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Transverse instability of a bunched beam with space charge and wake field

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Transverse coherent instability of a bunch in a ring accelerator is studied with space charge and short wake field taken into account. It is assumed that space charge tune shift significantly exceeds the synchrotron tune. The bunch spectrum, instability growth rate and effects of chromaticity are studied over a wide range of parameters. Fast instability caused by coupling of transverse modes is studied as well. It is shown that, for monotonic wakes, the TMCI is possible only with certain sign of the wake. Its threshold is calculated precisely for different bunch and wake forms.

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I. INTRODUCTION

Transverse instability of a bunched beam in a ring accelerator has been considered first by C. Pellegrini [1] and M. Sands [2] with synchrotron oscillations taken into account ("head-tail instability"). Coherent oscillations of a bunch were presented as a sum of uncoupled modes $\propto \exp(im\phi)$ where ϕ is synchrotron phase. After that, F. Sacherer [3] investigated the effect in depth including so called radial modes which describe dependence of the coherent displacement on synchrotron amplitude. Role of space charge effects has been studied first in Ref. [4] on the assumption that the contributed tune shift is small in comparison with synchrotron tune. It was shown that corresponding tune spread causes Landau damping which suppresses most of the head-tail modes. Similar conclusion was made later in Ref. [5] by an analysis of some comparatively simple models. The space charge influence at low synchrotron frequency was thoroughly investigated in recent articles [6] and [7]. Although very similar equations have been presented there, the authors have came to rather different conclusions.

According to Ref. [7], almost all transverse modes are prone to Landau damping until space charge tune shift is less then about synchrotron frequency. Only the lowest (rigid) mode is a general exception from the rule, and 1-2 next modes can be unstable additionally, in dependence on the bunch shape. However, the Landau damping is shut off when the space charge shift becomes more, lower modes being free from the decay first. Actually, it was stated that all eigentunes of the bunch should obtain an imaginary part in response to wake field of any nature and strength. However, no concrete wakes were investigated in this work.

In contrast to this, solutions with several specific wakes were presented in Ref. [6]. The conclusion was done that, at zero chromaticity, small wake cannot violate the bunch stability, and all its eigentunes remain real numbers. The wake should be sufficiently strong to overcome a threshold and to excite the Transverse Mode Coupling Instability due to merge of some eigentunes accompanied by essential change of the eigenfunctions (vanished TMCI, by the author).

The contradiction is resolved in this paper. It is shown that some small parts of basic equation were lost at limiting transition to zero frequency in Ref. [6] With the parts recovered, all the eigentunes prove to be complex numbers almost in all the cases, with rare exclusions due to special combinations of the parameters. The imaginary parts do increase when the TMCI arises; however, recalculated TMCI thresholds partly differ from those presented in Ref. [6] as it is marked and discussed in sections. The effects of chromaticity on the instability growth rate is studied as well.

II. BUNCH MODES EQUATION

Wake field of a beam produces transverse Lorenz force which can be written in the form (per unit charge):

$$\vec{G}^{(L)}(t,\theta^{(L)}) = -\int_0^t W(t-t')\vec{D}^{(L)}(t',\theta^{(L)})\,dt' \qquad (1)$$

where $D^{(L)}(t,\theta^{(L)})$ is linear density of the beam dipole moment in azimuth $\theta^{(L)}$. The equation refers to the laboratory frame, but corresponding rest frame is more convenient for our purpose. Therefore the following variables and definitions will be used hereafter:

$$\theta = \theta^{(L)} - \Omega_0 t, \qquad G(t,\theta) = G^{(L)}(t,\theta^{(L)}), \qquad \text{etc.} \quad (2)$$

where Ω_0 is the bunch angular velocity. Then Eq. (1) transforms to:

$$\vec{G}(t,\theta) = -\int_0^t W(t-t')\vec{D}(t',\theta+\Omega_0(t-t'))\,dt' \qquad (3)$$

With this force and space charge field taken into account, equation of betatron oscillations of a particle is:

$$\frac{d^2x}{dt^2} + \Omega^2 Q^2 x = \frac{eE_x\big(\theta, x-\bar{X}(t,\theta), y\big)}{m\gamma^3} + \frac{eG_x(t,\theta)}{m\gamma} \qquad (4)$$

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where Ω and Q are the particle angular velocity and tune, $\bar{X}(t,\theta)$ is the beam transverse displacement at azimuth θ ($\theta = 0$ in the bunch center). The space charge electric field E_x is, possibly, a nonlinear function of transverse coordinates but nonlinearity of external field is not included in the equation. Note as well that d/dt is the total derivative in time involving longitudinal motion.

An averaging of Eq. (4) over transverse phase space gives an equation for a function $X(t,\theta,p)$ which is transverse displacement of the bunch in the point (θ,p) of longitudinal phase space [7]:

$$\frac{\partial X}{\partial t} + \Omega_s \frac{\partial X}{\partial \phi} + i\,\Omega\, QX \simeq i\,\Omega \Delta Q\,(X - \bar{X}) + \frac{ieG_x(t,\theta)}{2m\gamma\Omega_0 Q_0} \tag{5}$$

Here ϕ and $\Omega_s = \Omega_0 Q_s$ are phase and frequency of synchrotron oscillations which are presumed to be linear. The functions $X(t,\theta,p)$ and $\bar{X}(t,\theta)$ are related by the equation:

$$\rho(\theta)\,\bar{X}(t,\theta) = \int_{-\infty}^{\infty} F(\theta,u)\,X(t,\theta,p)\,dp \tag{6}$$

where $\rho(\theta)$ is linear bunch density associated with longitudinal distribution function F:

$$\rho(\theta) = \int_{-\infty}^{\infty} F(\theta,p)\,dp. \tag{7}$$

Effective space charge tune shift does not depend on transverse coordinates in Eq. (5), demonstrating that nonlinearity of the beam field does not affect the coherent oscillations [8]:

$$\Delta Q(\theta) = \frac{e}{2m\gamma^3\Omega_0^2 Q_0}\int_{-\infty}^{\infty} \frac{\partial E_x}{\partial x}(\theta,x,y)\rho_\perp(x,y)\,dxdy. \tag{8}$$

where $\rho_\perp$ is normalized steady state transverse density of the beam which just produces the electric field $\vec{E}$.

