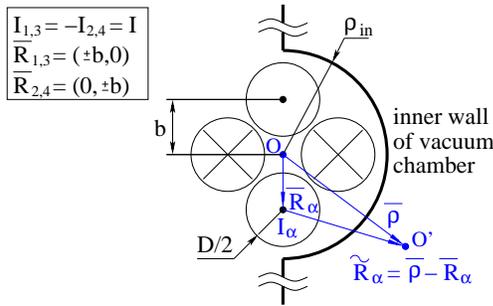


Figure 3: Schematic plot of the nonlinear lens geometry along with inner part of vacuum pipe. Points O and O' represents the origin of coordinates and the point inside the vacuum chamber where the field induced by the lens should be considered,  $\vec{\rho} = \vec{OO'}$ .

Indeed, in ordinary polar coordinates

$$\rho = \sqrt{x^2 + z^2}, \quad \theta = \arctan(z/x),$$

the longitudinal component of the magnetic vector potential  $\mathbf{A} = (0, A_s, 0)$  created by such a quadrupole structure can be easily calculated and expressed as a multipole series

$$\begin{aligned} A_s &= -\frac{\mu_0}{2\pi} \sum_{\alpha=1,2,3,4} I_{\alpha} \ln |\tilde{R}_{\alpha}| \\ &= \frac{\mu_0 I}{\pi} \sum_{n=0}^{\infty} \left(\frac{b}{\rho}\right)^{2+4n} \frac{\cos[(2+4n)\theta]}{1+2n} \\ &= \frac{\mu_0 I}{\pi} \left( b^2 \frac{\cos 2\theta}{\rho^2} + \frac{b^6 \cos 6\theta}{3 \rho^6} + O\left[\left(\frac{b}{\rho}\right)^{10}\right] \right), \end{aligned}$$

where  $\mu_0$  is a vacuum permeability. As one can see that first perturbing term corresponds to the dodecapole field which is significantly suppressed by 4 orders of magnitude in a smallness parameter ( $b/\rho$ ). It shows that the drawing apart of wires does not create a strong perturbation since all additional terms are rapidly decay with the growth of  $\rho$ . The nonlinear potential in the Hamiltonian is related to  $A_s$  as

$$V(\rho, \theta) = \frac{U(r, \theta)}{\beta(s)} = \frac{A \sin(2\theta + \varphi)}{\rho^2} = -\frac{eA_s}{P_{\text{eq}}},$$

which gives the expression for the field amplitude:

$$A = \frac{\mu_0 e I b^2}{\pi P_{\text{eq}}}.$$

Due to a presence of the physical aperture, the following inequality should be satisfied

$$\frac{\rho_{\text{in}}^4}{\beta^2(L/2)} < A < \frac{\rho_{\text{out}}^4}{\beta^2(L)}, \quad (1)$$

in order to have a suitable admittance. Among the transverse geometrical parameters of the lens there is only one independent since we want to pack all wires as tightly as we can. Thus for a given current density in a wire,  $\rho_I = 4I/\pi D^2$ , the optimization of  $A$  becomes a problem of a single parameter variation. For example, the value of the diameter of a wire allows to approximate the wire displacement and the inner radius of vacuum pipe as

$$b \approx D/\sqrt{2}, \quad \rho_{\text{in}} \approx (1 + \sqrt{2})D/2.$$

Possible choice of  $A$  is presented in Fig. 4.

For simulations we considered 2 possible scenarios: lens with water cooling and the superconducting one. Parameters for both cases are listed in a Table 2. It is worth noting that the use of high current is limited from technical limitations, while the use of low current lens leads to the shrink of admittance, which can be compensated by decreasing the energy of a beam.

In order to study the monochromatic beam motion in a lattice with perturbations, we decided to use the Poincaré

