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Dependence of Neutrino Mixing Angles and CP-violating Phase on Mixing Matrix Parametrizations

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We consider various neutrino mixing matrix parametrizations and the dependence of the mixing angles and CP-violating phase on the different parametrizations. The transformations of neutrino mixing parameters between various parametrizations are presented. Although the θ_{13} mixing angle is determined to be small in the conventional parametrization, we note that in several other parametrizations the values of θ_{13} are quite large. Should the value of θ_{13} turn out to be tiny in the conventional parametrization, this study suggests that other alternative mixing matrix representations would be more suitable for dertermining the value of the CP-violating phase.

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I. INTRODUCTION

Neutrino flavor oscillation has been well established by observations from experiments involving solar [1–7], reactor [8], atmospheric [9], and accelerator [10, 11] neutrinos. Central for describing neutrino oscillation phenomenology is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [12, 13]. The mixing matrix is an invariant quantity, but the parametrization of the mixing matrix can be of different forms [14–17]. The conventional parametrization for the mixing matrix of Dirac neutrinos follows the convention adopted for the quark mixing, proposed in 1984 [18] prior to the observation of neutrino oscillation.

For three active neutrinos with no sterile neutrino, the mixing matrix can be expressed as a product of three rotation matrices. We define

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix},$$

$$R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

and

$$W_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{-i\delta_{cp}} \\ 0 & -s_{23}e^{i\delta_{cp}} & c_{23} \end{pmatrix},$$

$$W_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{cp}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{cp}} & 0 & c_{13} \end{pmatrix},$$

$$W_{12} = \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_{cp}} & 0 \\ -s_{12}e^{i\delta_{cp}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where θ_{ab} and δ_{cp} are the mixing angles and CP phase, respectively, and $c_{ab} \equiv \cos\theta_{ab}$ and $s_{ab} \equiv \sin\theta_{ab}$. There are several different ways to place the δ_{cp} in the rotation matrices. We follow the standard CKM mixing matrix for the δ_{cp} placement in this work.

The conventional ordering of the mixing matrix has been taken to be the product of $R_{23} \times W_{13} \times R_{12}$. An important feature of such a parametrization is that the three mixing angles are almost decoupled for the solar, atmospheric, and reactor neutrino oscillation experiments. In particular, the solar neutrino experiments are mostly sensitive to θ_{12}, the atmospheric neutrino experiments are more susceptible to θ_{23}, and the θ_{13} angle is probed in the short-baseline reactor experiments. Another interesting feature of the conventional parametrization is that $\theta_{23} \sim 45^o$, corresponding to maximal mixing, while $\theta_{13} \sim 0^o$, signifying minimal mixing. Since this convention for parametrizing the neutrino mixing matrix was adopted prior to the extraction of mixing angles from neutrino oscillation experiments, it is interesting to examine whether or not these features would be preserved in other possible parametrizations. In this paper, we try to address several questions. First, how do the three mixing angles depend on the parametrization of the mixing matrix? Will the three mixing angles always be nicely decoupled for different oscillation experiments? How are the δ_{cp} values in different parametrizations related? Finally, is there any particular parametrization better suited for determining the value of the δ_{cp} phase?

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This paper is organized as follows. The parametrizations of the mixing matrix are presented in Section II for the case of $\delta_{cp} = 0$ and in Section III for the non-zero δ_{cp} case, respectively. The transformations of the three mixing angles and the one CP-violating phase from one parametrization to another one are also presented in Section III. A discussion on how the expressions for survival or transition probabilities depend on the mixing matrix parametrization is given in Section IV. We then present the relevant expressions for investigating δ_{cp} in various parametrizations in Section V. We show that certain parametrizations are more suitable for determining the δ_{cp} phase if θ_{13} has a very small value in the conventional parametrization. A conclusion is given in Section VI.

II. MIXING PARAMETRIZATIONS FOR $\delta_{cp} = 0$

The state of a neutrino can be expressed either in the flavor eigenstate basis, $|\nu_\alpha >$ $(\alpha = e, \mu, \tau)$, or in the mass eigenstate basis, $|\nu_k >$ $(k = 1, 2, 3)$. The transformation from mass eigenstates to flavor eigenstates is described by a unitary mixing matrix U:

$$|\nu_\alpha >= \sum_k U_{\alpha k} |\nu_k > \ . \tag{3}$$

The parametrizations of U can be acquired using three Euler angles $(\theta_{12}, \theta_{23}, \theta_{13})$ (denoted as θ_{ij}) and one CP-violating phase, δ_{cp}. One category of parametrizations for U involves rotations around three distinct axes sequentially: $R_{23}W_{13}R_{12}$, $R_{23}W_{12}R_{13}$, $R_{13}W_{23}R_{12}$, $R_{13}W_{12}R_{23}$, $R_{12}W_{23}R_{13}$, $R_{12}W_{13}R_{23}$, where $R_{23}W_{13}R_{12}$ is the conventional parametrization. In this paper, we will call these six different parametrizations as $\mathcal{A}$–$\mathcal{F}$ 'representations', respectively. The other category is for rotations around two distinct axes: $R_{12}W_{23}R_{12}$, $R_{12}W_{13}R_{12}$, $R_{13}W_{23}R_{13}$, $R_{13}W_{12}R_{13}$, $R_{23}W_{12}R_{23}$, $R_{23}W_{13}R_{23}$. It has been demonstrated that the second category can be reduced to only three independent parametrizations [15]. We here limit our consideration to the rotations around three distinct axes for mixing parametrizations.

Appendix A presents the expressions of U in terms of θ_{ij} in various representations for the special case of $\delta_{cp} = 0$. Given the elements of U, $U_{\alpha k}$, one can solve θ_{ij} for any representation, also presented in this appendix. Note that the values of θ_{ij} vary from one representation to another.

Taking the central values of the three neutrino mixing angles from [8] obtained from the conventional representation, we present the values of the three mixing angles in other representations in Table I. Again, the symbols $\mathcal{A}$–$\mathcal{F}$ denote various representations. It can be seen that parametrizations $R_{23}R_{13}R_{12}$, $R_{23}R_{12}R_{13}$, and $R_{13}R_{23}R_{12}$ produce small values of θ_{13} while parametrizations $R_{13}R_{12}R_{23}$, $R_{12}R_{23}R_{13}$, and $R_{12}R_{13}R_{23}$ generate significant non-zero values for θ_{13}. It is interesting to note that the existing neutrino oscillation data favor a small or zero value for θ_{13} when the representations $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are chosen. In contrast, large non-zero central values for θ_{13} would already have been obtained from existing data if representations $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ were chosen. The case with $\theta_{13} = 0$ and $\theta_{23} = 45^o$ in the conventional representation is also shown in Table I. Although the mixing angles in representations $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are identical in this case, they are very different when representations $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ are adopted.

III. MIXING PARAMETRIZATIONS FOR $\delta_{cp} \neq 0$

For the more general case of $\delta_{cp} \neq 0$, the transformations of $(\theta_{ij}, \delta_{cp})$ from one representation to another is more involved. In the following, we show how these transformations can be obtained.

A. Transformations between Representations

In general, the relation between two different representations, say $\mathcal{G}$ and $\mathcal{H}$ which can be any representation among $\mathcal{A}$–$\mathcal{F}$, can be expressed as

$$U = RWR(\theta_{ij}^{\mathcal{G}}, \delta_{cp}^{\mathcal{G}}) = D^L \cdot RWR(\theta_{ij}^{\mathcal{H}}, \delta_{cp}^{\mathcal{H}}) \cdot D^R. \tag{4}$$

D^L and D^R are diagonal unitary matrices given as

$$\begin{aligned} D^L &= diag\left(e^{i\Psi_{L1}}, e^{i\Psi_{L2}}, e^{i\Psi_{L3}}\right) \ , \\ D^R &= diag\left(e^{i\Psi_{R1}}, e^{i\Psi_{R2}}, e^{i\Psi_{R3}}\right) \ , \end{aligned} \tag{5}$$

where Ψ_{Li}'s and Ψ_{Rj}'s are six phases. Equations (4) and (5) show that one can obtain the elements of U in representation $\mathcal{G}$ by respectively multiplying Ψ_{Li} and Ψ_{Rj} to the i-th row and j-th column of the elements of U in representations $\mathcal{H}$. In reality, there are only five independent phases out of six. This can be done by factoring out one of the phases in D^L or in D^R and as a result only five phases remain. For instance, one can factor out $e^{i\Psi_{L2}}$ in D^L and simultaneously multiply the factor $e^{i\Psi_{L2}}$ in D^R. Consequently, D^L and D^R become

$$\begin{aligned} D^L &\Rightarrow diag\left(e^{i(\Psi_{L1}-\Psi_{L2})}, 1, e^{i(\Psi_{L3}-\Psi_{L2})}\right) \\ &\equiv diag\left(e^{i\Phi_{L1}}, 1, e^{i\Phi_{L3}}\right) \ , \\ D^R &\Rightarrow diag\left(e^{i(\Psi_{R1}+\Psi_{L2})}, e^{i(\Psi_{R2}+\Psi_{L2})}, e^{i(\Psi_{R3}+\Psi_{L2})}\right) \\ &\equiv diag\left(e^{i\Phi_{R1}}, e^{i\Phi_{R2}}, e^{i\Phi_{R3}}\right) \ . \end{aligned}$$

This is equivalent to applying a set of five arbitrary phases

$$\Phi = (\Phi_{L1}, 0, \Phi_{L3}, \Phi_{R1}, \Phi_{R2}, \Phi_{R3}) \ .$$

This suggests that only five independent phases are involved to transform $(\theta_{ij}^{\mathcal{H}}, \delta_{cp}^{\mathcal{H}})$ to $(\theta_{ij}^{\mathcal{G}}, \delta_{cp}^{\mathcal{G}})$. The solution for expressing $(\theta_{ij}^{\mathcal{G}}, \delta_{cp}^{\mathcal{G}})$ in terms of $(\theta_{ij}^{\mathcal{H}}, \delta_{cp}^{\mathcal{H}})$ is unique and is independent of the five phases. This will be illustrated in the next section using transformation from representations $\mathcal{A}$ to $\mathcal{D}$ as an example.

Rep. U	Mixing Matrix	θ_{23}	θ_{13}	θ_{12}	θ_{23}	θ_{13}	θ_{12}
$\mathcal{A}$	$R_{23}R_{13}R_{12}$	45.00	5.44	33.91	45.00	0.00	33.91
$\mathcal{B}$	$R_{23}R_{12}R_{13}$	48.65	6.55	33.74	45.00	0.00	33.91
$\mathcal{C}$	$R_{13}R_{23}R_{12}$	44.74	7.68	39.33	45.00	0.00	33.91
$\mathcal{D}$	$R_{13}R_{12}R_{23}$	52.03	-22.30	26.76	50.31	-25.43	23.24
$\mathcal{E}$	$R_{12}R_{23}R_{13}$	38.63	-25.71	45.31	35.93	-29.16	43.55
$\mathcal{F}$	$R_{12}R_{13}R_{23}$	41.57	-19.81	28.59	39.69	-23.24	25.43

TABLE I: Neutrino mixing angles in different representations for the case of $\delta_{cp} = 0$.