Laplace transformation in time will be applied to Eq. (5) to obtain a new variable defined by the relation

$$Y(\theta,p) = \exp(i\chi\theta)\int_0^{\infty} X(t,\theta,p)\exp(i\omega t)\,dt \tag{9}$$

where

$$\chi = \frac{d(\Omega Q)}{d\Omega} = Q_0 + \Omega_0 \frac{dQ/dp}{d\Omega/dp} = Q_0 + \zeta \tag{10}$$

($\zeta = -\xi/\eta$ is normalized chromaticity). The exponential factor is added before the integral to exclude dependence of the parameters Ω and Q on momentum in Eq. (5). With zero initial conditions, the equation transforms to

$$(\omega - \Omega_0 Q_0)Y + i\,\Omega_0 Q_s \frac{\partial Y}{\partial \phi} + \Omega_0 \Delta Q(\theta)\,(Y - \bar{Y}) \tag{11}$$

$$= \frac{e\,\exp(i\chi\theta)}{2m\gamma\Omega_0 Q_0}\int_0^{\infty} W(t')D_\omega(\theta + \Omega_0 t')\exp(i\omega t')\,dt'$$

where

$$D_\omega(\theta) = \int_0^{\infty} D_x(t,\theta)\exp(i\omega t)\,dt \tag{12}$$

Note that relation like Eq. (6) is valid for the functions $Y(\theta,p)$ and $\bar{Y}(\theta)$ as well.

Variable θ has a range $[-\pi,\,\pi]$ (one turn) whereas the actual bunch has a less range $[-\theta_0,\,\theta_0]$. Therefore it is more convenient to use normalized value

$$\tau = \frac{\theta}{\theta_0} \tag{13}$$

with a range $[-1,\,1]$ (it was denoted as θ in Ref. [7], but the symbol is occupied now). Then Eq. (6) and (7) hold true with formal replacement τ instead of θ, because normalization of the distribution function was not specified yet. Further we will apply the condition:

$$\int_{-1}^{1} \rho(\tau)\,d\tau = 1 \tag{14}$$

Then linear density of the dipole moment is:

$$D_\omega = \frac{e\bar{X}_\omega}{R}\frac{dN}{d\theta} = \frac{eN\bar{Y}(\tau)\rho(\tau)}{R\theta_0}\exp(-i\chi\theta_0\tau) \tag{15}$$

where R is the machine radius, and N in number of particles in the bunch. Designating

$$\frac{e^2 NW(t)}{2m\gamma R\Omega_0^2 Q_0} = 2\,\Omega_0 q(\Omega_0 t) \tag{16}$$

one can rewrite Eq. (11) in the form:

$$\Big(\frac{\omega}{\Omega_0} - Q_0\Big)Y + i\,Q_s\frac{\partial Y}{\partial\phi} + \Delta Q(\theta)\,(Y - \bar{Y}) = \tag{17}$$

$$2\int_0^{\infty} q(\theta_0\tau')\exp\Big(i\theta_0\tau'\big(\frac{\omega}{\Omega_0} - \chi\big)\Big)\bar{Y}(\tau'+\tau)\rho(\tau'+\tau)\,d\tau'$$

Without the right-hang part, the equation has a universal solution (rigid mode):

$$Y = \bar{Y} = 1, \qquad \frac{\omega}{\Omega_0} = Q_0 \tag{18}$$

It is an easy matter to add a contribution of a sufficiently weak wake field to the tune. In a simplest case (constant wake, low chromaticity, boxcar bunch), the shift is:

$$\frac{\Delta\omega}{\Omega_0} \simeq q\,\Big(1 - \frac{2i\theta_0\zeta}{3}\Big) \tag{19}$$

Hence, q coincides in order of value with addition to the lowest eigentune. Note that q is negative in many practical cases (e.g. resistive wall wake). According to Eq. (19), similar wakes should decrease eigentune of this mode and excite its instability at $\zeta > 0$. The statements hold true independently on ΔQ.

III. LOW SYNCHROTRON FREQUENCY LIMIT

The ultimate case $\mu = Q_s/\Delta Q \ll 1$ will be considered in next sections. Then, using the methods developed in Ref. [7], one can get equation for the function $\bar{Y}(\tau)$:

$$U^2 \frac{d^2\bar{Y}}{d\tau^2} - \left(\tau + \frac{U^2}{\nu\rho_c + \rho}\frac{d\rho}{d\tau}\right)\frac{d\bar{Y}}{d\tau} + \frac{\nu(\nu + \rho/\rho_c)}{\mu^2}\bar{Y} =$$

$$\frac{2\Delta Q_c \rho(\tau)}{Q_s^2 \rho_c}\int_0^\infty q(\theta_0\tau')\exp\Big(i\theta_0\tau'(\nu\Delta Q_c - \zeta)\Big) \times$$

$$\bar{Y}(\tau' + \tau)\rho(\tau' + \tau)\, d\tau' \qquad (20)$$

with

$$\nu = \frac{\omega - \Omega_0 Q_0}{\Omega_0 \Delta Q_c}, \qquad U^2 = \frac{1}{\rho(\tau)}\int F(\tau, u)u^2 du \qquad (21)$$

Here and after, subindex c marks the bunch center, u is reduced particle momentum: $u^2 = A^2 - \tau^2$, and $A = \tau_{max}$ is amplitude of synchrotron oscillation.