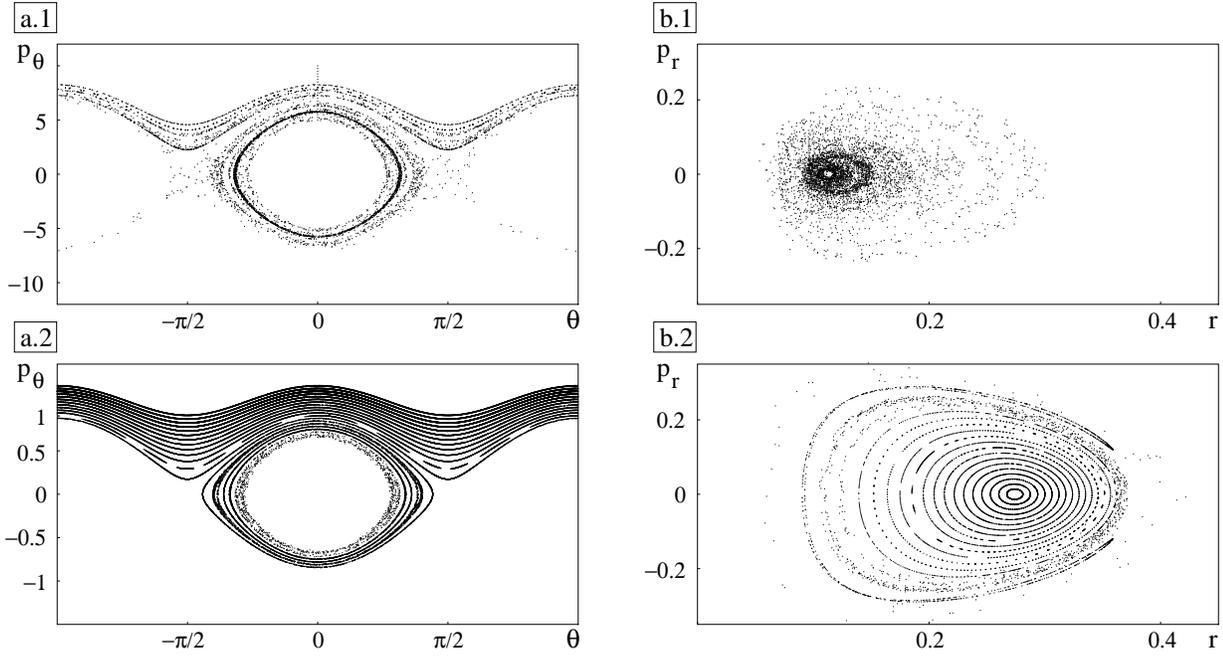


Figure 5: (a.1,2) and (b.1,2) show Poincaré surfaces of section for  $[r = r_0, p_r = 0]$  and  $[\theta, p_\theta = 0]$  respectively at the middle of nonlinear lens. Beam motion is simulated with nonlinear kick defined by 4 wires moved apart from each other. Linear lattice parameters are calculated for betatron frequency  $\nu_{x,z} = 0.44$ . Parameters of nonlinear lens are chosen in accordance to Fig. 4. Plots show presence of different types of cross sections in a phase space: (a,b.1) — strongly chaotic and (a,b.2) — close to integrable.

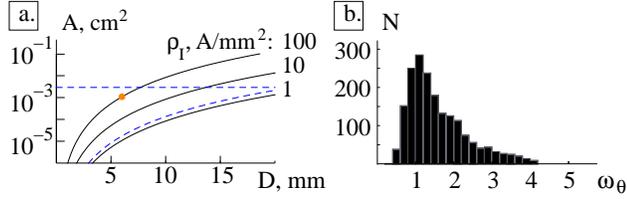


Figure 4: (a.) Nonlinear lens strength,  $A$ , dependence of the wire diameter,  $D$ , for a given values of the current density in logarithmic scale ( $\rho_I = 10 \text{ A/mm}^2$  requires water cooling of wires, while in order to keep  $\rho_I = 100 \text{ A/mm}^2$  lens should be superconducting). Blue dashed lines represent condition given by inequality from Eq. (1). Orange circle shows possible choice of  $A$  for superconducting scenario. (b.) Example of the distribution of angular frequencies in the beam filling the admittance, which determined by the choice on left figure.

surfaces of section in a phase space. A numerical example, when the perturbation of integrable optics caused by the moving apart wires in superconducting nonlinear lens, is presented in Fig. 5.

## REFERENCES

- [1] V. Danilov and S. Nagaitsev, “Nonlinear accelerator lattices with one and two analytic invariants,” *Phys. Rev. ST Accel. Beams* **13**, 084002 (2010)
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Table 2: Parameters of the nonlinear polar lens for two different values of current density: superconducting lens and the one with water cooling

	Supercond.	Water Cooling
Beam momentum		
$P_{eq}$ , (MeV/c)	60	30
Diameter of the wire		
$D$ , (mm)	6	7
Current density		
$\rho_I$ , (A/mm <sup>2</sup> )	100	10
Total current		
$I$ , (A)	2827	385
Inner radius of pipe		
$\rho_{in}$ , (cm)	0.85	1
Outer radius of pipe		
$\rho_{out}$ , (cm)	4	4
Strength of lens		
$A$ , (cm <sup>2</sup> )	$10.8 \times 10^{-4}$	$4 \times 10^{-4}$

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