B. $(\theta_{ij}^{\mathcal{D}}, \delta_{cp}^{\mathcal{D}})$ Solutions

Consider the case that $(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}})$ in the conventional representation $\mathcal{A}$ are known, and one would like to solve for $(\theta_{ij}^{\mathcal{D}}, \delta_{cp}^{\mathcal{D}})$ in representation $\mathcal{D}$. Equation (4) becomes

$$U = RWR(\theta_{ij}^{\mathcal{D}}, \delta_{cp}^{\mathcal{D}}) = D^L \cdot RWR(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}}) \cdot D^R \ . \quad (6)$$

Take the D^L and D^R matrices as

$$\begin{aligned} D^L &= diag\left(e^{i\Phi_{L1}^{\mathcal{D}}}, e^{i\Phi_{L2}^{\mathcal{D}}}, e^{i\Phi_{L3}^{\mathcal{D}}}\right) \ , \\ D^R &= diag\left(e^{i\Phi_{R1}^{\mathcal{D}}}, e^{i\Phi_{R2}^{\mathcal{D}}}, 1\right) \ . \end{aligned} \quad (7)$$

There are nine independent parameters in Eq.(6), five phases in D^L and D^R, three mixing angles and one δ_{cp} phase from the mixing matrix. The solutions for the nine parameters can be obtained as follows (for derivation, see Appendix B 1).

$$\Phi_{L1}^{\mathcal{D}} + \Phi_{R1}^{\mathcal{D}} = 0 \quad \text{and} \quad \Phi_{\mathrm{L2}}^{\mathcal{D}} = 0 \ . \quad (8)$$

$$\sin^2 \Phi_{R2}^{\mathcal{D}} = \frac{(a \ \sin\delta_{cp}^{\mathcal{A}})^2}{(a \ \sin\delta_{cp}^{\mathcal{A}})^2 + (a \ \cos\delta_{cp}^{\mathcal{A}} - b)^2} \ , \quad (9)$$

where $a = s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}}$ and $b = c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}}$. The sign of $\Phi_{R2}^{\mathcal{D}}$ can be determined through the equation leading to Eq.(9). (see Appendix B 1). We also have

$$\sin^2 \Phi_{34}^{\mathcal{D}} = \frac{(a^{'} \sin\delta_{cp}^{\mathcal{A}})^2}{(a^{'} \sin\delta_{cp}^{\mathcal{A}})^2 + (a^{'} \cos\delta_{cp}^{\mathcal{A}} - b^{'})^2} \ , \quad (10)$$

where $a^{'} = c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}}$, $b^{'} = s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}}$, and $\Phi_{34}^{\mathcal{D}} \equiv \Phi_{L3}^{\mathcal{D}} + \Phi_{R1}^{\mathcal{D}}$. Again, one can readily determine the sign of $\Phi_{34}^{\mathcal{D}}$ (see Appendix B 1).

After obtaining the phases $\Phi_{R2}^{\mathcal{D}}$ and $\Phi_{34}^{\mathcal{D}}$, the three mixing angles in representation $\mathcal{D}$ can be calculated:

$$\tan\theta_{23}^{\mathcal{D}} = \frac{s_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}}}{c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \cos\Phi_{R2}^{\mathcal{D}} - s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{R2}^{\mathcal{D}})}, \quad (11)$$

$$\begin{aligned} \cos\theta_{12}^{\mathcal{D}} &= \frac{s_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}}}{s_{23}^{\mathcal{D}}} \\ &= \frac{c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \cos\Phi_{R2}^{\mathcal{D}} - s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{R2}^{\mathcal{D}})}{c_{23}^{\mathcal{D}}}, \end{aligned} \quad (12)$$

$$\sin\theta_{13}^{\mathcal{D}} = \frac{c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{34}^{\mathcal{D}}) - s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} \cos\Phi_{34}^{\mathcal{D}}}{c_{12}^{\mathcal{D}}}. \quad (13)$$

The remaining parameters can be readily determined. First,

$$\cos(\delta_{cp}^{\mathcal{A}} - \Phi_{L1}^{\mathcal{D}}) = \frac{(c_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}})^2 + (s_{13}^{\mathcal{A}})^2 - (s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}})^2}{2 c_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}} s_{13}^{\mathcal{A}}}. \quad (14)$$

Thus $\Phi_{L1}^{\mathcal{D}}$ can be calculated. Subsequently, $\Phi_{R1}^{\mathcal{D}} = -\Phi_{L1}^{\mathcal{D}}$ and $\Phi_{L3}^{\mathcal{D}} = \Phi_{34}^{\mathcal{D}} - \Phi_{R1}^{\mathcal{D}}$ can be obtained. Finally, the CP-violating phase in representation $\mathcal{D}$, $\delta_{cp}^{\mathcal{D}}$, can be calculated using the expressions associated with $\sin\delta_{cp}^{\mathcal{D}}$ in Eq.(6), listed as follows:

$$\begin{aligned} &\sin\delta_{cp}^{\mathcal{D}} \\ &= \frac{s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \sin(\delta_{cp}^{\mathcal{A}} + \eta) + c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} \sin\eta}{s_{12}^{\mathcal{D}}} \\ &= \frac{c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \sin(\delta_{cp}^{\mathcal{A}} + \xi) + s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \sin\xi}{-c_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} s_{13}^{\mathcal{D}}} \\ &= \frac{-s_{12}^{\mathcal{A}} c_{13}^{\mathcal{A}} \sin(\Phi_{L1}^{\mathcal{D}} + \Phi_{R2}^{\mathcal{D}})}{c_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}}} \\ &= \frac{s_{13}^{\mathcal{A}} \sin(\delta_{cp}^{\mathcal{A}} - \Phi_{L1}^{\mathcal{D}})}{s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}}} \\ &= \frac{c_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}} \sin\Phi_{L3}^{\mathcal{D}}}{s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} s_{13}^{\mathcal{D}}} \ , \end{aligned} \quad (15)$$

where $\eta \equiv \Phi_{L2}^{\mathcal{D}} + \Phi_{R1}^{\mathcal{D}}$ and $\xi \equiv \Phi_{L3}^{\mathcal{D}} + \Phi_{R2}^{\mathcal{D}}$. Through these five different expressions of $\sin\delta_{cp}^{\mathcal{D}}$, the consistency of the $\delta_{cp}^{\mathcal{D}}$ values can be checked.

With the expressions of $\Phi^{\mathcal{D}}$'s, one can further express $(\theta_{ij}^{\mathcal{D}}, \delta_{cp}^{\mathcal{D}})$ in terms of $(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}})$ only. As a result, the relations between $(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}})$ and $(\theta_{ij}^{\mathcal{D}}, \delta_{cp}^{\mathcal{D}})$ are decoupled from $\Phi^{\mathcal{D}}$'s. In other words, the matrices D^L and D^R only act as a bridge in the transformation of $(\theta_{ij}, \delta_{cp})$ from one representation to another. Note that δ_{cp} also plays a role in the transformation, as discussed in the next section.

In Appendix B, a detailed derivation of transforming $(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}})$ to $(\theta_{ij}^{\mathcal{D}}, \delta_{cp}^{\mathcal{D}})$ is given followed by listings of the solutions for the nine parameters used in the transformations of $(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}})$ to $(\theta_{ij}, \delta_{cp})$ in the remaining representations.

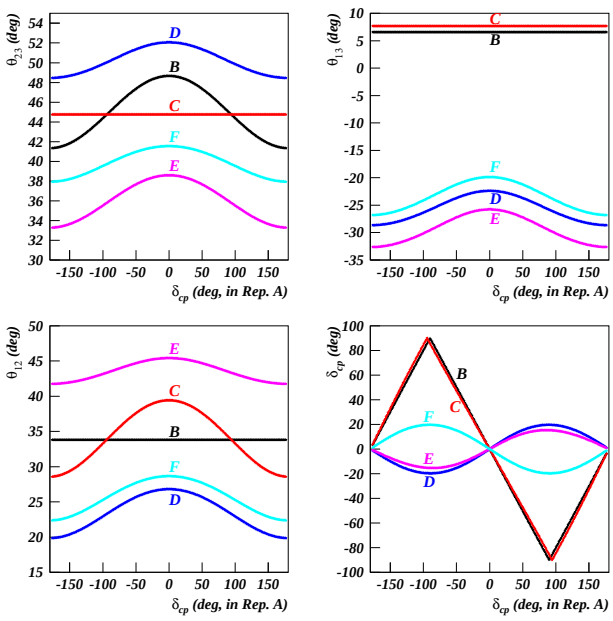

FIG. 1: Values of $(\theta_{ij},\, \delta_{cp})$ in different representations as a function of $\delta_{cp}^{\mathcal{A}}$, where $\theta_{12}^{\mathcal{A}} = 34.0^o$, $\theta_{23}^{\mathcal{A}} = 45^o$, and $\theta_{13}^{\mathcal{A}} = 5.44^o$ are assumed; curves in different colors indicate different representations.

C. Mixing Parameters in Neutrino Sector

Using the recent result of neutrino oscillation parameters [8] along with $\theta_{23}^{\mathcal{A}} = 45^o$, Fig. 1 shows how $\delta_{cp}^{\mathcal{A}}$ influences $(\theta_{ij},\, \delta_{cp})$ in other representations. As can be seen from Eqs.(B25) and (B34), $\theta_{13}^{\mathcal{B}}$ and $\theta_{13}^{\mathcal{C}}$ are independent of $\delta_{cp}^{\mathcal{A}}$. Furthermore, $\theta_{12}^{\mathcal{B}}$ and $\theta_{23}^{\mathcal{C}}$ become independent of $\delta_{cp}^{\mathcal{A}}$ after replacing the phases Φ's by $(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}})$ even though they are functions of $\delta_{cp}^{\mathcal{A}}$ explicitly [see Eqs.(B24) and (B33)]. It is interesting to note that the dependence of the new θ_{12}, θ_{13}, and θ_{23} on the old $\delta_{cp}^{\mathcal{A}}$ is symmetric with respect to $\delta_{cp}^{\mathcal{A}} = 0$, while the new δ_{cp} has an odd-dependence on the old $\delta_{cp}^{\mathcal{A}}$.

It is worth noting that the uncertainties of the mixing parameters in other representations cannot be evaluated through error propagation using the uncertainties of the mixing parameters obtained in the conventional representation. This is because the covariances of any two mixing parameters in the conventional representation are not provided. One way to obtain the uncertainties of the mixing parameters in any representation is through global fits of the experimental data using various representations. This will be reported in a separate article.

IV. TRANSITION PROBABILITY

Neutrinos and anti-neutrinos are related by a CP transformation which transforms a left-handed neutrino into a right-handed anti-neutrino. Recall Eq.(3) that characterizes the neutrino oscillation in the flavor eigenstate linking to the mass eigenstate through a unitary mixing matrix. The coefficients of the massive anti-neutrino components are simply related to the corresponding coefficients of the massive neutrino components by complex conjugation. The anti-neutrinos can thus be described by

$$\bar{\nu}_\alpha >= \sum_k U_{\alpha k}^\star |\bar{\nu}_k > \ . \tag{16}$$

The expressions for the transition probabilities of channels $\alpha \to \beta$ in vacuum for neutrinos and anti-neutrinos can be found in references, for example [17], which are listed as follows:

$$\begin{aligned} & P_{\nu_\alpha \to \nu_\beta}(L, E_\nu) = \delta_{\alpha\beta} \\ & -4 \sum_{k>j} Re\left[U_{\alpha k} U_{\beta k}^\star U_{\alpha j}^\star U_{\beta j}\right] \sin^2 \Delta_{kj} \\ & +2 \sum_{k>j} Im\left[U_{\alpha k} U_{\beta k}^\star U_{\alpha j}^\star U_{\beta j}\right] \sin 2\Delta_{kj} \ , \end{aligned} \tag{17}$$

$$\begin{aligned} & P_{\bar{\nu}_\alpha \to \bar{\nu}_\beta}(L, E_\nu) = \delta_{\alpha\beta} \\ & -4 \sum_{k>j} Re\left[U_{\alpha k} U_{\beta k}^\star U_{\alpha j}^\star U_{\beta j}\right] \sin^2 \Delta_{kj} \\ & -2 \sum_{k>j} Im\left[U_{\alpha k} U_{\beta k}^\star U_{\alpha j}^\star U_{\beta j}\right] \sin 2\Delta_{kj} \ , \end{aligned} \tag{18}$$

where $\Delta_{kj} = \Delta m_{kj}^2 L/4E_\nu \equiv (m_k^2 - m_j^2)L/4E_\nu$ with L and E_ν being the distance traveled and energy of neutrinos, respectively. The difference between these two transition probabilities appears only in the sign of the imaginary parts that are quartic products of the elements of the mixing matrix.