Eq. (20) without the wake is obtained in Ref. [7] where details of the derivation are provided as well. With an additional assumption $\nu \propto \mu^2 \to 0$, the equation coincides with Eq. (60) of Ref. [6]. Such limiting transition is quite legitimate in the left-hand side of Eq. (20) which bears only real coefficients. However, it totally annihilates the imaginary part of the right-hand side at $\zeta = 0$. Such was indeed in case in Ref. [6] when the conclusion was made that, at zero chromaticity, all eigentunes are real numbers as long as the wake reaches the TMCI threshold. However, it is seen that the right hand part of Eq. (20) is a complex value even at $\zeta = 0$. Purely real eigentunes of similar equations are possible only in exceptional cases. This and other discrepancies will be considered below in detail.

So the equation to be investigated is:

$$U^2 \frac{d^2\bar{Y}}{d\tau^2} = R(\tau) \qquad (22)$$

with

$$R(\tau) \;=\; \left(\tau + \frac{U^2}{\rho}\frac{d\rho}{d\tau}\right)\frac{d\bar{Y}}{d\tau} - \frac{\nu\rho\bar{Y}}{\mu^2\rho_c} \;+ \qquad (23)$$

$$\frac{2\Delta Q_c \rho(\tau)}{Q_s^2 \rho_c}\int_\tau^1 q\big(\theta_0(\tau' - \tau)\big)\bar{Y}(\tau')\rho(\tau')\exp(i\varphi)\, d\tau'$$

and

$$\varphi = \theta_0(\nu\Delta Q_c - \zeta)(\tau' - \tau) \qquad (24)$$

Boundary conditions are evident just from the equation itself, because of the relation $U^2(\pm 1) = 0$ which follows right from definition (21) and will be reinforced by examples in the subsequent text. Therefore any appropriate solution of Eq. (22) should satisfy the boundary conditions:

$$R(\pm 1) = 0. \qquad (25)$$

All of them are regular functions, that is Landau damping plays no part at $Q_s \ll \Delta Q_c$ [7].

Actually the solution will be performed in two steps. Firstly, it will be solved at $\varphi = 0$ without additional assumptions or approximations like perturbation theory, independently on the wake amplitude. Thereafter the obtained solution will be used to find imaginary additions to the frequency due to non-zero φ. Rather serious restriction $|\zeta\theta_0| <\sim 1$ will be actually applied in this stage. However, it allows to define at least sign of the chromaticity required to depress the instability. More details are given below by specific examples.

IV. BOXCAR MODEL, RECTANGULAR WAKE

A constant wake within a bunch of constant density is considered in this section:

$$q = q_0, \quad \rho(\tau) = \frac{1}{2}, \quad U^2 = \frac{1 - \tau^2}{2} \quad \text{at} \quad |\tau| \le 1 \qquad (26)$$

This simple model is very pertinent to demonstrate both characteristic features of the phenomenon and procedure of the solving to be used.

A. Solution with $\varphi = 0$

First, Eq. (22) is investigated under the assumption $\varphi = 0$ what means, in particular, a negligible chromaticity. To satisfy boundary condition (25) in the bunch head, we have to take at $\tau = 1$:

$$\bar{Y} = 1, \qquad \frac{d\bar{Y}}{d\tau} = \frac{(\omega/\Omega_0 - Q_0)\Delta Q_c}{Q_s^2} \qquad (27)$$

With arbitrary trial ω and these initial conditions, solution of Eq. (22) can be found to know the function $R(\tau)$ everywhere, including the bunch tail $\tau = -1$. Applying boundary condition (25) to this point, one can select appropriate values of ω which just are the bunch eigentunes. In practice, it is a good way to get hight precision real eigentunes including the TMCI thresholds in a designated area. Complex eigentunes could be found by this method as well, but it would require essentially more widespread sorting over the trial frequencies. Therefore, other methods will be used below to calculate the TMCI growth rate.

The method is illustrated by Fig. 1 where the value $R(-1)$ (misalignment) is plotted against the real frequency at different wakes being presented in terms of the parameters h and K:

$$h(h+1) = \frac{2(\omega/\Omega_0 - Q_0)\Delta Q_c}{Q_s^2}, \qquad \mathrm{K} = \frac{\mathrm{q_0 \Delta Q_c}}{\mathrm{Q_s^2}} \qquad (28)$$

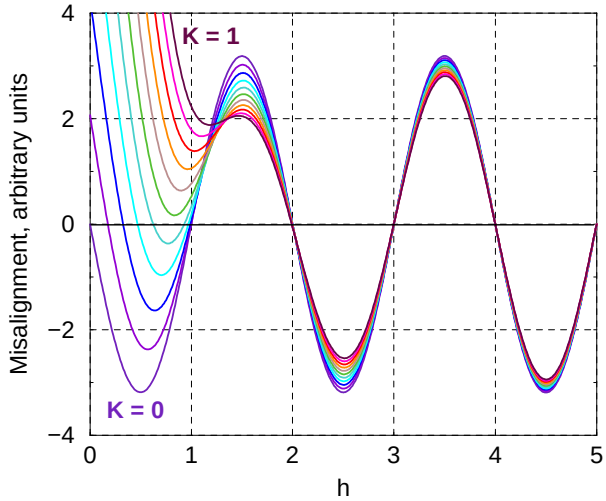

FIG. 1: Misalignment $R(-1)$ vs tune at different positive wakes (boxcar model, rectangular wake).

