

# Dirac Branes and Anomalies/Chern-Simons terms in any $D$

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## Abstract

The Dirac quantization procedure of a magnetic monopole can be used to derive the coefficient of the  $D = 3$  Chern-Simons term through a self-consistency argument, which can be readily generalized to any odd  $D$ . This yields consistent and covariant axial anomaly coefficients on a  $D - 1$  boundary, and Chern-Simons term coefficients in  $D$ . In  $D = 3$  magnetic monopoles *cannot exist* if the Chern-Simons  $AdA$  term is present. The Dirac solenoid then becomes a physical closed string carrying electric current. The charge carriers on the string must be consistent with the charge used to quantize the Dirac solenoidal flux. This yields the Chern-Simons term coefficient. In higher odd  $D$  the intersection of  $(D - 1)/2$  Dirac branes yields a charged world-line permitting the consistency argument. The covariant anomaly coefficients follow readily from generalizing the counterterm. This purely bosonic derivation of anomalies is quite simple, involving semiclassical evaluation of exact integrals, like  $\int dAdA\dots dA$ , in the brane intersections.

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# 1 Introduction

Two of the most fundamental results in quantum field theory are the scale anomaly and axial anomaly. Both effects begin at order  $\hbar$  (one-loop), and represent true quantum breaking of classical symmetries. The scale anomaly is equivalent to the renormalization group  $\beta$ -function and running of the coupling constant and through it quantum field theory can establish a fundamental mass scale by dimensional transmutation, as happens in the case of QCD (for a review see [1] and references therein).

The axial anomaly [2, 3, 4, 5, 6] arises in even  $\hat{D}$  (generally we'll use  $\hat{\phantom{x}}$  to denote even dimension). Weyl spinor loops avoid many ambiguities and yield the “consistent anomaly,” which has the form in a  $U(1)$  theory,  $(e/2\pi)^{\hat{D}/2}/((\hat{D}+2)/2)!\epsilon_{\mu\nu\dots\rho\sigma}\partial^\mu A^\nu\dots\partial^\rho A^\sigma$ . Fermion loop calculations that yield this simple result, however, are rather cumbersome [7]. This result is equivalent to the Chern-Simons (CS) term coefficient one dimension higher, since the  $\hat{D} + 1$  CS term generates the anomaly on a boundary under a gauge transformation. There are various arguments to fix the coefficients of the CS term, often in the context of normalizing the charges of solitons arising in the nonabelian case. We presently seek a transparent argument for the general case within a  $U(1)$  gauge theory.

We will give presently a simple illustration as to how the axial anomaly arises directly from Dirac's construction of the magnetic monopole in  $D = 3$  for pure electrodynamics. First we note that Dirac monopoles *do not exist when the Chern-Simons term is present*. The conserved Chern-Simons current requires that Dirac solenoids, which carry quantized magnetic flux, must then become closed loops. These loops carry electric current, or in  $D = 1+2$ , the solenoids become the world-lines of charged particles. The Dirac solenoid in any  $D$ -odd generalizes to a  $D-2$  dimensional hypersurface, or Dirac brane, first considered by Teitelboim [8], to which a quantized electromagnetic flux is attached. The intersection of  $(D-1)/2$  Dirac branes becomes a charged particle world-line when the Chern-Simons term is turned on.

Our essential trick, therefore, is to note that in any  $D$ -odd there is always a configuration of electromagnetic fields (not necessarily a solution to equations of motion), typically an intersection of Dirac branes, that forms a charged particle world-line in the presence of the CS term. The resulting electric charge of this special configuration must then be consistently set equal to the original charge used to quantize the Dirac flux. This “bootstrapping” condition then dictates the coefficient of the CS term in any odd  $D$ . The result depends only upon exact integrals, *e.g.*, the “core structure” of a Dirac

brane is irrelevant. The boundary of the CS term under a gauge transformation yields the consistent anomaly. We furthermore obtain the “covariant anomaly” by generalizing the Adler-Bardeen counterterm.

This analysis is carried out in a  $U(1)$  theory, but the result is general to any nonabelian theory. The result is completely bosonic in origin, requiring no fermion loop calculation. It virtually reduces to a “back of an envelope” computation. In nonabelian theories the CS term controls the properties of various solitons, such as instantonic vortices in  $D = 5$ . To us, the present result illuminates why the  $D$ -odd CS term in a  $U(1)$  theory,  $\epsilon_{\lambda\mu\nu\dots\rho\sigma} A^\lambda \partial^\mu A^\nu \dots \partial^\rho A^\sigma$ , exists at all and what it is physically measuring, *i.e.*, the world-line intersection of Dirac branes.

## 2 Dirac Monopoles and the Chern-Simons Term

### 2.1 The Dirac Monopole

The intertwining of quantum physics and topology begins with Dirac’s construction of the magnetic monopole in  $D = 3$  (see reviews of [9], [10]). We give a quick review in this section.