Various neutrino oscillation experiments utilize different sources of neutrinos and measure survival or transition probabilities. Solar and reactor neutrino experiments observe survival probabilities of ν_e or $\bar{\nu}_e$, whereas atmospheric and accelerator neutrino experiments study ν_μ disappearance, ν_e appearance [19, 20], and ν_τ appearance [21]. Given a neutrino oscillation channel, some mixing parametrizations support simple probability forms, while others give complicated expressions. For the channel of $\nu_e \to \nu_e$, the survival probability in representation $\mathcal{A}$ is

$$\begin{aligned} & P_{\nu_e \to \nu_e}^{\mathcal{A}} = \\ & 1 - (c_{13}^{\mathcal{A}})^4 \sin^2 2\theta_{12}^{\mathcal{A}} \sin^2 \Delta_{21} - (c_{12}^{\mathcal{A}})^2 \sin^2 2\theta_{13}^{\mathcal{A}} \sin^2 \Delta_{31} \\ & -(s_{12}^{\mathcal{A}})^2 \sin^2 2\theta_{13}^{\mathcal{A}} \sin^2 \Delta_{32} \ , \end{aligned} \tag{19}$$

while the survival probability in representation $\mathcal{D}$, for example, is

$$\begin{aligned} & P_{\nu_e \to \nu_e}^{\mathcal{D}} = 1 - \\ & 4\left\{(c_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}} - s_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}})^2 (c_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}})^2 \sin^2 \Delta_{21}\right. \\ & +(s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}} + c_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}})^2 (c_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}} - s_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}})^2 \sin^2 \Delta_{32} \\ & \left.+(s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}} + c_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}})^2 (c_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}})^2 \sin^2 \Delta_{31}\right\} , \end{aligned} \tag{20}$$

which is considerably more complicated than Eq.(19). Another example is the channel $\nu_\mu \to \nu_\mu$ studied in the atmospheric or accelerator experiments. In representation $\mathcal{D}$,

$$P^{\mathcal{D}}_{\nu_\mu\to\nu_\mu} = 1 - (c^{\mathcal{D}}_{23})^2 \sin^2 2\theta^{\mathcal{D}}_{12} \sin^2\Delta_{21} - (s^{\mathcal{D}}_{23})^2 \sin^2 2\theta^{\mathcal{D}}_{12} \sin^2\Delta_{31} - (c^{\mathcal{D}}_{12})^4 \sin^2 2\theta^{\mathcal{D}}_{23} \sin^2\Delta_{32}, \quad (21)$$

which is a simple expression, while in representation $\mathcal{A}$,

$$P^{\mathcal{A}}_{\nu_\mu\to\nu_\mu} = 1 - 4\left\{(s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13} - c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12})^2(s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13} + c^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12})^2 \sin^2\Delta_{21} + (s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13} - c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12})^2(s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{13})^2 \sin^2\Delta_{32} + (s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13} + c^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12})^2(s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{13})^2 \sin^2\Delta_{31}\right\}, \quad (22)$$

which is more complicated than that in representation $\mathcal{D}$. However, in the limit of $\Delta_{31} \sim \Delta_{32} \gg \Delta_{21}$ and $\theta^{\mathcal{A}}_{13} \sim 0$, Eq.(22) can be reduced to

$$P^{\mathcal{A}}_{\nu_\mu\to\nu_\mu} = 1 - \sin^2 2\theta^{\mathcal{A}}_{23} \sin^2\Delta_{31} \ . \quad (23)$$

In the limit of $\delta_{cp} = 0$, Table II lists representations in which the survival or transition probabilities possess simpler forms in vacuum, whose full expressions are given in Appendix C.

Probability	Representations
$P(\nu_e \to \nu_e)$	$\mathcal{A}$, $\mathcal{B}$
$P(\nu_\mu \to \nu_\mu)$	$\mathcal{C}$, $\mathcal{D}$
$P(\nu_\tau \to \nu_\tau)$	$\mathcal{E}$, $\mathcal{F}$
$P(\nu_\mu \to \nu_e)$	$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$
$P(\nu_\mu \to \nu_\tau)$	$\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$

TABLE II: List of representations in which the survival or transition probabilities possess simpler forms in vacuum.

V. CP ASYMMETRY

For the case of $\delta_{cp} \neq 0$, the mixing matrix is complex and leads to a violation of CP symmetry. Such a violation can be revealed by probing the CP asymmetry, $A^{cp}_{\alpha\beta}$, in neutrino oscillation experiments:

$$A^{cp}_{\alpha\beta} = P_{\nu_\alpha\to\nu_\beta} - P_{\bar\nu_\alpha\to\bar\nu_\beta} \ . \quad (24)$$

According to Eqs.(17) and (18), the CP asymmetry can be acquired readily:

$$A^{cp}_{\alpha\beta} = 4\sum_{k>j} Im\left[U_{\alpha k}U^\star_{\beta k}U^\star_{\alpha j}U_{\beta j}\right] \sin 2\Delta_{kj} \ . \quad (25)$$

This expression confirms that the CP asymmetry can be probed only in the transitions between different flavors since the imaginary part in Eq.(25) vanishes if $\alpha = \beta$. For the oscillation channels $\nu_\mu \to \nu_e$ and $\nu_\mu \to \nu_\tau$, the CP asymmetry in representation, say $\mathcal{S}$, can be formulated as

$\theta^{\mathcal{A}}_{13}$	$\delta^{\mathcal{A}}_{cp}$	$\delta^{\mathcal{B}}_{cp}$	$\delta^{\mathcal{C}}_{cp}$	$\delta^{\mathcal{D}}_{cp}$	$\delta^{\mathcal{E}}_{cp}$	$\delta^{\mathcal{F}}_{cp}$
1.0	5.0	-5.001	-4.933	0.319	0.251	-0.317
	50.0	-50.008	-49.398	2.798	2.189	-2.790
	-35.0	35.007	34.554	-2.096	-1.643	2.089
5.44	5.0	-5.041	-4.722	1.778	1.460	-1.737
	50.0	-50.229	-47.256	15.358	12.384	-15.128
	-35.0	35.219	33.060	-11.590	-9.431	11.371
9.44	5.0	-5.123	-4.638	3.209	2.725	-3.087
	50.0	-50.685	-46.015	26.882	22.212	-26.197
	-35.0	35.658	32.324	-20.559	-17.225	19.902
15.0	5.0	-5.310	-4.661	5.556	4.907	-5.246
	50.0	-51.699	-45.232	43.239	36.772	-41.540
	-35.0	36.646	32.108	-34.047	-29.508	32.401

TABLE III: Expected δ_{cp} values in various representatsions for given $(\theta^{\mathcal{A}}_{13}, \delta^{\mathcal{A}}_{cp})$, where $\theta^{\mathcal{A}}_{12} = 33.91$ and $\theta^{\mathcal{A}}_{23} = 45$ are used. Note that all numbers in this table are in degrees.

$$A^{cp,\mathcal{S}}_{\alpha,\beta} = \kappa \cdot \frac{1}{2}\mathcal{J}'^{\mathcal{S}} \cdot \sin\delta^{\mathcal{S}}_{cp} \cdot \chi \ , \quad (26)$$

where

$$\mathcal{J}'^{\mathcal{S}} = \sin 2\theta^{\mathcal{S}}_{23} \cdot \sin 2\theta^{\mathcal{S}}_{12} \cdot \sin 2\theta^{\mathcal{S}}_{13} \cdot \cos\theta^{\mathcal{S}}_{jk} \ ,$$

and

$$\chi = \sin\left(\frac{\Delta m^2_{32}L}{2E_\nu}\right) + \sin\left(\frac{\Delta m^2_{13}L}{2E_\nu}\right) + \sin\left(\frac{\Delta m^2_{21}L}{2E_\nu}\right).$$

For $\nu_\mu \to \nu_e$,

$$\kappa = \begin{cases} +1 & \text{for } \mathcal{S} = \mathcal{A} \text{ or } \mathcal{F} \\ -1 & \text{for } \mathcal{S} = \mathcal{B}, \mathcal{C}, \mathcal{D}, \text{ or } \mathcal{E} \end{cases} ,$$

while for $\nu_\mu \to \nu_\tau$,

$$\kappa = \begin{cases} -1 & \text{for } \mathcal{S} = \mathcal{A} \text{ or } \mathcal{F} \\ +1 & \text{for } \mathcal{S} = \mathcal{B}, \mathcal{C}, \mathcal{D}, \text{ or } \mathcal{E} \end{cases} .$$

In Eq.(26), the θ_{jk} appearing in $\cos\theta_{jk}$ refers to the mixing angle situated in the middle of the matrix product, $R_{ij}W_{jk}R_{ik}$.

The CP asymmetry is a physical observable and has been verified to be invariant in all representations. Given $(\theta^{\mathcal{A}}_{13}, \delta^{\mathcal{A}}_{cp})$ in the conventional representation $\mathcal{A}$, Table III shows the expected δ_{cp} values in other representations. From Table I, it can be seen that the three mixing angles in representations $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ are far away from zero unlike those in representations $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ in which θ_{13} is small. Given that the CP asymmetry is invariant and the values of $\mathcal{J}'$ are larger in representations $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ than those in representations $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, the δ_{cp} values in representations $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, would be smaller than those in representations $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. However, if θ_{13} should turn out to be so small that the upper limit can only be obtained from experiments, this would not allow the determination of δ_{cp} in

the conventional representation because there are two unknown parameters (i.e., θ_{13} and δ_{cp}) in the CP-violation observable [see Eq.(26)]. On the other hand, representations $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ produce large values of θ_{ij} and thus there is only one unknown parameter (i.e., δ_{cp}), which makes it possible to determine δ_{cp}.

VI. CONCLUSIONS

We have studied several different parametrizations for the neutrino mixing matrix corresponding to different mixing angles and CP-violating phases. For both cases of $\delta_{cp} = 0$ and $\delta_{cp} \neq 0$, the transformation of $(\theta_{ij}, \delta_{cp})$ between two representations is derived. For the $\delta_{cp} = 0$ case, we present the predicted θ_{ij} values in various representations. For the $\delta_{cp} \neq 0$ case, we show how δ_{cp} can impact on the transformation of $(\theta_{ij}, \delta_{cp})$ from one representation to another. Solving for θ_{ij} in the various mixing parameterizations has shown that representations $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ produce significant non-zero θ_{ij}. This suggests that these three representations are more suitable for probing δ_{cp}. We have also examined how the survival and transition probabilities depend on the mixing matrix representation, and identified the representations and oscillation channels for which simpler expressions exist.

In conclusion, the mixing matrix describing the neutrino oscillation is unique, but the structure of each element of the mixing matrix vary, depending on the parametrizations. That is, the $(\theta_{ij}, \delta_{cp})$ values vary from one representation to another. In the conventional representation, θ_{13} is believed to be small or zero. This work reports alternative parametrizations for the mixing matrix that can produce significant non-zero mixing parameters and thus provides an easier way for probing δ_{cp}.