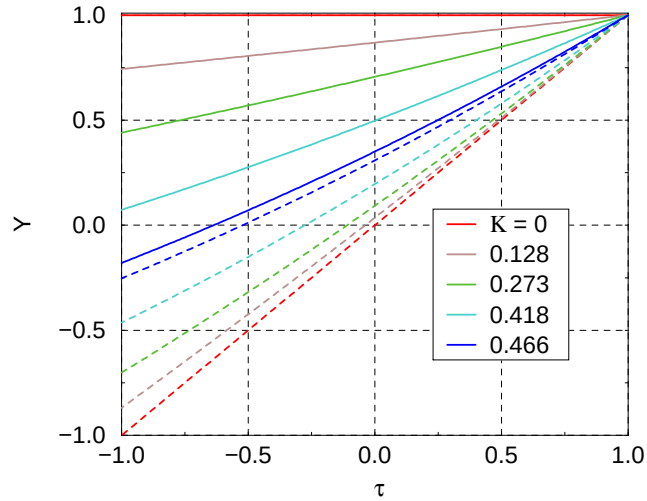

FIG. 3: Eigenfunctions of 0th and 1st modes (solid and dashed) at different positive K (boxcar, rectangular wake).

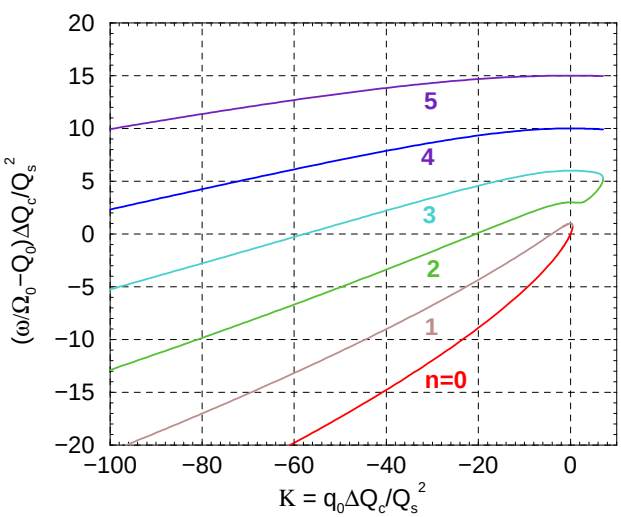

FIG. 2: Eigentunes of several modes vs modified wake strength (boxcar model, rectangular wake).

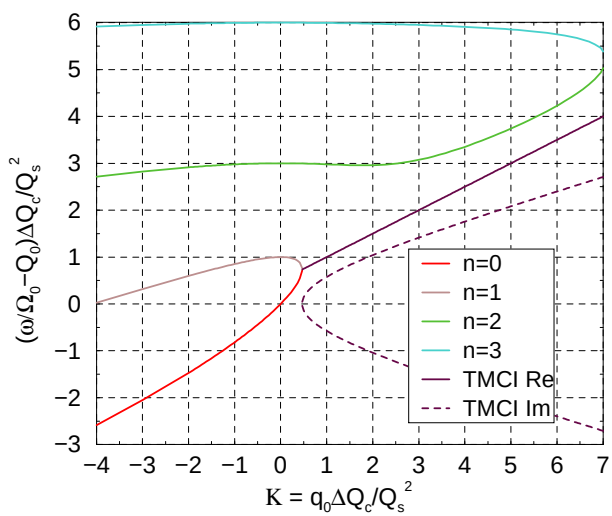

FIG. 4: Eigentunes of lower modes vs the wake strength. Zoom of Fig. 2 + TMCI (boxcar model, rectangular wake).

Convenience of such a parameterization is that all the eigennumbers are integer at $\mathrm{K} = 0$: $h_n = n$. By Fig. 1, most of them almost do not depend on K in considered interval. There are only two essential exclusions: the eigentunes h_0 and h_1 approach each other when the wake increases, and merge at $\mathrm{K} = 0.467$ what is the TMCI threshold. Extended picture is shown in Fig. 2 where several eigentunes are presented in a wider range including negative wakes. It is seen that second TMCI threshold appears at $\mathrm{K} \simeq 7$ by merge of 2nd and 3rd modes. Other TMCI modes can appear at higher K but they never been observed with negative K.