Dirac imagined an idealized solenoid in three space dimensions carrying a magnetic flux  $\Phi = \int_S B \cdot d(\text{area})$  where the integral is over a cross-section of the solenoid. The mass per length of the solenoid is neglected. The solenoid can be viewed as an infinitely long ray terminating at a point in space,  $\vec{x}_0$ . At  $\vec{x}_0$  the magnetic flux emerges from the open end of the solenoid. The magnetic charge of the monopole is given by Gauss’ law:

$$4\pi g_m = \Phi \tag{2.1}$$

Dirac asked how to make the solenoid undetectable in electrodynamics? Classically one can hide the solenoid arbitrarily well by making it have an arbitrarily small cross-sectional area. However, the infinitesimal solenoid remains detectable at large distances in quantum mechanics. External to the solenoid there is a circumferential vector potential. By Stoke’s theorem:

$$\Phi = \oint \vec{A} \cdot d\vec{z} \tag{2.2}$$

The line integral over  $\vec{A}$  determines the phase shift,  $\phi$ , of an electron wave-function as the electron, of charge  $e$ , loops the solenoid (the Aharonov-Bohm phase):

$$\phi = e \oint \vec{A} \cdot d\vec{z} / \hbar \tag{2.3}$$

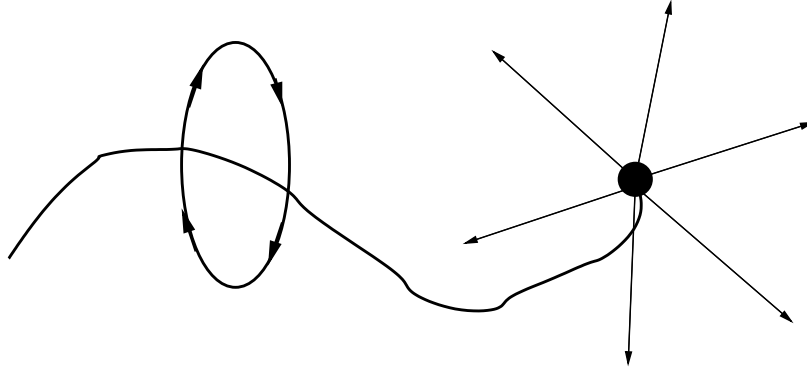


Figure 1: Dirac monopole construction with solenoid. The flux is quantized such that the Aharonov-Bohm phase for an encircling electron is a multiple of  $2\pi$ .

For arbitrary  $\phi$  the solenoid is observable as a diffraction pattern of an impinging, long wave-length electron beam. However, if  $\phi = 2\pi N$  then this phase shift is unobservable and the solenoid is “cloaked.” This implies that the magnetic flux must be quantized as  $\Phi = \hbar\phi/e = 2\pi\hbar N/e$ . Thus the magnetic charge of a Dirac monopole is then quantized:

$$g_m = \frac{N\hbar}{2e} \quad (2.4)$$

In the following we will consider the case  $N = 1$ .

The solenoid part of the construction can be removed by describing a single monopole with two vector potentials, respectively on the left (right) hemisphere of a monopole with Dirac solenoid running in from the right (left). Wu and Yang [11] demand that these two potentials are gauge equivalent in an overlapping region. This latter consistency condition then enforces the monopole quantization condition. In this sense the solenoidal becomes an artificial component of the construction.

Our present perspective, however, is exactly the opposite: we keep the solenoid in  $D = 3$ , and in fact, we *discard the monopole*. In fact, *we must discard the monopole* when the CS term is incorporated into the theory, as we will now discuss. The fate of the solenoid then becomes interesting: The Dirac solenoids can be taken to be closed loops of string, which resemble closed bosonic strings, or infinite time-like world-lines in  $D = 1 + 2$  Minkowski space.

## 2.2 D=3 Electrodynamics with Chern-Simons Term

Let us now incorporate the CS term into the action of  $D = 3$  electrodynamics:

$$S_{CS} = -\kappa \int d^3x \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho = -\kappa \int AdA = -\kappa \int \vec{A} \cdot (\vec{\nabla} \times \vec{A}) \quad (2.5)$$

The CS term depends explicitly upon the vector potential and forces the parameterization of the electromagnetic field,  $F_{\mu\nu}$ , to be determined by it. We also include the kinetic term  $(-1/4) \int F_{\mu\nu} F^{\mu\nu} \rightarrow (1/2) \int \vec{A} \cdot (\nabla^2 \vec{A} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A}))$  into the action.

The resulting Maxwell's equations are modified by the presence of the CS term,

$$\nabla^2 \vec{A} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = 2\kappa \vec{B} \quad (2.6)$$

where  $\vec{B} = \vec{\nabla} \times \vec{A}$ . Crossing the Maxwell equation with  $\vec{\nabla}$  we have,  $\nabla^2(\vec{\nabla} \times \vec{A}) = 2\kappa \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = 2\kappa(\nabla^2 \vec{A} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A}))$  and, using the Maxwell equation again on the *rhs*, we thus have:

$$\nabla^2 \vec{B} = 4\kappa^2 \vec{B} \quad (2.7)$$

This latter form displays the fact that the CS term induces a mass for the photon [12, 13] (in  $D = 5$  or higher the CS term induces interactions amongst KK-modes that typically violate T-parities [14]).

However, with the magnetic field defined as usual,  $\vec{B} = \vec{\nabla} \times \vec{A}$ , we have everywhere outside the solenoid:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (2.8)$$

From the equation of motion eq.(2.7) a Dirac monopole, with nonzero  $\kappa$ , would have to produce a radial magnetic field that attenuates in the Yukawa form:  $\vec{B} \propto \vec{\nabla} \phi$  where  $\phi = (\exp(-2\kappa r)/r)$ . This would require a corresponding nonzero  $\nabla \cdot B \propto \vec{\nabla}^2 \phi = 4\kappa^2 \phi$  everywhere, violating eq.(2.8). Hence, a radial magnetic field, or a corresponding magnetic charge cannot exist with  $\kappa \neq 0$ ! The CS term requires that solenoids not terminate with open ends. Solenoids thus become closed loops carrying electric current.