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Appendix A: Solutions of θ_{ij} for $\delta_{cp} = 0$ Case

Below presents the mixng matrix, U, in different representations for the case of $\delta_{cp} = 0$. In the conventional representation $\mathcal{A}$,

$$U = R_x(\theta_{23}^{\mathcal{A}}) R_y(\theta_{13}^{\mathcal{A}}) R_z(\theta_{12}^{\mathcal{A}}) = \begin{pmatrix} c_{12}^{\mathcal{A}} c_{13}^{\mathcal{A}} & s_{12}^{\mathcal{A}} c_{13}^{\mathcal{A}} & s_{13}^{\mathcal{A}} \\ -c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} - s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} & c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} - s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} & s_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}} \\ s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} - c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} & -s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} - c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} & c_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}} \end{pmatrix}.$$

In the representation $\mathcal{B}$,

$$U = R_x(\theta_{23}^{\mathcal{B}}) R_z(\theta_{12}^{\mathcal{B}}) R_y(\theta_{13}^{\mathcal{B}}) = \begin{pmatrix} c_{12}^{\mathcal{B}} c_{13}^{\mathcal{B}} & s_{12}^{\mathcal{B}} & c_{12}^{\mathcal{B}} s_{13}^{\mathcal{B}} \\ -s_{23}^{\mathcal{B}} s_{13}^{\mathcal{B}} - c_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} c_{13}^{\mathcal{B}} & c_{23}^{\mathcal{B}} c_{12}^{\mathcal{B}} & s_{23}^{\mathcal{B}} c_{13}^{\mathcal{B}} - c_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} s_{13}^{\mathcal{B}} \\ -c_{23}^{\mathcal{B}} s_{13}^{\mathcal{B}} + s_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} c_{13}^{\mathcal{B}} & -s_{23}^{\mathcal{B}} c_{12}^{\mathcal{B}} & c_{23}^{\mathcal{B}} c_{13}^{\mathcal{B}} + s_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} s_{13}^{\mathcal{B}} \end{pmatrix}.$$

In the representation $\mathcal{C}$,

$$U = R_y(\theta_{13}^{\mathcal{C}}) R_x(\theta_{23}^{\mathcal{C}}) R_z(\theta_{12}^{\mathcal{C}}) = \begin{pmatrix} c_{12}^{\mathcal{C}} c_{13}^{\mathcal{C}} + s_{23}^{\mathcal{C}} s_{12}^{\mathcal{C}} s_{13}^{\mathcal{C}} & s_{12}^{\mathcal{C}} c_{13}^{\mathcal{C}} - s_{23}^{\mathcal{C}} c_{12}^{\mathcal{C}} s_{13}^{\mathcal{C}} & c_{23}^{\mathcal{C}} s_{13}^{\mathcal{C}} \\ -c_{23}^{\mathcal{C}} s_{12}^{\mathcal{C}} & c_{23}^{\mathcal{C}} c_{12}^{\mathcal{C}} & s_{23}^{\mathcal{C}} \\ -c_{12}^{\mathcal{C}} s_{13}^{\mathcal{C}} + s_{23}^{\mathcal{C}} s_{12}^{\mathcal{C}} c_{13}^{\mathcal{C}} & -s_{12}^{\mathcal{C}} s_{13}^{\mathcal{C}} - s_{23}^{\mathcal{C}} c_{12}^{\mathcal{C}} c_{13}^{\mathcal{C}} & c_{23}^{\mathcal{C}} c_{13}^{\mathcal{C}} \end{pmatrix}.$$

In the representation $\mathcal{D}$,

$$U = R_y(\theta_{13}^{\mathcal{D}}) R_z(\theta_{12}^{\mathcal{D}}) R_x(\theta_{23}^{\mathcal{D}}) = \begin{pmatrix} c_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}} & -s_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}} + c_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}} & c_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}} + s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}} \\ -s_{12}^{\mathcal{D}} & c_{23}^{\mathcal{D}} c_{12}^{\mathcal{D}} & s_{23}^{\mathcal{D}} c_{12}^{\mathcal{D}} \\ -c_{12}^{\mathcal{D}} s_{13}^{\mathcal{D}} & -s_{23}^{\mathcal{D}} c_{13}^{\mathcal{D}} - c_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} s_{13}^{\mathcal{D}} & c_{23}^{\mathcal{D}} c_{13}^{\mathcal{D}} - s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} s_{13}^{\mathcal{D}} \end{pmatrix}.$$

In the representation $\mathcal{E}$,

$$U = R_z(\theta_{12}^{\mathcal{E}}) R_x(\theta_{23}^{\mathcal{E}}) R_y(\theta_{13}^{\mathcal{E}}) = \begin{pmatrix} c_{12}^{\mathcal{E}} c_{13}^{\mathcal{E}} - s_{23}^{\mathcal{E}} s_{12}^{\mathcal{E}} s_{13}^{\mathcal{E}} & c_{23}^{\mathcal{E}} s_{12}^{\mathcal{E}} & c_{12}^{\mathcal{E}} s_{13}^{\mathcal{E}} + s_{23}^{\mathcal{E}} s_{12}^{\mathcal{E}} c_{13}^{\mathcal{E}} \\ -s_{12}^{\mathcal{E}} c_{13}^{\mathcal{E}} - s_{23}^{\mathcal{E}} c_{12}^{\mathcal{E}} s_{13}^{\mathcal{E}} & c_{23}^{\mathcal{E}} c_{12}^{\mathcal{E}} & -s_{12}^{\mathcal{E}} s_{13}^{\mathcal{E}} + s_{23}^{\mathcal{E}} c_{12}^{\mathcal{E}} c_{13}^{\mathcal{E}} \\ -c_{23}^{\mathcal{E}} s_{13}^{\mathcal{E}} & -s_{23}^{\mathcal{E}} & c_{23}^{\mathcal{E}} c_{13}^{\mathcal{E}} \end{pmatrix}.$$

In the representation $\mathcal{F}$,

$$U = R_z(\theta_{12}^{\mathcal{F}}) R_y(\theta_{13}^{\mathcal{F}}) R_x(\theta_{23}^{\mathcal{F}}) = \begin{pmatrix} c_{12}^{\mathcal{F}} c_{13}^{\mathcal{F}} & c_{23}^{\mathcal{F}} s_{12}^{\mathcal{F}} - s_{23}^{\mathcal{F}} c_{12}^{\mathcal{F}} s_{13}^{\mathcal{F}} & s_{23}^{\mathcal{F}} s_{12}^{\mathcal{F}} + c_{23}^{\mathcal{F}} c_{12}^{\mathcal{F}} s_{13}^{\mathcal{F}} \\ -s_{12}^{\mathcal{F}} c_{13}^{\mathcal{F}} & c_{23}^{\mathcal{F}} c_{12}^{\mathcal{F}} + s_{23}^{\mathcal{F}} s_{12}^{\mathcal{F}} s_{13}^{\mathcal{F}} & s_{23}^{\mathcal{F}} c_{12}^{\mathcal{F}} - c_{23}^{\mathcal{F}} s_{12}^{\mathcal{F}} s_{13}^{\mathcal{F}} \\ -s_{13}^{\mathcal{F}} & -s_{23}^{\mathcal{F}} c_{13}^{\mathcal{F}} & c_{23}^{\mathcal{F}} c_{13}^{\mathcal{F}} \end{pmatrix}.$$

Denote the unitary matrix as

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}.$$

Through the elements of the unitary matrix, one can solve the three mixing angles in each representation, as shown in Table IV.

Appendix B: Solutions of $(\theta_{ij}, \delta_{cp})$ for $\delta_{cp} \neq 0$ Case

The procedure of obtaining $(\theta_{ij}, \delta_{cp})$ for each representation as expressed in terms of $(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}})$ is described in details in Section B 1 using transformation from representation $\mathcal{A}$ to $\mathcal{D}$. Following the procedure presented in Section B 1, the transformation from representation $\mathcal{A}$ to other representations is briefly summarized in this appendix by only listing the equations of the nine parameters.

Rep.	Solutions	
$\mathcal{A}$	$s^{\mathcal{A}}_{13} = U_{e3} \Rightarrow \theta^{\mathcal{A}}_{13}$;	$s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{13} = U_{\mu 3} \Rightarrow \theta^{\mathcal{A}}_{23}$;
		$s^{\mathcal{A}}_{12}c^{\mathcal{A}}_{13} = U_{e2} \Rightarrow \theta^{\mathcal{A}}_{12}$
$\mathcal{B}$	$s^{\mathcal{B}}_{12} = U_{e2} \Rightarrow \theta^{\mathcal{B}}_{12}$;	$s^{\mathcal{B}}_{23}c^{\mathcal{B}}_{12} = -U_{\tau 2} \Rightarrow \theta^{\mathcal{B}}_{23}$;
		$c^{\mathcal{B}}_{12}s^{\mathcal{B}}_{13} = U_{e3} \Rightarrow \theta^{\mathcal{B}}_{13}$
$\mathcal{C}$	$s^{\mathcal{C}}_{23} = U_{\mu 3} \Rightarrow \theta^{\mathcal{C}}_{23}$;	$c^{\mathcal{C}}_{23}s^{\mathcal{C}}_{13} = U_{e3} \Rightarrow \theta^{\mathcal{C}}_{13}$;
		$c^{\mathcal{C}}_{23}s^{\mathcal{C}}_{12} = -U_{\mu 1} \Rightarrow \theta^{\mathcal{C}}_{12}$
$\mathcal{D}$	$s^{\mathcal{D}}_{12} = -U_{\mu 1} \Rightarrow \theta^{\mathcal{D}}_{12}$;	$s^{\mathcal{D}}_{23}c^{\mathcal{D}}_{12} = U_{\mu 3} \Rightarrow \theta^{\mathcal{D}}_{23}$;
		$c^{\mathcal{D}}_{12}s^{\mathcal{D}}_{13} = -U_{\tau 1} \Rightarrow \theta^{\mathcal{D}}_{13}$
$\mathcal{E}$	$s^{\mathcal{E}}_{23} = -U_{\tau 2} \Rightarrow \theta^{\mathcal{E}}_{23}$;	$c^{\mathcal{E}}_{23}s^{\mathcal{E}}_{12} = U_{e2} \Rightarrow \theta^{\mathcal{E}}_{12}$;
		$c^{\mathcal{E}}_{23}s^{\mathcal{E}}_{13} = -U_{\tau 1} \Rightarrow \theta^{\mathcal{E}}_{13}$
$\mathcal{F}$	$s^{\mathcal{F}}_{13} = -U_{\tau 1} \Rightarrow \theta^{\mathcal{F}}_{13}$;	$s^{\mathcal{F}}_{23}c^{\mathcal{F}}_{13} = -U_{\tau 2} \Rightarrow \theta^{\mathcal{F}}_{23}$;
		$s^{\mathcal{F}}_{12}c^{\mathcal{F}}_{13} = -U_{\mu 1} \Rightarrow \theta^{\mathcal{F}}_{12}$

TABLE IV: Solutions of the three mixing angles in different representatsions for the case of $\delta_{cp} = 0$.