The basic ($\mathrm{K} = 0$) eigenfunctions of the boxcar bunch are Legendre polynomials [3, 7]. Eigenfunctions of 0th and 1st modes are plotted in Fig. 3 at different positive K below the first TMCI threshold. They are about linear functions that is can be presented as linear combinations of corresponding basic eigenmodes. With this assumption, the eigenfunctions and eigentunes are:

$$\bar{Y}(\tau) = 1 - \frac{2\mathrm{K}\tau}{1 - \mathrm{K} \mp \mathrm{S}}; \quad \frac{\omega}{\Omega_0} - Q_0 = \frac{Q_s^2}{2\Delta Q_c}(1 + \mathrm{K} \pm \mathrm{S}) \tag{29}$$

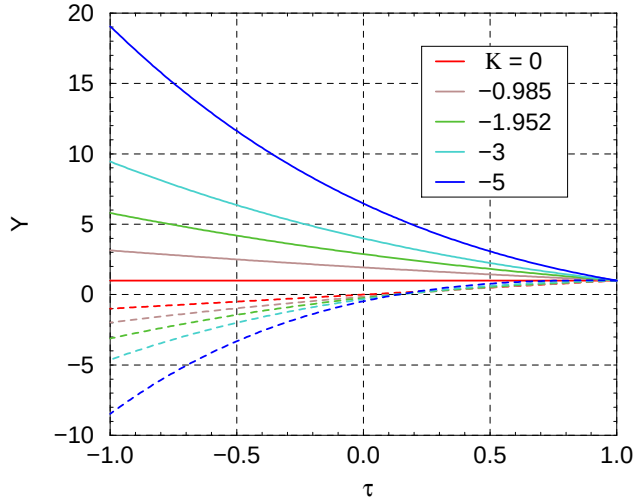

FIG. 5: Eigenfunctions of 0th and 1st modes (solid and dashed lines) with negative K (boxcar model, rectangular wake).

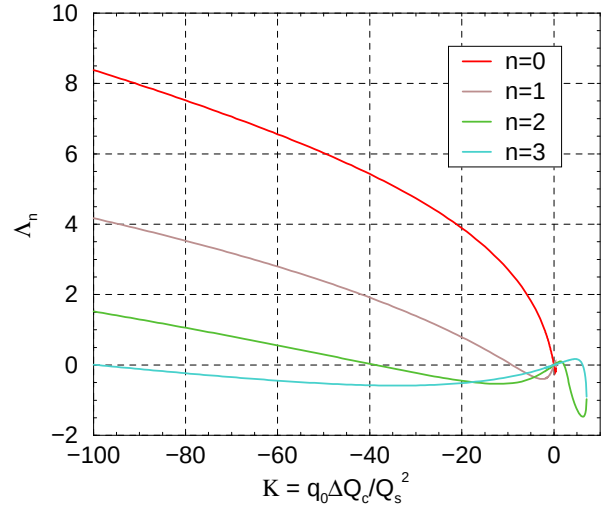

FIG. 6: Chromaticity factor $\Lambda_n(B)$ of different modes against the wake strength (boxcar model, rectangular wake).

where

$$S = \sqrt{1 - 2\mathrm{K} - \frac{\mathrm{K}^2}{3}} \tag{30}$$

All of them are real numbers at $\mathrm{K} < 0.467$ which boundary almost coincides with the numerically calculated TMCI threshold. Being continued beyond the threshold, the formulae results in complex tunes which are presented in Fig. 4 where part of Fig. 2 is also replicated on a larger scale.

Different behavior demonstrate eigenfunctions of the bunch with negative wakes. Examples are given in Fig. 5 where the lowest functions are plotted at several $\mathrm{K} < 0$. In contrast with Fig. 3, the bunch deviation accrues about exponentially from the bunch head to tail, and the curves diverge faster at stronger wake. Higher modes have similar behavior leading to the conclusion that TMCI is impossible at $\mathrm{K} < 0$. Essential dissimilarity of the eigenfunctions from the basic ones (Legendre polynomials) at $|\mathrm{K}| \gg 1$ deserves an attention, too.

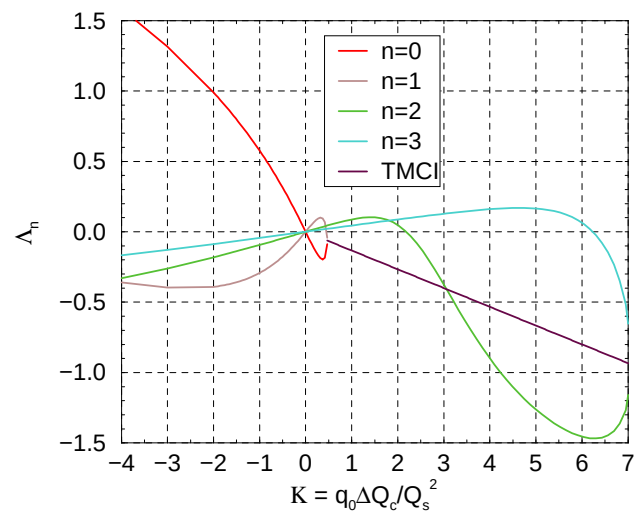

FIG. 7: The same as in Fig. 6, zoom and 1st TMCI mode added (boxcar model, rectangular wake).

B. Effect of chromaticity

Considering Eq. (22) and (23) with $\varphi \neq 0$, we will treat its effect as a small perturbation of the above given eigenmodes. The assumption imposes the condition $|\zeta\theta_0| < 1$, at least, which may be a rather rigorous restriction in practice. However, it allows to determine sign of chromaticity required to depress the instability, at least.

Imaginary addition to the tune is the most important effect which can be described by the formula:

$$\mathrm{Im}\,\frac{\Delta\omega_n \Delta Q_c}{\Omega_0 Q_s^2} = \Lambda_n(\mathrm{K})\,\theta_0\Big[\zeta - \mathrm{Re}\Big(\frac{\omega_\mathrm{n}}{\Omega_0} - \mathrm{Q}_0\Big)\Big] \tag{31}$$

Contribution of the last term may be small numerically, but it shows that, with rare exclusions, the eigentunes are complex numbers even the chromaticity is zero.