Consider the Dirac solenoid loop in Fig.(2). Let us describe the loop by coordinate  $\vec{x}(\tau)$  parameterized by  $\tau$ . The solenoid has a circumferential vector potential,  $\vec{A}_{solenoid}$ , and a ‘‘core’’ field  $\vec{B}_{core} = \vec{\nabla} \times \vec{A}_{solenoid}$ .  $\vec{B}_{core}$  is a singular field attached to the solenoid loop. Consider the coupling to an external ‘‘photon,’’  $\vec{A}^{ext}$ , by the solenoid loop. Here we effectively evaluate the matrix element of  $AdA = \vec{A} \cdot (\vec{\nabla} \times \vec{A}) = \vec{A} \cdot \vec{B}$  in a coherent state containing the solenoidal field and external field:  $|\vec{B}_{core}, \vec{A}^{ext}\rangle$ . The matrix element

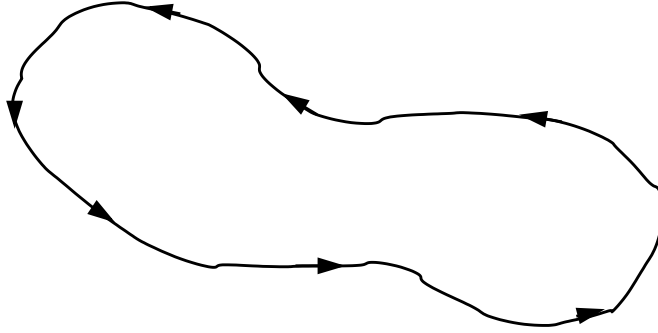


Figure 2: The solenoid loop becomes a current loop when the CS term is included into the action. Demanding that the current carrier charge is  $e$  fixes the coefficient of the CS term:  $\kappa = e^2/4\pi$ .

$\langle \vec{B}_{core}, \vec{A}^{ext} | AdA | \vec{B}_{core}, \vec{A}^{ext} \rangle$  becomes  $2\vec{A}^{ext} \cdot \vec{B}_{core}$ , where the factor of 2 arises because we can contract the fields in  $AdA$  with the internal flux and external photon in two ways (see discussion in section 3.1 on coherent states).

If we integrate over volume, we integrate out the transverse dimensions (the solenoid cross-section), using  $\int d(area) \vec{B}_{core} = (2\pi\hbar/e)d\vec{x}(\tau)/d\tau$ , and the CS term takes the form:

$$-\frac{4\pi\kappa}{e} \int d\tau A_{\mu}^{ext} \frac{dx^{\mu}}{d\tau} \quad (2.9)$$

This is just the action of a charged classical current loop, carrying charge  $q = 4\pi\kappa/e$ . The details of the singular core magnetic field have disappeared by integrating over the transverse dimensions of the solenoid and the result depends only upon the quantized flux  $\Phi$ . We can apply this result to 1 + 2 dimensions where the solenoid string is stretched out to become a timelike world-line (returning to the past as an antiparticle). The CS term on the world-line becomes:

$$-\kappa \int d^3x \langle \vec{B}_{core}, \vec{A}^{ext} | \epsilon_{\mu\nu\rho} A^{\mu} \partial^{\nu} A^{\rho} | \vec{B}_{core}, \vec{A}^{ext} \rangle \rightarrow -\frac{4\pi\kappa}{e} \int dx^{\mu} A_{\mu}^{ext} \quad (2.10)$$

Consistency demands that we equate the induced electric charge  $q$  of the solenoid world-line to the same value,  $e$ , that is the defining charge of Dirac's quantization condition. Hence we must have:

$$q = e \quad \text{or,} \quad \kappa = \frac{e^2}{4\pi} \quad (2.11)$$

This bootstrapping condition fixes the value of the CS term coefficient,  $\kappa$ .

Note that the CS current is obtained from the action by varying *wrt*  $A_\mu$ :

$$J_\mu = -\frac{\delta}{\delta A_\mu} S_{CS} = 2\kappa \epsilon_{\mu\nu\rho} \partial^\nu A^\rho \quad (2.12)$$

and we see that automatically:

$$\partial_\mu J^\mu = 0 \quad (2.13)$$

From eq.(2.6) we see that the conserved CS current is the source term for the modified Maxwell's equation. Of course, the CS current is just the magnetic field and the conservation law is just  $\vec{\nabla} \cdot \vec{B} = 0$ . The conservation law of the CS current is forcing the solenoid to be a closed loop with the conservation of electric charge.

Another way of getting the quantization of the solenoid charge is to note that solenoid strings have nontrivial Gauss-linking. If we link two closed loops we can view one loop as the test particle used to measure the flux in the other. The result involves the contributions to each loop from the other and thus takes the form:

$$\begin{aligned} \kappa \int d^3x \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho &\rightarrow \kappa \int A_1 dA_2 + A_2 dA_1 \\ &= 2\kappa \times \left(\frac{2\pi}{e}\right)^2 \\ &= 2\pi \end{aligned} \quad (2.14)$$

Thus, the CS term with the Dirac quantization condition implies that the action shifts by (a multiple of)  $2\pi$  under linking. This enforces locality of the interactions of the strings. That is, the only observable consequence of linking a pair of loops would be seen at the point of intersection, and not in a nonlocal overall arbitrary phase shift of the path integral. Gauss linking is another way to argue for the consistency condition for the Chern-Simons term coefficient (see an alternative recent derivation of Witten, [15]).