1. $(\theta^{\mathcal{D}}_{ij}, \delta^{\mathcal{D}}_{cp})$ Solutions

Rotations undertaken with the matrices presented in Eq.(7) will have nine independent parameters: five phases in D^L and D^R, three mixing angles and one δ_{cp} phase from the mixing matrix. To solve for $(\theta^{\mathcal{D}}_{ij}, \delta^{\mathcal{D}}_{cp})$ in representation $\mathcal{D}$ where the $(\theta^{\mathcal{A}}_{ij}, \delta^{\mathcal{A}}_{cp})$ are known for representation $\mathcal{A}$, start with Eq.(4):

$$U = RWR(\theta^{\mathcal{D}}_{ij}, \delta^{\mathcal{D}}_{cp}) = D^L \cdot RWR(\theta^{\mathcal{A}}_{ij}, \delta^{\mathcal{A}}_{cp}) \cdot D^R \ , \quad \text{(B1)}$$

where

$$\begin{aligned} D^L(\Phi^{\mathcal{D}}_{Li}) &= diag\left(e^{i\Phi^{\mathcal{D}}_{L1}}, e^{i\Phi^{\mathcal{D}}_{L2}}, e^{i\Phi^{\mathcal{D}}_{L3}}\right) \ , \\ D^R(\Phi_{Ri}) &= diag\left(e^{i\Phi^{\mathcal{D}}_{R1}}, e^{i\Phi^{\mathcal{D}}_{R2}}, 1\right) \ . \end{aligned} \quad \text{(B2)}$$

The nine real parts and the nine imaginary parts of Equation (B 1) are listed in Eqs.(B3) through (B11). For element (1,1):

$$\begin{aligned} c^{\mathcal{D}}_{13}c^{\mathcal{D}}_{12} &= c^{\mathcal{A}}_{13}c^{\mathcal{A}}_{12}\cos(\Phi^{\mathcal{D}}_{L1} + \Phi^{\mathcal{D}}_{R1}), \\ 0 &= c^{\mathcal{A}}_{13}c^{\mathcal{A}}_{12}\sin(\Phi^{\mathcal{D}}_{L1} + \Phi^{\mathcal{D}}_{R1}). \end{aligned} \quad \text{(B3)}$$

For element (2,1):

$$\begin{aligned} s^{\mathcal{D}}_{12}\cos\delta^{\mathcal{D}}_{cp} &= c^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}\cos(\Phi^{\mathcal{D}}_{L2} + \Phi^{\mathcal{D}}_{R1}) \\ &\quad + s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{L2} + \Phi^{\mathcal{D}}_{R1}), \\ s^{\mathcal{D}}_{12}\sin\delta^{\mathcal{D}}_{cp} &= c^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}\sin(\Phi^{\mathcal{D}}_{L2} + \Phi^{\mathcal{D}}_{R1}) \\ &\quad + s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\sin(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{L2} + \Phi^{\mathcal{D}}_{R1}). \end{aligned} \quad \text{(B4)}$$

For element (3,1):

$$\begin{aligned} -s^{\mathcal{D}}_{13}c^{\mathcal{D}}_{12} &= s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}\cos(\Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R1}) \\ &\quad - c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R1}), \\ 0 &= s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}\sin(\Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R1}) \\ &\quad - c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\sin(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R1}). \end{aligned} \quad \text{(B5)}$$

For element (1,2):

$$\begin{aligned} c^{\mathcal{D}}_{23}c^{\mathcal{D}}_{13}s^{\mathcal{D}}_{12}\cos\delta^{\mathcal{D}}_{cp} - s^{\mathcal{D}}_{23}s^{\mathcal{D}}_{13} &= s^{\mathcal{A}}_{12}c^{\mathcal{A}}_{13}\cos(\Phi^{\mathcal{D}}_{L1} + \Phi^{\mathcal{D}}_{R2}), \\ -c^{\mathcal{D}}_{23}c^{\mathcal{D}}_{13}s^{\mathcal{D}}_{12}\sin\delta^{\mathcal{D}}_{cp} &= s^{\mathcal{A}}_{12}c^{\mathcal{A}}_{13}\sin(\Phi^{\mathcal{D}}_{L1} + \Phi^{\mathcal{D}}_{R2}). \end{aligned} \quad \text{(B6)}$$

For element (2,2):

$$\begin{aligned} c^{\mathcal{D}}_{23}c^{\mathcal{D}}_{12} &= c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}\cos(\Phi^{\mathcal{D}}_{L2} + \Phi^{\mathcal{D}}_{R2}) \\ &\quad - s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{L2} + \Phi^{\mathcal{D}}_{R2}), \\ 0 &= c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}\sin(\Phi^{\mathcal{D}}_{L2} + \Phi^{\mathcal{D}}_{R2}) \\ &\quad - s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\sin(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{L2} + \Phi^{\mathcal{D}}_{R2}). \end{aligned} \quad \text{(B7)}$$

For element (3,2):

$$\begin{aligned} s^{\mathcal{D}}_{23}c^{\mathcal{D}}_{13} + c^{\mathcal{D}}_{23}s^{\mathcal{D}}_{12}s^{\mathcal{D}}_{13}\cos\delta^{\mathcal{D}}_{cp} &= s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}\cos(\Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R2}) \\ &\quad + c^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R2}), \\ -c^{\mathcal{D}}_{23}s^{\mathcal{D}}_{12}s^{\mathcal{D}}_{13}\sin\delta^{\mathcal{D}}_{cp} &= s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}\sin(\Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R2}) \\ &\quad + c^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R2}). \end{aligned} \quad \text{(B8)}$$

For element (1,3):

$$\begin{aligned} c^{\mathcal{D}}_{23}s^{\mathcal{D}}_{13} + s^{\mathcal{D}}_{23}s^{\mathcal{D}}_{12}c^{\mathcal{D}}_{13}\cos\delta^{\mathcal{D}}_{cp} &= s^{\mathcal{A}}_{13}\cos(\delta^{\mathcal{A}}_{cp} - \Phi^{\mathcal{D}}_{L1}), \\ -s^{\mathcal{D}}_{23}s^{\mathcal{D}}_{12}c^{\mathcal{D}}_{13}\sin\delta^{\mathcal{D}}_{cp} &= -s^{\mathcal{A}}_{13}\sin(\delta^{\mathcal{A}}_{cp} - \Phi^{\mathcal{D}}_{L1}). \end{aligned} \quad \text{(B9)}$$

For element (2,3):

$$\begin{aligned} s^{\mathcal{D}}_{23}c^{\mathcal{D}}_{12} &= s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{13}\cos\Phi^{\mathcal{D}}_{L2}, \\ 0 &= s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{13}\sin\Phi^{\mathcal{D}}_{L2}. \end{aligned} \quad \text{(B10)}$$

For element (3,3):

$$\begin{aligned} c^{\mathcal{D}}_{23}c^{\mathcal{D}}_{13} - s^{\mathcal{D}}_{23}s^{\mathcal{D}}_{12}s^{\mathcal{D}}_{13}\cos\delta^{\mathcal{D}}_{cp} &= c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{13}\cos\Phi^{\mathcal{D}}_{L3}, \\ s^{\mathcal{D}}_{23}s^{\mathcal{D}}_{12}s^{\mathcal{D}}_{13}\sin\delta^{\mathcal{D}}_{cp} &= c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{13}\sin\Phi^{\mathcal{D}}_{L3}. \end{aligned} \quad \text{(B11)}$$

Based on Eqs.(B3)–(B11), one can solve $(\theta^{\mathcal{D}}_{ij}, \delta^{\mathcal{D}}_{cp})$ in terms of $(\theta^{\mathcal{A}}_{ij}, \delta^{\mathcal{A}}_{cp})$ as detailed in the following. From the imaginary parts of elements (1,1) and (2,3), one has

$$\begin{aligned} \Phi^{\mathcal{D}}_{L1} + \Phi^{\mathcal{D}}_{R1} &= 0, \\ \Phi^{\mathcal{D}}_{L2} &= 0. \end{aligned} \quad \text{(B12)}$$

From the imaginary part of element (2,2), one has

$$\sin^2\Phi^{\mathcal{D}}_{R2} = \frac{(a\ \sin\delta^{\mathcal{A}}_{cp})^2}{(a\ \sin\delta^{\mathcal{A}}_{cp})^2 + (a\ \cos\delta^{\mathcal{A}}_{cp} - b)^2} \ , \quad \text{(B13)}$$

where $a = s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}$ and $b = c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}$. The sign of $\Phi^{\mathcal{D}}_{R2}$ can be easily determined by the original equation, Eq.(B7). From the imaginary part of element (3,1), one has

$$\sin^2\Phi^{\mathcal{D}}_{34} = \frac{(a^{'}\ \sin\delta^{\mathcal{A}}_{cp})^2}{(a^{'}\sin\delta^{\mathcal{A}}_{cp})^2 + (a^{'}\cos\delta^{\mathcal{A}}_{cp} - b^{'})^2} \ , \quad \text{(B14)}$$

where $a^{'} = c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}$, $b^{'} = s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}$, and $\Phi^{\mathcal{D}}_{34} \equiv \Phi^{\mathcal{D}}_{L3} + \Phi^{\mathcal{D}}_{R1}$. Again, one can readily determine the sign of $\Phi^{\mathcal{D}}_{34}$ using the original equation, Eq.(B5), that is used for obtaining Eq.(B14). With these phases, the three mixing angles in representation $\mathcal{D}$ can be extracted using the real parts of elements (1,1), (3,1), (2,2), and (2,3):

$$\tan\theta^{\mathcal{D}}_{23} = \frac{s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{13}}{c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}\cos\Phi^{\mathcal{D}}_{R2} - s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}\cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{D}}_{R2})} \quad \text{(B15)}$$

$$\cos\theta_{12}^{\mathcal{D}} = \frac{s_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}}}{s_{23}^{\mathcal{D}}} = \frac{c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \cos\Phi_{R2}^{\mathcal{D}} - s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{R2}^{\mathcal{D}})}{c_{23}^{\mathcal{D}}}, \quad \text{(B16)}$$

$$\sin\theta_{13}^{\mathcal{D}} = \frac{c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{34}^{\mathcal{D}}) - s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} \cos\Phi_{34}^{\mathcal{D}}}{c_{12}^{\mathcal{D}}}. \quad \text{(B17)}$$

The remaining parameters thus can be determined. From the real and imaginary parts of element (1,3), one has

$$\cos(\delta_{cp}^{\mathcal{A}} - \Phi_{L1}^{\mathcal{D}}) = \frac{(c_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}})^2 + (s_{13}^{\mathcal{A}})^2 - (s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}})^2}{2 c_{23}^{\mathcal{D}} s_{13}^{\mathcal{D}} s_{13}^{\mathcal{A}}}. \quad \text{(B18)}$$

As before, the sign of $\delta_{cp}^{\mathcal{A}} - \Phi_{L1}^{\mathcal{D}}$ can be determined by the real and imaginary parts of Eq.(B5) or (B6), and thus $\Phi_{L1}^{\mathcal{D}}$ can be determined. Subsequently, $\Phi_{R1}^{\mathcal{D}} = -\Phi_{L1}^{\mathcal{D}}$ and $\Phi_{L3}^{\mathcal{D}} = \Phi_{34}^{\mathcal{D}} - \Phi_{R1}^{\mathcal{D}}$ can be therefore determined. Finally, the CP-violating phase in representation $\mathcal{D}$, $\delta_{cp}^{\mathcal{D}}$, can be resolved using those conditions associated with $\sin\delta_{cp}^{\mathcal{D}}$ in elements (2,1), (1,2), (3,2), (1,3), or (3,3), which are listed as follows:

$$\begin{aligned}
&\sin\delta_{cp}^{\mathcal{D}} \\
&= \frac{s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \sin(\delta_{cp}^{\mathcal{A}} + \eta) + c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} \sin\eta}{s_{12}^{\mathcal{D}}} \\
&= \frac{c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \sin(\delta_{cp}^{\mathcal{A}} + \xi) + s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \sin\xi}{-c_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} s_{13}^{\mathcal{D}}} \\
&= \frac{-s_{12}^{\mathcal{A}} c_{13}^{\mathcal{A}} \sin(\Phi_{L1}^{\mathcal{D}} + \Phi_{R2}^{\mathcal{D}})}{c_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}}} \\
&= \frac{s_{13}^{\mathcal{A}} \sin(\delta_{cp}^{\mathcal{A}} - \Phi_{L1}^{\mathcal{D}})}{s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} c_{13}^{\mathcal{D}}} \\
&= \frac{c_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}} \sin\Phi_{L3}^{\mathcal{D}}}{s_{23}^{\mathcal{D}} s_{12}^{\mathcal{D}} s_{13}^{\mathcal{D}}} ,
\end{aligned} \quad \text{(B19)}$$

where $\eta \equiv \Phi_{L2}^{\mathcal{D}} + \Phi_{R1}^{\mathcal{D}}$ and $\xi \equiv \Phi_{L3}^{\mathcal{D}} + \Phi_{R2}^{\mathcal{D}}$. Consistency in the value of $\sin\delta_{cp}^{\mathcal{D}}$ calculated from the five different expressions in Eq.(B19) serves as a means to check whether the values of the nine parameters are correct. Furthermore, by looking at Eq.(B 1), 18 conditions are formed, nine from real parts and the other nine from imaginary parts. With the solutions presented in Eqs.(B12) through (B19), the consistency among the 18 conditions has been checked.