Parameter Λ_n is plotted against wake strength in Fig. 6 and 7. According to them, opposite signs of chromaticity are needed to suppress 0-th or any other mode, when the parameter $|\mathrm{K}|$ is sufficiently small. Similar situation is very known in the case of "usual" (no space charge) head-tail instability [1, 2]. However, it is seen that more and more modes can be depressed along with 0th one at higher $|\mathrm{K}|$, if sign of chromaticity coincides with the wake sign.

V. GAUSSIAN BUNCH, RECTANGULAR WAKE

Constant wake within a truncated Gaussian bunch is considered in this section:

$$F \propto \exp\frac{1-A^2}{2\sigma^2} \qquad \text{at} \qquad A \le 1 \tag{32}$$

Characteristic functions of the bunch for Eq. (22)-(23) are at $|\tau| \le 1$:

$$\rho(\tau) = C\left[\sigma\sqrt{\frac{\pi}{2}}\exp\left(\frac{1-\tau^2}{2\sigma^2}\right)\mathrm{erf}\left(\frac{\sqrt{1-\tau^2}}{\sigma\sqrt{2}}\right) - \sqrt{1-\tau^2}\right] \tag{33}$$

and

$$U^2 = \sigma^2 - C\,\frac{(1-\tau^2)^{3/2}}{3\rho(\tau)} \tag{34}$$

where $C \simeq \exp(-1/2\sigma^2)/(\pi\sigma^2)$ is a normalizing coefficient. Note that at $|\tau| \simeq 1$

$$\rho(\tau) \simeq \frac{C(1-\tau^2)^{3/2}}{3\sigma^2}\left[1+\frac{1-\tau^2}{5\sigma^2}\right], \qquad U^2 \simeq \frac{1-\tau^2}{5} \tag{35}$$

It is asserted in Ref. [6] that TMCI can arise in non-truncated Gaussian bunch at zero chromaticity and negative wake corresponding $\mathrm{K} < -180$, the instability being provoked by merge of 2nd and 3rd base modes. Positive wakes are not presented in the work.

Our calculations lead to the conclusions that Gaussian bunch is very similar to the boxcar one in behavior. In particular, in both cases TMCI can be caused only by positive wake, appears as a merge of 0th and 1st modes, and has a threshold at $\mathrm{K} \simeq 1$.

Examples are given below with $\sigma = 1/3$ what means 3σ truncation. The coefficient C is about 0.0339 in this case. Method of solution is the same as in the boxcar case with even simpler initial conditions:

$$\bar{Y}(1) = 1, \qquad \bar{Y}'(1) = 0. \tag{36}$$

Dependence of eigentunes on the wake strength is presented in Fig. 8 and 9 at zero chromaticity. The plots look much like Fig. 2 and 4 presenting the boxcar model. With negative wake, the lines diverge more and more when the wake enhances, so that TMCI is not observed in this region. The statement is sustained by Fig. 10 where the lowest eigenfunctions are plotted for several negative wakes. Like previous, the curves in the graph come apart more and more giving no way to expect the TMCI appears.

Very different behavior is demonstrated by the eigenfunctions with positive wakes as it is shown in Fig. 11. A reciprocal approach of 0th and 1st eigenmodes occurs before the wake parameter reaches threshold value $\mathrm{K} = 0.612$. Therefore, in the threshold vicinity, the eigenfunctions are adequately representable as a linear combination of corresponding basic eigenfunctions (red lines in Fig. 11). Other basic modes give an unessential contribution because their tunes differ at least by several units. TMCI growth rate can be obtained by applying of this assumption above the threshold as it is shown in Fig. 9. Second TMCI threshold appears at $\mathrm{K} = 9.78$ by merge of 2nd and 3rd eigenmodes.

Chromaticity effects are described by Eq. (31) with coefficients Λ_n which are shown in Fig. 12 and 13 being very similar to Fig. 6-7.

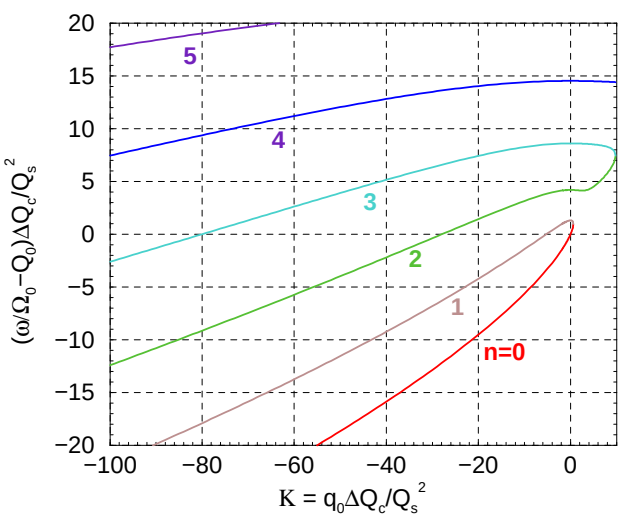

FIG. 8: Eigentunes of several modes against wake strength (Gaussian bunch, rectangular wake).