We see, however, that if two stationary solenoidal world-lines are exchanged in spatial position, the corresponding phase shift is exactly half of a Gauss linking of two loops. The action must therefore shift by  $\pi$ . Ergo, Dirac world-line solenoids are fermions. If we relaxed the consistency condition that enforces  $\kappa = e^2/4\pi$ , then the exchange of solenoid particles leads to an arbitrary shift in the action. The particles become “anyons” with arbitrary statistics [16]. From another perspective, if we have electrons that already have spinor (fermionic) statistics, and we turn on the CS term, the electrons can become bosons. This raises various intriguing possibilities, *e.g.*, that electrons in a 1 + 2 dimensional

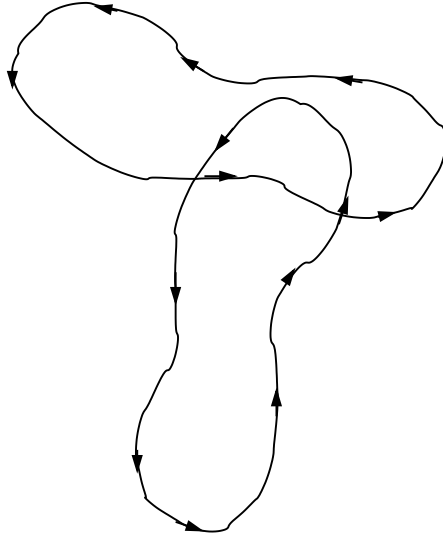


Figure 3: Two solenoid loops are linked and the action shifts by a multiple of  $2\pi$  with the quantized CS term coefficient.

systems could undergo Bose condensation without Cooper pairing. The interplay of the CS term suggests various components of a model of high- $T_c$  superconductivity.

Note that the profile of the Dirac solenoid on a  $\hat{D} = 2$  spacelike hypersurface is the current loop containing  $\epsilon_{ij}F^{ij}$ , *i.e.*, it is an instanton. Instantons in even  $\hat{D}$  can be viewed as events in which solenoids (or instanton vortices in nonabelian models) in  $\hat{D} + 1$  are created or destroyed on boundary branes. The discussion of instantons is beyond our present scope, involving the exchange of solenoids (instantonic vortices) between boundary branes containing chiral fermions. However, this construction leads to the correct normalization of the instanton, *i.e.*, the Pontryagin index, and we can show that this is related directly to the covariant anomaly.

As an aside, this raises the question of what the solenoid core structure is? We emphasize that nothing in our present results depend upon the core. In a Nielsen-Olesen flux-tube vortex, the Higgs field develops a nontrivial radial profile that approaches a constant as one moves away from the vortex, and carries a circulating current density. The vacuum thus has a distributed charge and current density which supports the core magnetic field structure, and the magnetic field is confined in the core with a radial profile. The Higgs field at infinity carries a phase factor that is proportional to the azimuthal angle and thus maps the  $U(1)$  gauge group onto the circle at infinity. In this way the Nielsen-Olesen flux tube is an element of the homotopy group  $\pi_1(U(1))$  which is the



topological definition of a vortex. In the present case, the loop of the gauge field,  $\epsilon_{ijk}\partial^j\vec{A}^k$  is promoted to a physical circulating current by the CS term. Thus the  $U(1)$  gauge group is mapped onto the circle at infinity by the Wilson line containing the circulating  $\vec{A}$ . In this way the Dirac solenoid is an element of the homotopy group  $\pi_1(U(1))$ , and behaves as a vortex. In the case of the Dirac solenoid, however, the core has collapsed. Indeed, the core singularity is not a solution of eq.(2.7) either, and we must invoke new short-distance physics to support the core flux. This is analogous to the Skyrme in the absence of the Skyrme term. The Skyrme is on a  $D = 3$  space-like hypersurface, defined by  $U = \exp(if(r)\hat{r} \cdot \vec{\tau}/2)$  with a radial core profile such that  $f(0) = 0$  and  $f(\infty) = 2\pi$ . It thus is both an element of  $\pi_3(SU(2))$ , mapping the  $SU(2)$  gauge group onto the full manifold, and it is also an element of  $\pi_2(SU(2))$ , mapping  $SU(2)$  onto the surface  $S_2$  at infinity. When the core of the Skyrme collapses,  $f(r) \propto \theta(r)$ , it ceases to be an element of  $\pi_3(SU(2))$ , but remains an element of  $\pi_2(SU(2))$ . Since our discussion is mainly focused on the large distance topological aspects of the solenoid, we will presently sidestep the issue of the Dirac solenoid core structure, and assume that we can model it by suitable extensions of the theory (*e.g.*, in analogy to adding the Skyrme term; perhaps gravity is involved).