2. $(\theta_{ij}^{\mathcal{B}}, \delta_{cp}^{\mathcal{B}})$ Solutions

For the rotation matrices in Eq.(7) to transform $(\theta_{ij}^{\mathcal{A}}, \delta_{cp}^{\mathcal{A}})$ from representations $\mathcal{A}$ to $\mathcal{B}$, the solutions to the nine parameters are listed as follows.

$$\Phi_{R1}^{\mathcal{B}} = -\delta_{cp}^{\mathcal{A}} \quad \text{and} \quad \Phi_{\rm L1}^{\mathcal{B}} = \delta_{\rm cp}^{\mathcal{A}} . \quad \text{(B20)}$$

$$\sin^2\Phi_{25}^{\mathcal{B}} = \frac{(a\ \sin\delta_{cp}^{\mathcal{A}})^2}{(a\ \sin\delta_{cp}^{\mathcal{A}})^2 + (a\ \cos\delta_{cp}^{\mathcal{A}} - b)^2} , \quad \text{(B21)}$$

where $a = s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}}$, $b = c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}}$, and $\Phi_{25}^{\mathcal{B}} = \Phi_{L2}^{\mathcal{B}} + \Phi_{R2}^{\mathcal{B}}$.

$$\sin^2\Phi_{35}^{\mathcal{B}} = \frac{(a'\ \sin\delta_{cp}^{\mathcal{A}})^2}{(a'\ \sin\delta_{cp}^{\mathcal{A}})^2 + (a'\ \cos\delta_{cp}^{\mathcal{A}} + b')^2} , \quad \text{(B22)}$$

where $a' = c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}}$, $b' = s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}}$, and $\Phi_{35}^{\mathcal{B}} \equiv \Phi_{L3}^{\mathcal{B}} + \Phi_{R2}^{\mathcal{B}}$. The three mixing angles can thus be obtained in representation $\mathcal{B}$.

$$\tan\theta_{23}^{\mathcal{B}} = \frac{s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \cos\Phi_{35}^{\mathcal{B}} + c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{35}^{\mathcal{B}})}{c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \cos\Phi_{25}^{\mathcal{B}} - s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{25}^{\mathcal{B}})}, \quad \text{(B23)}$$

$$\cos\theta_{12}^{\mathcal{B}} = \frac{s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \cos\Phi_{35}^{\mathcal{B}} + c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{35}^{\mathcal{B}})}{s_{23}^{\mathcal{B}}} = \frac{c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} \cos\Phi_{25}^{\mathcal{B}} - s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \cos(\delta_{cp}^{\mathcal{A}} + \Phi_{25}^{\mathcal{B}})}{c_{23}^{\mathcal{B}}}, \quad \text{(B24)}$$

$$\sin\theta_{13}^{\mathcal{B}} = \frac{s_{13}^{\mathcal{A}}}{c_{12}^{\mathcal{B}}} . \quad \text{(B25)}$$

The remaining parameters thus can be determined as follows.

$$\cos\Phi_{L2}^{\mathcal{B}} = \frac{(s_{23}^{\mathcal{B}} c_{13}^{\mathcal{B}})^2 + (s_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}})^2 - (c_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} s_{13}^{\mathcal{B}})^2}{2 s_{23}^{\mathcal{B}} c_{13}^{\mathcal{B}} s_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}}} . \quad \text{(B26)}$$

Therefore, $\Phi_{R2}^{\mathcal{B}} = \Phi_{25}^{\mathcal{B}} - \Phi_{L2}^{\mathcal{B}}$ and $\Phi_{L3}^{\mathcal{B}} = \Phi_{35}^{\mathcal{B}} - \Phi_{R2}^{\mathcal{B}}$ can be determined. Finally, the CP-violating phase in representation $\mathcal{B}$, $\delta_{cp}^{\mathcal{B}}$, can be obtained using those conditions associated with $\sin\delta_{cp}^{\mathcal{B}}$:

$$\begin{aligned}
&\sin\delta_{cp}^{\mathcal{B}} \\
&= \frac{s_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \sin(\delta_{cp}^{\mathcal{A}} + \eta^{\mathcal{B}}) + c_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} \sin\eta^{\mathcal{B}}}{c_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} c_{13}^{\mathcal{B}}} \\
&= \frac{-c_{23}^{\mathcal{A}} c_{12}^{\mathcal{A}} s_{13}^{\mathcal{A}} \sin(\delta_{cp}^{\mathcal{A}} + \xi^{\mathcal{B}}) + s_{23}^{\mathcal{A}} s_{12}^{\mathcal{A}} \sin\xi^{\mathcal{B}}}{s_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} c_{13}^{\mathcal{B}}} \\
&= \frac{-s_{12}^{\mathcal{A}} c_{13}^{\mathcal{A}} \sin(\Phi_{L1}^{\mathcal{B}} + \Phi_{R2}^{\mathcal{B}})}{s_{12}^{\mathcal{B}}} \\
&= \frac{-s_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}} \sin\Phi_{L2}^{\mathcal{B}}}{c_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} s_{13}^{\mathcal{B}}} \\
&= \frac{c_{23}^{\mathcal{A}} c_{13}^{\mathcal{A}} \sin\Phi_{L3}^{\mathcal{B}}}{s_{23}^{\mathcal{B}} s_{12}^{\mathcal{B}} s_{13}^{\mathcal{B}}} ,
\end{aligned} \quad \text{(B27)}$$

where $\eta^{\mathcal{B}} \equiv \Phi_{L2}^{\mathcal{B}} + \Phi_{R1}^{\mathcal{B}}$ and $\xi^{\mathcal{B}} \equiv \Phi_{L3}^{\mathcal{B}} + \Phi_{R1}^{\mathcal{B}}$.

3. $(\theta_{ij}^{\mathcal{C}}, \delta_{cp}^{\mathcal{C}})$ Solutions

For this case, the rotation matrices,

$$\begin{aligned}
D^L &= diag\left(e^{i\Phi_{L1}^{\mathcal{C}}}, e^{i\Phi_{L2}^{\mathcal{C}}}, e^{i\Phi_{L3}^{\mathcal{C}}}\right), \\
D^R &= diag\left(1, e^{i\Phi_{R2}^{\mathcal{C}}}, e^{i\Phi_{R3}^{\mathcal{C}}}\right),
\end{aligned} \quad \text{(B28)}$$

are applied to the transformation of $(\theta_{ij}, \delta_{cp})$ from representations $\mathcal{A}$ to $\mathcal{C}$. The solutions of the nine parameters are listed as follows.

$$\Phi^{\mathcal{C}}_{L1} + \Phi^{\mathcal{C}}_{R3} = \delta^{\mathcal{A}}_{cp} \quad \text{and} \quad \Phi^{\mathcal{C}}_{\rm L3} + \Phi^{\mathcal{C}}_{\rm R3} = 0 \,. \tag{B29}$$

$$\sin^2 \Phi^{\mathcal{C}}_{L2} = \frac{(a\ \sin\delta^{\mathcal{A}}_{cp})^2}{(a\ \sin\delta^{\mathcal{A}}_{cp})^2 + (a\ \cos\delta^{\mathcal{A}}_{cp} + b)^2} \,, \tag{B30}$$

where $a = s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13}$ and $b = c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12}$.

$$\sin^2 \Phi^{\mathcal{C}}_{25} = \frac{(a^{'}\ \sin\delta^{\mathcal{A}}_{cp})^2}{(a^{'}\ \sin\delta^{\mathcal{A}}_{cp})^2 + (a^{'}\ \cos\delta^{\mathcal{A}}_{cp} - b^{'})^2} \,, \tag{B31}$$

where $a^{'} = s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13}$, $b^{'} = c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12}$, and $\Phi^{\mathcal{C}}_{25} \equiv \Phi^{\mathcal{C}}_{L2} + \Phi^{\mathcal{C}}_{R2}$. The three mixing angles can thus be obtained in representation $\mathcal{C}$.

$$\tan\theta^{\mathcal{C}}_{12} = \frac{c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{C}}_{L2} + s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{C}}_{L2})}{c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{C}}_{25} - s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{C}}_{25})} , \tag{B32}$$

$$\begin{aligned} \cos\theta^{\mathcal{C}}_{23} &= \frac{c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{C}}_{L2} + s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{C}}_{L2})}{s^{\mathcal{C}}_{12}} \\ &= \frac{c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{C}}_{25} - s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{C}}_{25})}{c^{\mathcal{C}}_{12}} , \end{aligned} \tag{B33}$$

$$\sin\theta^{\mathcal{C}}_{13} = \frac{s^{\mathcal{A}}_{13}}{c^{\mathcal{C}}_{23}} \,. \tag{B34}$$

The remaining parameters thus can be determined, as shown below.

$$\cos\Phi^{\mathcal{C}}_{L1} = \frac{(c^{\mathcal{C}}_{12} c^{\mathcal{C}}_{13})^2 + (c^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13})^2 - (s^{\mathcal{C}}_{23} s^{\mathcal{C}}_{12} s^{\mathcal{C}}_{13})^2}{2 c^{\mathcal{C}}_{12} c^{\mathcal{C}}_{13} c^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13}} . \tag{B35}$$

Therefore, $\Phi^{\mathcal{C}}_{R3} = \delta^{\mathcal{A}}_{cp} - \Phi^{\mathcal{C}}_{L1}$ and $\Phi^{\mathcal{C}}_{L3} = -\Phi^{\mathcal{C}}_{R3}$ can be determined. Finally, the CP-violating phase in representation $\mathcal{C}$, $\delta^{\mathcal{C}}_{cp}$, can be obtained using those conditions associated with $\sin\delta^{\mathcal{C}}_{cp}$:

$$\begin{aligned} &\sin\delta^{\mathcal{C}}_{cp} \\ &= \frac{s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} \sin\eta^{\mathcal{C}} + c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \sin(\delta^{\mathcal{A}}_{cp} + \eta^{\mathcal{C}})}{s^{\mathcal{C}}_{23} c^{\mathcal{C}}_{12} c^{\mathcal{C}}_{13}} \\ &= \frac{s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} \sin\Phi^{\mathcal{C}}_{L3} - c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \sin(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{C}}_{L3})}{s^{\mathcal{C}}_{23} s^{\mathcal{C}}_{12} c^{\mathcal{C}}_{13}} \\ &= \frac{-s^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13} \sin(\Phi^{\mathcal{C}}_{L1} + \Phi^{\mathcal{C}}_{R2})}{s^{\mathcal{C}}_{23} c^{\mathcal{C}}_{12} s^{\mathcal{C}}_{13}} \\ &= \frac{-s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{13} \sin(\Phi^{\mathcal{C}}_{L2} + \Phi^{\mathcal{C}}_{R3})}{s^{\mathcal{C}}_{23}} \\ &= \frac{c^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13} \sin\Phi^{\mathcal{C}}_{L1}}{s^{\mathcal{C}}_{23} s^{\mathcal{C}}_{12} s^{\mathcal{C}}_{13}} \,, \end{aligned} \tag{B36}$$

where $\eta^{\mathcal{C}} \equiv \Phi^{\mathcal{C}}_{L3} + \Phi^{\mathcal{C}}_{R2}$.