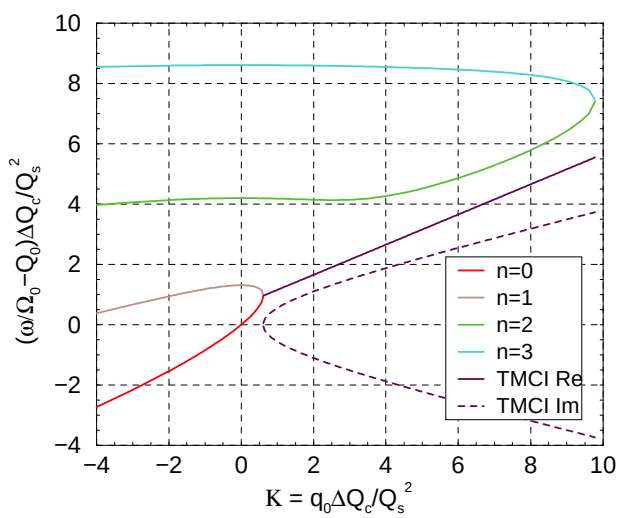

FIG. 9: Eigentunes of lower modes against the wake strength. Zoom of Fig. 8 + TMCI (Gaussian bunch, rectangular wake).

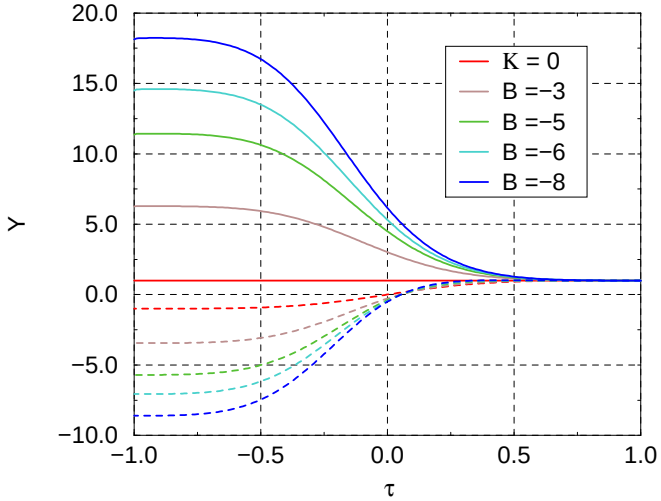

FIG. 10: Eigenfunctions of 0th and 1st modes (solid and dashed lines). Gaussian bunch, negative rectangular wake.

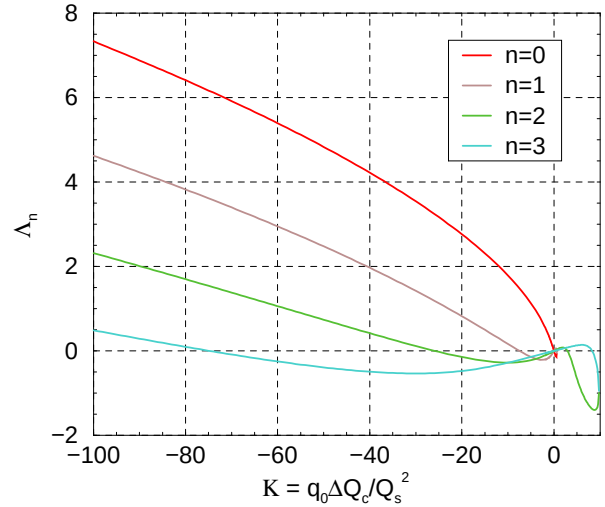

FIG. 12: Chromaticity factor $\Lambda_n(\mathrm{K})$ for several modes against the wake strength (Gaussian bunch, rectangular wake).

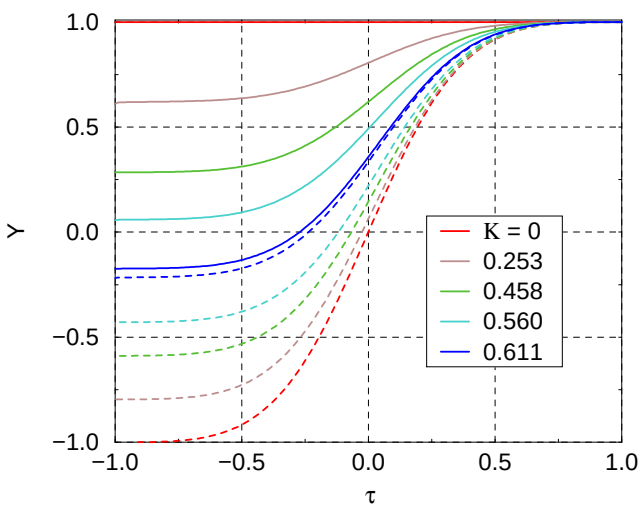

FIG. 11: Eigenfunctions of 0th and 1st modes (solid and dashed lines) before the TMCI appears. Gaussian bunch, positive rectangular wake.

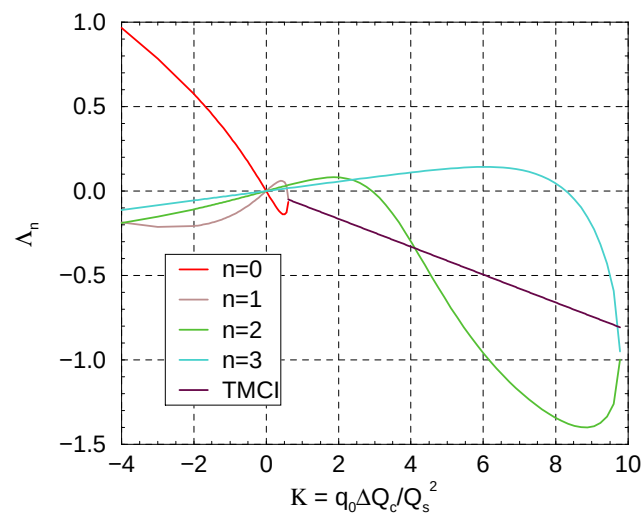

FIG. 13: Zoom of Fig. 12 to show two TMCI thresholds (Gaussian bunch, rectangular wake). Chromaticity contribution above first TMCI threshold is shown as well.