### 2.3 Fermionic Anomalies on Boundaries

This section deals with compactification of  $D = 3$  and chiral delocalization of fermions on the boundary of  $\hat{D} = 2$  (the  $D = 5$  to  $\hat{D} = 4$  case is treated in ref.([14])), and it can be skipped on a first reading. The purpose is to illustrate the interplay of fermionic anomalies with the CS term to maintain an overall gauge invariant theory. The CS term provides a counterterm which converts the axial anomaly into the ‘‘covariant’’ form automatically. We return to the mainline discussion of Dirac brane arguments in higher  $D$  in section 3.

We can compactify  $D = 3$  to  $\hat{D} = 2$  by sandwiching pure  $D = 3$  QED between branes separated by  $R$  (*e.g.*, ‘‘capacitor plates,’’ in an older lexicon). The vector potential exists in the bulk, and performing a local gauge transformation,  $\delta A_\mu = -\partial_\mu\theta$ , we see that the CS term generates a surface term on the branes (for convenience, we set  $e = 1$ ):

$$\delta S_{CS} = \int d^2x \frac{\theta(x_\mu, x^3)}{4\pi} \epsilon_{\mu\nu} \partial^\mu A^\nu(x^\mu, x^3) \Big|_{x^3=0}^{x^3=R}. \quad (2.15)$$

The CS term shifts the action on the boundaries and thus yields anomalies on the  $\hat{D} = 2$  branes.

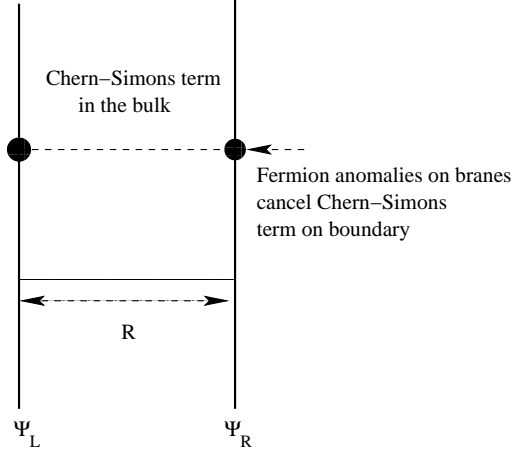


Figure 4: Orbifold with split, anomalous fermions (electrons).  $\psi_L$  ( $\psi_R$ ) is attached to the  $\hat{D} = 2$  left-brane, (right-brane). Gauge fields propagate in the  $D = 3$  bulk, which has a compactification scale  $R$ . The bulk contains a CS term, and the branes produce loop diagram amplitudes in the effective action. The anomalies from the CS term cancel the anomalies from the triangle diagrams on the respective branes so the overall theory is anomaly free.

We introduce chiral fermions that are constrained to the boundary branes:

$$\begin{aligned} \int d^2x \bar{\psi}_L (i\cancel{\partial} - A_L) \psi_L & \quad \text{brane I} \\ \int d^2x \bar{\psi}_R (i\cancel{\partial} - A_R) \psi_R & \quad \text{brane II} \end{aligned} \quad (2.16)$$

where chiral projections are  $\psi_{L,R} = \frac{1}{2}(1 \mp \gamma^3)$ . Here we have the gauge fields on the boundary branes:

$$A_L(x_\mu) = A(x_1, x_2, x^3 = 0) \quad A_R(x_\mu) = A(x_1, x_2, x^3 = R) \quad (2.17)$$

The anomaly is readily computed for massless Weyl fermions in  $\hat{D} = 2$  by performing the one-loop current bubble with an external gauge field (see the Appendix):

$$\partial_\mu j_L^\mu = -\frac{1}{4\pi} \epsilon_{\mu\nu} \partial^\mu A_L^\nu \quad \partial_\mu j_R^\mu = \frac{1}{4\pi} \epsilon_{\mu\nu} \partial^\mu A_R^\nu \quad (2.18)$$

(recall,  $j_{L,R} = -\delta S / \delta A_{L,R} = \bar{\psi} \gamma_\mu (1 \mp \gamma^3) \psi / 2$ ).

Classically the theory is invariant if the fermions transform on the boundary as  $\psi_L \rightarrow e^{i\theta(0)} \psi_L$  and  $\psi_R \rightarrow e^{i\theta(R)} \psi_R$ , when in the bulk,  $\delta A_\mu = -\partial_\mu \theta$ . However, the action on the branes also shifts due to the fermionic anomalies, under  $\delta A_\mu = -\partial_\mu \theta$ , as:

$$\begin{aligned} \delta S_{\delta A} &= -\int d^2x \theta(0) \partial^\mu \bar{\psi}_L \gamma_\mu \psi_L - \int d^2x \theta(R) \partial^\mu \bar{\psi}_R \gamma_\mu \psi_R \\ &= -\int d^2x \frac{\theta(x_\mu, x^3)}{4\pi} \epsilon_{ab} \partial^a A^b \Big|_{x^3=0}^{x^3=R}. \end{aligned} \quad (2.19)$$

whence:

$$\delta S_{CS} + \delta S_{\delta A} = 0 \quad (2.20)$$

We have thus arranged a cancellation of the CS anomalies of eq.(2.18) by chiral fermions on the boundaries. This is a ‘‘chiral delocalization’’ in  $D = 3$  leading to a compactified theory with a Dirac spinor in  $\hat{D} = 2$ .<sup>1</sup>

It is convenient to write the anomalies in the ‘‘VA’’ form using:

$$A_L = V - A, \quad A_R = V + A, \quad j_V = j_R + j_L, \quad j_A = j_R - j_L \quad (2.21)$$

whence:

$$\partial_\mu j_V^\mu = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial^\mu A^\nu \quad \partial_\mu j_A^\mu = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial^\mu V^\nu \quad (2.22)$$

These are called the ‘‘consistent anomalies.’’ In a nonabelian theory they are generally not gauge invariant operators but satisfy the Wess-Zumino consistency conditions.