4. $(\theta^{\mathcal{E}}_{ij}, \delta^{\mathcal{E}}_{cp})$ Solutions

For this case, the rotation matrices,

$$\begin{aligned} D^L &= diag\left(e^{i\Phi^{\mathcal{E}}_{L1}}, e^{i\Phi^{\mathcal{E}}_{L2}}, e^{i\Phi^{\mathcal{E}}_{L3}}\right), \\ D^R &= diag\left(1, e^{i\Phi^{\mathcal{E}}_{R2}}, e^{i\Phi^{\mathcal{E}}_{R3}}\right), \end{aligned} \tag{B37}$$

are employed to transform $(\theta^{\mathcal{A}}_{ij}, \delta^{\mathcal{A}}_{cp})$ to $(\theta^{\mathcal{E}}_{ij}, \delta^{\mathcal{E}}_{cp})$. Below presents the solutions of the nine parameters.

$$\Phi^{\mathcal{E}}_{L1} + \Phi^{\mathcal{E}}_{R2} = 0 \quad \text{and} \quad \Phi^{\mathcal{E}}_{\rm L3} + \Phi^{\mathcal{E}}_{\rm R3} = 0 \,. \tag{B38}$$

$$\sin^2 \Phi^{\mathcal{E}}_{L3} = \frac{(a\ \sin\delta^{\mathcal{A}}_{cp})^2}{(a\ \sin\delta^{\mathcal{A}}_{cp})^2 + (a\ \cos\delta^{\mathcal{A}}_{cp} - b)^2} \,, \tag{B39}$$

where $a = c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13}$ and $b = s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12}$.

$$\sin^2 \Phi^{\mathcal{E}}_{25} = \frac{(a^{'}\ \sin\delta^{\mathcal{A}}_{cp})^2}{(a^{'}\ \sin\delta^{\mathcal{A}}_{cp})^2 + (a^{'}\ \cos\delta^{\mathcal{A}}_{cp} - b^{'})^2} \,, \tag{B40}$$

where $a^{'} = s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13}$, $b^{'} = c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12}$, and $\Phi^{\mathcal{E}}_{25} \equiv \Phi^{\mathcal{E}}_{L2} + \Phi^{\mathcal{E}}_{R2}$. The three mixing angles can thus be acquired in representation $\mathcal{E}$.

$$\tan\theta^{\mathcal{E}}_{12} = \frac{s^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13}}{c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{E}}_{25} - s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{E}}_{25})} , \tag{B41}$$

$$\begin{aligned} \cos\theta^{\mathcal{E}}_{23} &= \frac{s^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13}}{s^{\mathcal{E}}_{12}} \\ &= \frac{c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{E}}_{25} - s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{E}}_{25})}{c^{\mathcal{E}}_{12}} , \end{aligned} \tag{B42}$$

$$\sin\theta^{\mathcal{E}}_{13} = \frac{c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{E}}_{L3}) - s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{E}}_{L3}}{c^{\mathcal{E}}_{23}} . \tag{B43}$$

The remaining parameters thus can be determined, which are shown below.

$$\cos\Phi^{\mathcal{E}}_{L1} = \frac{(c^{\mathcal{E}}_{12} c^{\mathcal{E}}_{13})^2 + (c^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13})^2 - (s^{\mathcal{E}}_{23} s^{\mathcal{E}}_{12} s^{\mathcal{E}}_{13})^2}{2 c^{\mathcal{E}}_{12} c^{\mathcal{E}}_{13} c^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13}} . \tag{B44}$$

Therefore, $\Phi^{\mathcal{E}}_{R2} = -\Phi^{\mathcal{E}}_{L1}$, $\Phi^{\mathcal{E}}_{L2} = \Phi^{\mathcal{E}}_{25} - \Phi^{\mathcal{E}}_{R2}$, and $\Phi^{\mathcal{E}}_{R3} = -\Phi^{\mathcal{E}}_{L3}$ can be determined. Finally, the CP-violating phase in representation $\mathcal{E}$, $\delta^{\mathcal{E}}_{cp}$, can be derived using those conditions associated with $\sin\delta^{\mathcal{E}}_{cp}$:

$$\begin{aligned} &\sin\delta^{\mathcal{E}}_{cp} \\ &= \frac{-s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} \sin\eta^{\mathcal{E}} - c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \sin(\delta^{\mathcal{A}}_{cp} + \eta^{\mathcal{E}})}{-s^{\mathcal{E}}_{23}} \\ &= \frac{-c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} \sin\Phi^{\mathcal{E}}_{L2} - s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \sin(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{E}}_{L2})}{s^{\mathcal{E}}_{23} c^{\mathcal{E}}_{12} s^{\mathcal{E}}_{13}} \\ &= \frac{s^{\mathcal{A}}_{13} \sin(\delta^{\mathcal{A}}_{cp} - \Phi^{E}_{L1} - \Phi^{E}_{R3})}{s^{\mathcal{E}}_{23} s^{\mathcal{E}}_{12} c^{\mathcal{E}}_{13}} \\ &= \frac{-s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{13} \sin(\Phi^{\mathcal{E}}_{L2} + \Phi^{\mathcal{E}}_{R3})}{s^{\mathcal{E}}_{23} c^{\mathcal{E}}_{12} c^{\mathcal{E}}_{13}} \\ &= \frac{c^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13} \sin\Phi^{\mathcal{E}}_{L1}}{s^{\mathcal{E}}_{23} s^{\mathcal{E}}_{12} s^{\mathcal{E}}_{13}} \,, \end{aligned} \tag{B45}$$

where $\eta^{\mathcal{E}} \equiv \Phi^{\mathcal{E}}_{L3} + \Phi^{\mathcal{E}}_{R2}$.

5. $(\theta^{\mathcal{F}}_{ij}, \delta^{\mathcal{F}}_{cp})$ Solutions

For this case, the rotation matrices,

$$\begin{aligned} D^L &= diag\left(e^{i\Phi^{\mathcal{F}}_{L1}}, e^{i\Phi^{\mathcal{F}}_{L2}}, e^{i\Phi^{\mathcal{F}}_{L3}}\right), \\ D^R &= diag\left(e^{i\Phi^{\mathcal{F}}_{R1}}, 1, e^{i\Phi^{\mathcal{F}}_{R3}}\right), \end{aligned} \tag{B46}$$

are applied to the transform $(\theta^{\mathcal{A}}_{ij}, \delta^{\mathcal{A}}_{cp})$ to $(\theta^{\mathcal{F}}_{ij}, \delta^{\mathcal{F}}_{cp})$. The solutions of the nine parameters are presented as follows.

$$\Phi^{\mathcal{F}}_{L1} + \Phi^{\mathcal{F}}_{R1} = 0 \quad \text{and} \quad \Phi^{\mathcal{F}}_{\mathrm{L3}} + \Phi^{\mathcal{F}}_{\mathrm{R3}} = 0 \,. \tag{B47}$$

$$\sin^2 \Phi^{\mathcal{F}}_{24} = \frac{(a\ \sin\delta^{\mathcal{A}}_{cp})^2}{(a\ \sin\delta^{\mathcal{A}}_{cp})^2 + (a\ \cos\delta^{\mathcal{A}}_{cp} + b)^2} \,, \tag{B48}$$

where $a = s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13}$, $b = c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12}$, and $\Phi^{\mathcal{F}}_{24} \equiv \Phi^{\mathcal{F}}_{L2} + \Phi^{\mathcal{F}}_{R1}$.

$$\sin^2 \Phi^{\mathcal{F}}_{L3} = \frac{(a^{'}\ \sin\delta^{\mathcal{A}}_{cp})^2}{(a'\ \sin\delta^{\mathcal{A}}_{cp})^2 + (a'\ \cos\delta^{\mathcal{A}}_{cp} + b')^2} \,, \tag{B49}$$

where $a^{'} = c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13}$ and $b^{'} = s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12}$. The three mixing angles can thus be obtained in representation $\mathcal{F}$.

$$\tan\theta^{\mathcal{F}}_{12} = \frac{c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{F}}_{24} + s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{F}}_{24})}{c^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13}}, \tag{B50}$$

$$\begin{aligned} \cos\theta^{\mathcal{F}}_{13} &= \frac{c^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13}}{c^{\mathcal{F}}_{12}} \\ &= \frac{c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{F}}_{24} + s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{F}}_{24})}{s^{\mathcal{F}}_{12}}, \end{aligned} \tag{B51}$$

$$\sin\theta^{\mathcal{F}}_{23} = \frac{s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} \cos\Phi^{\mathcal{F}}_{L3} + c^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \cos(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{F}}_{L3})}{c^{\mathcal{F}}_{13}}. \tag{B52}$$

The remaining parameters thus can be determined as shown below.

$$\cos\Phi^{\mathcal{F}}_{L1} = \frac{(c^{\mathcal{F}}_{23} s^{\mathcal{F}}_{12})^2 + (s^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13})^2 - (s^{\mathcal{F}}_{23} c^{\mathcal{F}}_{12} s^{\mathcal{F}}_{13})^2}{2 c^{\mathcal{F}}_{23} s^{\mathcal{F}}_{12} s^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13}} \,. \tag{B53}$$

Therefore, $\Phi^{\mathcal{F}}_{R1} = -\Phi^{\mathcal{F}}_{L1}$, $\Phi^{\mathcal{F}}_{L2} = \Phi^{\mathcal{F}}_{24} - \Phi^{\mathcal{F}}_{R1}$, and $\Phi^{\mathcal{F}}_{R3} = -\Phi^{\mathcal{F}}_{L3}$ can be determined. Finally, the CP-violating phase in representation $\mathcal{F}$, $\delta^{\mathcal{F}}_{cp}$, can be obtained using those conditions associated with $\sin\delta^{\mathcal{F}}_{cp}$:

$$\begin{aligned} &\sin\delta^{\mathcal{F}}_{cp} \\ &= \frac{-s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} \sin\eta^{\mathcal{F}} + c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \sin(\delta^{\mathcal{A}}_{cp} + \eta^{\mathcal{F}})}{s^{\mathcal{F}}_{13}} \\ &= \frac{-c^{\mathcal{A}}_{23} c^{\mathcal{A}}_{12} \sin\Phi^{\mathcal{F}}_{L2} + s^{\mathcal{A}}_{23} s^{\mathcal{A}}_{12} s^{\mathcal{A}}_{13} \sin(\delta^{\mathcal{A}}_{cp} + \Phi^{\mathcal{F}}_{L2})}{s^{\mathcal{F}}_{23} s^{\mathcal{F}}_{12} s^{\mathcal{F}}_{13}} \\ &= \frac{s^{\mathcal{A}}_{13} \sin(\delta^{\mathcal{A}}_{cp} - \Phi^{\mathcal{F}}_{L1} - \Phi^{\mathcal{F}}_{R3})}{c^{\mathcal{F}}_{23} c^{\mathcal{F}}_{12} s^{\mathcal{F}}_{13}} \\ &= \frac{s^{\mathcal{A}}_{23} c^{\mathcal{A}}_{13} \sin(\Phi^{\mathcal{F}}_{L2} + \Phi^{\mathcal{F}}_{R3})}{c^{\mathcal{F}}_{23} s^{\mathcal{F}}_{12} s^{\mathcal{F}}_{13}} \\ &= \frac{s^{\mathcal{A}}_{12} c^{\mathcal{A}}_{13} \sin\Phi^{\mathcal{F}}_{L1}}{s^{\mathcal{F}}_{23} c^{\mathcal{F}}_{12} s^{\mathcal{F}}_{13}} \,, \end{aligned} \tag{B54}$$

where $\eta^{\mathcal{F}} \equiv \Phi^{\mathcal{F}}_{L3} + \Phi^{\mathcal{F}}_{R1}$.