VI. GAUSSIAN BUNCH, EXPONENTIAL WAKE

Gaussian bunch from previous section is combined here with the exponential wake:

$$W(t) \propto \exp\Big(-\frac{t}{t_0}\Big) \tag{37}$$

that is $q = q_0 \exp(-\alpha\tau)$ where $\alpha = \theta_0/(\Omega_0 t_0)$. The same method of solution is adaptable and effective in this case because the wake field factor breaks up in the equation into independent parts:

$$q\big(\theta_0(\tau' - \tau)\big) = q_0 \exp(\alpha\tau)\exp(-\alpha\tau') \tag{38}$$

The results are presented at $\alpha = 1$ in Fig. (14) and (15) being very similar to the case of rectangular wake. As before, TMCI is possible only with positive wake though the TMCI thresholds are a bit higher: 1.03 instead of 0.612 and 14.4 instead of 9.8 (Fig. 14). Depression factor 1.5-1.7 is just consistent with what is expected from average wake decay in the range of the bunch. Chromaticity effects are decreased by about by same factors (Fig. 15).

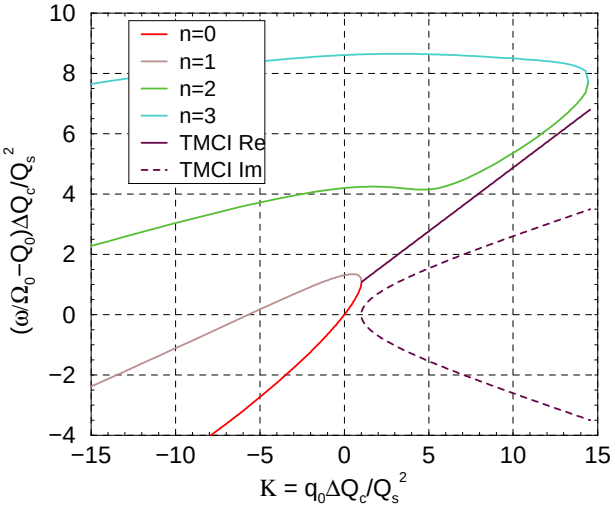

FIG. 14: Eigentunes of lower modes including TMCI with positive wake (Gaussian bunch, exponential wake).

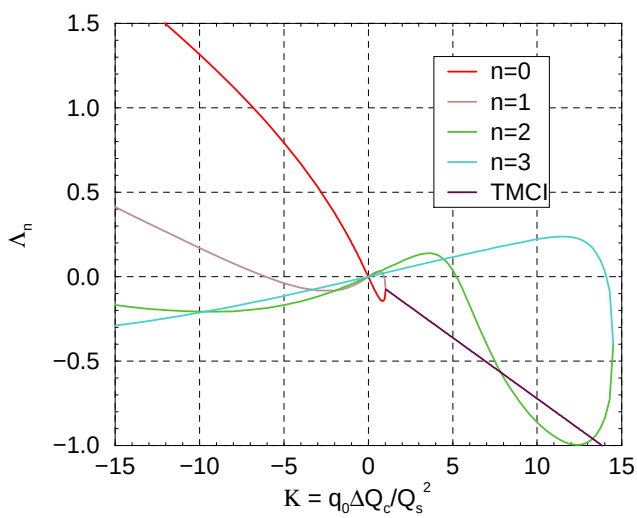

FIG. 15: The same as in Fig. 6 at positive wake. Two TMCI thresholds are seen (Gaussian bunch, exponential wake).

VII. SUMMARY AND DISCUSSION

Basic equation for transverse bunch oscillations is obtained in the article using two general assumptions:

$$Q_s \ll \Delta Q_c \qquad \text{and} \qquad \left|\frac{\omega}{\Omega_0} - Q_0\right| \ll \Delta Q_c$$

As it shown in the paper, second of them is equivalent to the condition:

$$|q| \ll \Delta Q_c$$

which is valid for the lower eigenmodes, at least. It means that space charge dominates in the impedance budget, whereas the wake is relatively weak. In spite of this, the wake can cause a radical change of the bunch spectrum, a circumstance which seriously restricts applicability of the perturbation methods for solution of the equation. Such methods are certainly valid at the additional condition

$$|\mathrm{K}| = \frac{|q|\Delta Q_c}{Q_s^2} < 1$$

but a special analysis is required if it is violated.

Most of presented results are obtained with no using of the perturbation methods. It is shown that, with zero chromaticity and monotonically decreasing wakes, only positive wakes can excite Transverse Mode Coupling Instability by a confluence of two lowest eigenmodes. Its lowest threshold is located at $\mathrm{K} \simeq 1$ so the perturbation theory can be used to determine the instability growth rate. Effects of chromaticity are studies as well with additional assumption that contributed phase advance is ~ 1 or less within the bunch.

Doubtless, resistive wall wake $q \propto -1/\sqrt{\tau}$ is of most interest in practice. The wake decays very slowly so that multi-bunch and multi-turns effects should be necessarily taken into account for its full investigation. Nevertheless, it can be predicted right now that this *negative* wake does not cause a merge or approach of the eigenmodes producing TMCI like effects.

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