In the  $D = 3$  bulk, let us consider only the three lowest KK modes, corresponding to a vector zero mode, and axial vector cosine mode, and a pseudoscalar, sine mode ( $x^\mu = (t, x)$ ,  $x^3 = y$ ):

$$\hat{A}_\mu(x^\mu, y) = V_\mu(x^\mu) - A_\mu(x^\mu) \cos(\pi y/R) \quad \hat{A}_3(x^\mu, y) = \phi \sin(\pi y/R) \quad (2.23)$$

The sign of  $A_\mu$  relative to  $V_\mu$  is chosen so that eq.(2.21) is satisfied for  $A_{L,R}$ . Orbifold boundary conditions equivalently follow from the assumption that  $F_{\mu 3}(x^\mu, y = 0) = F_{\mu 3}(x^\mu, y = R) = 0$ .

Evaluating the CS term in the truncated  $\hat{A}_\mu(x^\mu, y)$  yields:

$$S_{CS} = -\frac{1}{4\pi} \int d^2x \int dy \epsilon_{\mu\nu\rho} \hat{A}^\mu \partial^\nu \hat{A}^\rho = \frac{1}{2\pi} \int d^2x \epsilon_{\mu\nu} V^\mu A^\nu + \frac{1}{2\pi f} \int d^2x \phi \epsilon_{\mu\nu} \partial^\mu V^\nu \quad (2.24)$$

where  $f = \pi/4R$ , and note in the first term on the *rhs* a tricky sign coming from  $\epsilon_{\mu 3\nu} = -\epsilon_{\mu\nu}$ . The ‘‘pion’’  $\phi$  couples anomalously to the vector field, but this simple orbifold model  $\phi$  is eaten by  $A$ , and we define  $\tilde{A} = A + \partial\phi/f$ . The remaining term,  $\sim \epsilon_{\mu\nu} V^\mu \tilde{A}^\nu$ , acts as a counterterm in the action. Its presence modifies the currents:

$$\delta j_V^\mu = -\frac{\delta S_{CS}}{\delta V_\mu} = -\frac{1}{2\pi} \epsilon_{\mu\nu} \tilde{A}^\nu \quad \delta j_A^\mu = -\frac{\delta S_{CS}}{\delta \tilde{A}_\mu} = \frac{1}{2\pi} \epsilon_{\mu\nu} V^\nu \quad (2.25)$$

and we thus define the full currents,

$$\tilde{j}_V = j_V + \delta j_V \quad \tilde{j}_A = j_A + \delta j_A \quad (2.26)$$

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<sup>1</sup>Note that we can give the  $D = 3$  fermions a mass by way of a bilocal operator  $m \bar{\psi}_L(0) W \psi_R(R)$  where  $W$  is a Wilson line connecting the branes (see [14])

and we find:

$$\partial_\mu \tilde{j}_V^\mu = 0 \quad \partial_\mu \tilde{j}_A^\mu = \frac{1}{\pi} \epsilon_{\mu\nu} \partial^\mu V^\nu \quad (2.27)$$

These latter forms are the “covariant” anomalies. The vector current is now conserved reflecting the overall gauge invariance with the fermions on branes and CS term in the bulk. The corresponding analysis in compactifying  $D = 5$  QED to  $\hat{D} = 4$  QED is given in [14] with more detail on Wilson line fermion masses and the full KK-tower anomaly structure. The main lesson is that the  $D = \hat{D} + 1$  CS term becomes the  $\hat{D}$  counterterm, and it generally impacts the physics of the  $\hat{D}$  theory.

### 3 Generalization to Dirac Branes in any odd D

To construct a solenoid in  $D = 3$  we effectively “stack”  $(xy)$  plaquette current loops along the  $z$  axis. The loop integral of  $A_\mu$  in the  $(xy)$  plane circumnavigating the stack is then Dirac quantized. By Stoke’s theorem, the surface integral spanning the loop, hence the solenoidal flux, is likewise Dirac quantized.

To generalize this construction, let us first consider the special case of  $D = 5$ . An  $(xy)$  plaquette bounded by a current loop can be “stacked” simultaneously along all of the three orthogonal axes,  $(zwt)$ . We thus start with one “kernel”  $(xy)$  plaquette and we stack with three new  $(xy)$  plaquettes taking infinitesimal steps in the  $z$ ,  $w$ , and  $t$  directions respectively. We then iterate the procedure, generating nine more plaquettes, etc. Therefore, the resulting stack of current plaquettes spans a 3-dimensional hypersurface. This hypersurface carries the field strength dual to  $F_{xy}$ , *i.e.*,  $F_{zwt}^* = F_{xy}$ . This is a Dirac brane flux in the  $zwt$  hypersurface.