Appendix C: Neutrino Oscillation Probability

For oscillation channel $\nu_e \to \nu_e$, representations $\mathcal{A}$ and $\mathcal{B}$ have simpler forms of survival probabilities:

$$\begin{aligned} &P^{\mathcal{A}}_{\nu_e \to \nu_e} = \\ &1 - (c^{\mathcal{A}}_{13})^4 \sin^2 2\theta^{\mathcal{A}}_{12} \sin^2\Delta_{21} - (c^{\mathcal{A}}_{12})^2 \sin^2 2\theta^{\mathcal{A}}_{13} \sin^2\Delta_{31} \\ &-(s^{\mathcal{A}}_{12})^2 \sin^2 2\theta^{\mathcal{A}}_{13} \sin^2\Delta_{32} \,, \end{aligned} \tag{C1}$$

$$\begin{aligned} &P^{\mathcal{B}}_{\nu_e \to \nu_e} = \\ &1 - (c^{\mathcal{B}}_{13})^2 \sin^2 2\theta^{\mathcal{B}}_{12} \sin^2\Delta_{21} - (c^{\mathcal{B}}_{12})^4 \sin^2 2\theta^{\mathcal{B}}_{13} \sin^2\Delta_{31} \\ &-(s^{\mathcal{B}}_{13})^2 \sin^2 2\theta^{\mathcal{B}}_{12} \sin^2\Delta_{32} \,. \end{aligned} \tag{C2}$$

while all other representations have very complicated expressions. The survival probabilities of $\nu_\mu \to \nu_\mu$ in representations $\mathcal{C}$ and $\mathcal{D}$ are of simpler forms than other representations:

$$\begin{aligned} &P^{\mathcal{C}}_{\nu_\mu \to \nu_\mu} = \\ &1 - (c^{\mathcal{C}}_{23})^4 \sin^2 2\theta^{\mathcal{C}}_{12} \sin^2\Delta_{21} - (s^{\mathcal{C}}_{12})^2 \sin^2 2\theta^{\mathcal{C}}_{23} \sin^2\Delta_{31} \\ &-(c^{\mathcal{C}}_{12})^2 \sin^2 2\theta^{\mathcal{C}}_{23} \sin^2\Delta_{32} \,, \end{aligned} \tag{C3}$$

$$\begin{aligned} &P^{\mathcal{D}}_{\nu_\mu \to \nu_\mu} = \\ &1 - (c^{\mathcal{D}}_{23})^2 \sin^2 2\theta^{\mathcal{D}}_{12} \sin^2\Delta_{21} - (s^{\mathcal{D}}_{23})^2 \sin^2 2\theta^{\mathcal{D}}_{12} \sin^2\Delta_{31} \\ &-(c^{\mathcal{D}}_{12})^4 \sin^2 2\theta^{\mathcal{D}}_{23} \sin^2\Delta_{32} \,. \end{aligned} \tag{C4}$$

Furthermore, representations $\mathcal{E}$ and $\mathcal{F}$ have simpler forms for survival probabilities of $\nu_\tau \to \nu_\tau$:

$$\begin{aligned} &P^{\mathcal{E}}_{\nu_\tau \to \nu_\tau} = \\ &1 - (s^{\mathcal{E}}_{13})^2 \sin^2 2\theta^{\mathcal{E}}_{23} \sin^2\Delta_{21} - (c^{\mathcal{E}}_{23})^4 \sin^2 2\theta^{\mathcal{E}}_{13} \sin^2\Delta_{31} \\ &-(c^{\mathcal{E}}_{13})^2 \sin^2 2\theta^{\mathcal{E}}_{23} \sin^2\Delta_{32} \,, \end{aligned} \tag{C5}$$

$$\begin{aligned} &P^{\mathcal{F}}_{\nu_\tau \to \nu_\tau} = \\ &1 - (s^{\mathcal{F}}_{23})^2 \sin^2 2\theta^{\mathcal{F}}_{13} \sin^2\Delta_{21} - (c^{\mathcal{F}}_{23})^2 \sin^2 2\theta^{\mathcal{F}}_{13} \sin^2\Delta_{31} \\ &-(c^{\mathcal{F}}_{13})^4 \sin^2 2\theta^{\mathcal{F}}_{23} \sin^2\Delta_{32} \,. \end{aligned} \tag{C6}$$

The oscillation channel $\nu_\mu \to \nu_e$ does not have a simpler form if compared to the survival probabilities of $\nu_\alpha \to \nu_\alpha$. One, however, still can work out some simpler expressions in representations $\mathcal{A}$–$\mathcal{D}$ for the limit of $\Delta_{31} \sim \Delta_{32}$, which are

$$P^{\mathcal{A}}_{\nu_\mu\to\nu_e} = (s^{\mathcal{A}}_{23})^2 \sin^2 2\theta^{\mathcal{A}}_{13} \sin^2 \Delta_{31} + 2(c^{\mathcal{A}}_{13})^2 \sin 2\theta^{\mathcal{A}}_{12} \cdot (c^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12} - s^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13})(c^{\mathcal{A}}_{23}s^{\mathcal{A}}_{12} + s^{\mathcal{A}}_{23}c^{\mathcal{A}}_{12}s^{\mathcal{A}}_{13}) \sin^2 \Delta_{21} \ , \quad \text{(C7)}$$

$$P^{\mathcal{B}}_{\nu_\mu\to\nu_e} = \left[(s^{\mathcal{B}}_{13})^2 c^{\mathcal{B}}_{23} \sin 2\theta^{\mathcal{B}}_{12} - s^{\mathcal{B}}_{23}c^{\mathcal{B}}_{12} \sin 2\theta^{\mathcal{B}}_{13}\right]^2 \sin^2 \Delta_{31} + \left[(c^{\mathcal{B}}_{23})^2 (c^{\mathcal{B}}_{13})^2 \sin^2 2\theta^{\mathcal{B}}_{12} + \frac{1}{2} c^{\mathcal{B}}_{12} \sin 2\theta^{\mathcal{B}}_{23} \sin 2\theta^{\mathcal{B}}_{12} \sin 2\theta^{\mathcal{B}}_{13}\right] \sin^2 \Delta_{21} \ , \quad \text{(C8)}$$

$$P^{\mathcal{C}}_{\nu_\mu\to\nu_e} = (s^{\mathcal{C}}_{13})^2 \sin^2 2\theta^{\mathcal{C}}_{23} \sin^2 \Delta_{31} + 2(c^{\mathcal{C}}_{23})^2 \sin 2\theta^{\mathcal{C}}_{12} \cdot (s^{\mathcal{C}}_{12}c^{\mathcal{C}}_{13} - s^{\mathcal{C}}_{23}c^{\mathcal{C}}_{12}s^{\mathcal{C}}_{13})(c^{\mathcal{C}}_{12}c^{\mathcal{C}}_{13} + s^{\mathcal{C}}_{23}s^{\mathcal{C}}_{12}s^{\mathcal{C}}_{13}) \sin^2 \Delta_{21} \ , \quad \text{(C9)}$$

$$P^{\mathcal{D}}_{\nu_\mu\to\nu_e} = \left[(s^{\mathcal{D}}_{23})^2 c^{\mathcal{D}}_{13} \sin 2\theta^{\mathcal{D}}_{12} + c^{\mathcal{D}}_{12}s^{\mathcal{D}}_{13} \sin 2\theta^{\mathcal{D}}_{23}\right]^2 \sin^2 \Delta_{31} + \left[(c^{\mathcal{D}}_{23})^2 (c^{\mathcal{D}}_{13})^2 \sin^2 2\theta^{\mathcal{D}}_{12} - \frac{1}{2} c^{\mathcal{D}}_{12} \sin 2\theta^{\mathcal{D}}_{23} \sin 2\theta^{\mathcal{D}}_{12} \sin 2\theta^{\mathcal{D}}_{13}\right] \sin^2 \Delta_{21} \ . \quad \text{(C10)}$$

The oscillation channel $\nu_\mu \to \nu_\tau$, on the other hand, has simpler forms in representations $\mathcal{C}$–$\mathcal{F}$ in the limit of $\Delta_{31} \sim \Delta_{32}$, which are

$$P^{\mathcal{C}}_{\nu_\mu\to\nu_\tau} = (c^{\mathcal{C}}_{13})^2 \sin^2 2\theta^{\mathcal{C}}_{23} \sin^2 \Delta_{31} + 2(c^{\mathcal{C}}_{23})^2 \sin 2\theta^{\mathcal{C}}_{12} \cdot (s^{\mathcal{C}}_{12}s^{\mathcal{C}}_{13} + s^{\mathcal{C}}_{23}c^{\mathcal{C}}_{12}c^{\mathcal{C}}_{13})(c^{\mathcal{C}}_{12}s^{\mathcal{C}}_{13} - s^{\mathcal{C}}_{23}s^{\mathcal{C}}_{12}c^{\mathcal{C}}_{13}) sin^2 \Delta_{21} \ , \quad \text{(C11)}$$

$$P^{\mathcal{D}}_{\nu_\mu\to\nu_\tau} = \left[(s^{\mathcal{D}}_{23})^2 s^{\mathcal{D}}_{13} \sin 2\theta^{\mathcal{D}}_{12} - c^{\mathcal{D}}_{12}c^{\mathcal{D}}_{13} \sin 2\theta^{\mathcal{D}}_{23}\right]^2 \sin^2 \Delta_{31} + \left[(c^{\mathcal{D}}_{23})^2 (s^{\mathcal{D}}_{13})^2 \sin^2 2\theta^{\mathcal{D}}_{12} + \frac{1}{2} c^{\mathcal{D}}_{12} \sin 2\theta^{\mathcal{D}}_{23} \sin 2\theta^{\mathcal{D}}_{12} \sin 2\theta^{\mathcal{D}}_{13}\right] \sin^2 \Delta_{21} \ . \quad \text{(C12)}$$

$$P^{\mathcal{E}}_{\nu_\mu\to\nu_\tau} = \left[(c^{\mathcal{E}}_{13})^2 c^{\mathcal{E}}_{12} \sin 2\theta^{\mathcal{E}}_{23} - c^{\mathcal{E}}_{23}s^{\mathcal{E}}_{12} \sin 2\theta^{\mathcal{E}}_{13}\right]^2 \sin^2 \Delta_{31} + \left[(c^{\mathcal{E}}_{12})^2 (s^{\mathcal{E}}_{13})^2 \sin^2 2\theta^{\mathcal{E}}_{23} + \frac{1}{2} c^{\mathcal{E}}_{23} \sin 2\theta^{\mathcal{E}}_{23} \sin 2\theta^{\mathcal{E}}_{12} \sin 2\theta^{\mathcal{E}}_{13}\right] \sin^2 \Delta_{21} \ . \quad \text{(C13)}$$

$$P^{\mathcal{F}}_{\nu_\mu\to\nu_\tau} = \left[(c^{\mathcal{F}}_{23})^2 s^{\mathcal{F}}_{12} \sin 2\theta^{\mathcal{F}}_{13} - c^{\mathcal{F}}_{12}c^{\mathcal{F}}_{13} \sin 2\theta^{\mathcal{F}}_{23}\right]^2 \sin^2 \Delta_{31} + \left[(s^{\mathcal{F}}_{23})^2 (s^{\mathcal{F}}_{12})^2 \sin^2 2\theta^{\mathcal{F}}_{13} + \frac{1}{2} c^{\mathcal{F}}_{13} \sin 2\theta^{\mathcal{F}}_{23} \sin 2\theta^{\mathcal{F}}_{12} \sin 2\theta^{\mathcal{F}}_{13}\right] \sin^2 \Delta_{21} \ . \quad \text{(C14)}$$

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