In the  $xy$  plane the resulting 3-dimensional hypersurface can be encircled in  $D = 5$  by a closed loop and the flux in the hypersurface,  $F_{zwt}^*$ , can be Dirac quantized. We have, integrating over a loop in the  $(xy)$  plane (bounding a disk):

$$\oint dx^\mu A_\mu = \int_{disk} dx dy F_{xy} = \int_{disk} dx dy F_{zwt}^* = \frac{2\pi}{e} \quad (3.28)$$

This gives the generalization of the cloaked Dirac solenoid. Charged particles circumnavigating the brane undergo a phase shift of  $2\pi$ .

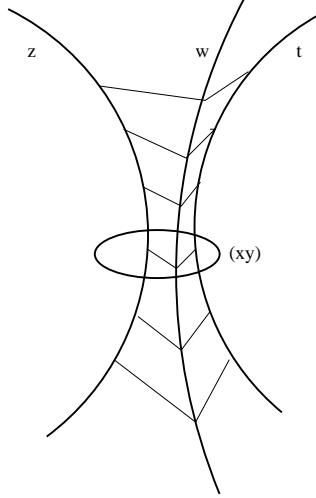


Figure 5: Construction of Dirac brane in  $D = 5$ . The  $(zwt)$  hypersurface is encircled by the  $(xy)$  current loop. The Aharonov-Bohm phase of an electron in the  $(xy)$  loop is quantized to  $2\pi$ , which quantizes the flux  $F_{zwt}^*$  on the brane.

### 3.1 D=5

Now consider the CS term in  $D = 5$ :  $\kappa \epsilon_{ABCDE} A^A \partial^B A^C \partial^D A^E$ , which we note is third order in the vector potential  $A^A$ . Recalling that the CS term induces an electric charge for the solenoid in  $D = 3$ , we are led to consider a field configuration that is second order in  $A^A$  in  $D = 5$ . We thus consider the intersection of two Dirac branes in  $D = 5$ . Let us view the 3-dimensional hypersurface dual to the  $(xy)$  current plaquettes, carrying the flux  $F_{zwt}^*$ , as a *coherent state* of photons. We denote this state by  $|\Phi_{xy}\rangle$ .

Recall the construction of coherent states: Let  $\phi(x)$  be a quantum field with canonical momentum,  $\pi(x)$ , and let  $\phi_c$  be a classical background field. Then the coherent state on a spacelike hypersurface is defined as:

$$|\phi_c\rangle = \exp\left(i \int \pi \phi_c\right) |0\rangle \quad \text{whence,} \quad \langle \phi_c | \phi | \phi_c \rangle = \phi_c \quad (3.29)$$

Note that if we have two classical configurations,  $\phi_c^1$  and  $\phi_c^2$  superimposed, then:

$$|\phi_c^1 \phi_c^2\rangle = \exp\left(i \int \pi (\phi_c^1 + \phi_c^2)\right) |0\rangle \quad (3.30)$$

hence,

$$\langle \phi_c | : \phi^2 : | \phi_c \rangle = 2\phi_c^1 \phi_c^2 + (\phi_c^1)^2 + (\phi_c^2)^2 \quad (3.31)$$

In what follows we are only interested in the cross-term for the superposition of  $N$  fields, since all other contributions will be zero by the  $\epsilon$ -symbol. For  $N$  superimposed configurations the cross-term coefficient of  $\phi_c^1 \phi_c^2 \dots \phi_c^N$  is just  $N!$ .

Technically the coherent state of photons requires a Coulomb gauge for the vector potential,  $\vec{A}$ , which then has a well-defined canonical momentum  $\partial\vec{A}/\partial t$ . This is always possible for our world-lines on a spacelike hypersurface. The field strength operator,  $F_{\mu\nu}$ , then has a classical expectation value in the coherent state, on the brane which is a singular local form,

$$\langle\Phi_{xy}|F_{\mu\nu}|\Phi_{xy}\rangle = \frac{2\pi}{e}(g_{x\mu}g_{y\nu} - g_{y\mu}g_{x\nu})\delta(x)\delta(y) \quad (3.32)$$

However, all that is relevant for our present considerations is that this expectation value satisfies eq.(3.28), so we need deal only with exact integrals away from the singularity:

$$\int dx dy \langle\Phi_{xy}|F_{xy}|\Phi_{xy}\rangle = \frac{2\pi}{e}. \quad (3.33)$$

We now form the intersection of two Dirac branes constructed of  $(xy)$  and  $(zw)$  current plaquettes. We describe this as a coherent state  $|\Phi_{xy}\Phi_{wz}\rangle$ . We can again compute the expectation value of local operators in this coherent state, taking care to count the contractions of field operators with photons in the state. We thus have:

$$\epsilon_{ABCDE} \int dx dy dz dw \langle\Phi_{xy}\Phi_{wz}|\partial^B A^C \partial^D A^E |\Phi_{xy}\Phi_{wz}\rangle = 2 \left(\frac{2\pi}{e}\right)^2 g_{At} \quad (3.34)$$

The prefactor of 2 comes from the two possible contractions of  $dA$  with either  $(xy)$  or  $(wz)$  with the internal coherent fields *i.e.*, it is just the 2! in the coherent state of two superimposed classical fields as in eq.(3.31). The exact orthogonality of the surfaces  $(xy)$  and  $(wz)$  does not affect this result, but the result would be zero if the hypersurfaces were degenerate, *e.g.*,  $|\Phi_{xy}\Phi_{xz}\rangle$  yields a vanishing expectation value for the above operator owing to the  $\epsilon$  symbol in the CS term.

Note that the intersection of the branes defines a world-line in the  $t$  direction. It can thus be viewed as a particle world-line in 1 + 4 dimensions. If we now turn on the CS term, the intersecting brane world-line develops an electric charge. We can compute the coupling of the world-line to an external coherent photon field  $A_\mu^{ext}$  as:

$$\kappa \epsilon_{ABCDE} \int d^5x \langle\Phi_{xy}\Phi_{wz}, A_\mu^{ext} | A^A \partial^B A^C \partial^D A^E |\Phi_{xy}\Phi_{wz}, A_\mu^{ext}\rangle = 3 \times 2\kappa \left(\frac{2\pi}{e}\right)^2 \int A_0^{ext} dt \quad (3.35)$$

here the extra factor of 3 counts the number of contractions with the external radiated photon field  $A^{ext}$  by the vector potentials in the CS term, where we have integrated by parts to remove the external photon momentum.

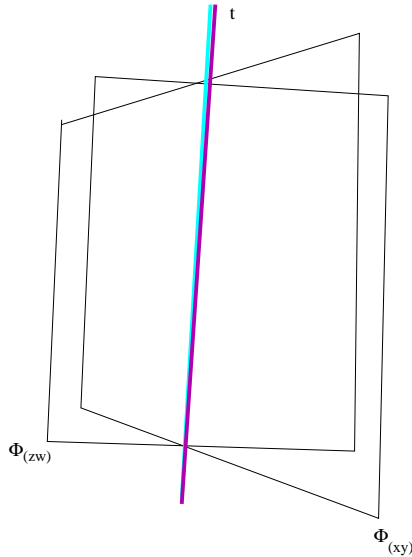


Figure 6: Intersection of Dirac branes in  $D = 5$ . The Dirac brane carrying flux  $\Phi_{xy}$  intersects  $\Phi_{zw}$  to produce a charged worldline  $t$ . The charge is determined by the CS term,  $AdAdA$  and must equal the charge in the Dirac quantization procedure.

We now enforce the self-consistency condition. We demand that the induced world-line electric charge have the same value as the original defining electric charge,  $e$ , used to quantize the brane flux. We thus have:

$$e \equiv 3 \times 2\kappa \left(\frac{2\pi}{e}\right)^2; \quad \text{hence,} \quad \kappa = \frac{e^3}{24\pi^2} \quad (3.36)$$

We have thus determined the CS term coefficient  $\kappa$ . It agrees with the consistent anomaly coefficient obtained from pure Weyl spinor triangle diagrams in  $D = 4$ , or equivalently Bardeen's LR symmetric form of the anomaly, [5, 7, 14, 17].

Note also that the intersection of the world-line by an  $(xyzw)$  spacelike hypersurface is an instanton, *i.e.*, it produces a nonzero value of the matrix element in the intersecting brane coherent state,  $\int d^4x \langle \Phi_{xy}\Phi_{wz} | F_{\mu\nu} F^{*\mu\nu} | \Phi_{xy}\Phi_{wz} \rangle = 16\pi^2/e^2$  where  $F_{\mu\nu}^* = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$  and  $F = 2dA$ . The intersecting brane configuration mimics features of the solenoid in  $D = 3$ .

### 3.2 Generalization to any $D$

The construction we have described generalizes readily to any odd  $D$ . Current loops in the  $(xy)$  plane are stacked to generate a  $D - 2$  hypersurface carrying the flux dual to  $F_{xy}$ .

We again define this to be the coherent state  $|\Phi_{xy}\rangle$  with the identical result of eq.(3.33), satisfying eq.(3.28).

We now build the intersection of  $(D - 1)/2$  Dirac branes. This is described by the coherent state:

$$|\Phi_{xy}\Phi_{uv}\dots\Phi_{zw}\rangle \quad (3.37)$$

This contains  $(D - 1)/2$ , 2-forms,  $F_{ij}$  and is dual to a  $D - 1$  dimensional object, *i.e.*, it can be taken to be a time-like world-line. We compute the matrix element for a photon emission in the presence of the CS term:

$$\begin{aligned} & \kappa \epsilon_{ABC\dots DE} \int d^5x \langle \Phi_{xy}\dots\Phi_{zw}, A_\mu^{ext} | A^A \partial^B A^C \dots \partial^D A^E | \Phi_{xy}\dots\Phi_{zw}, A_\mu^{ext} \rangle \\ &= [((D + 1)/2)!] \kappa \left( \frac{2\pi}{e} \right)^{(D-1)/2} \int A_0 dx^0 \end{aligned} \quad (3.38)$$

The prefactor  $((D + 1)/2)!$  is the generalization of the  $3!$  in the  $D = 5$  case counting the  $((D - 1)/2)!$  contractions with the brane intersection coherent state, and  $((D + 1)/2)$  contractions with the external photon. The factor  $\left(\frac{2\pi}{e}\right)^{(D-1)/2}$  is just the  $(D - 1)/2$  brane flux factors of  $2\pi/e$ . Again, though we chose orthogonal branes for simplicity, the result applies to any nondegenerate intersection of  $(D - 1)/2$  branes.

Demanding the consistency condition, *i.e.*, that induced electric charge for the intersection be equal to the defining Dirac charge  $e$ , determines the CS term coefficient:

$$\kappa = \frac{e^{(D+1)/2}}{[((D + 1)/2)!] (2\pi)^{(D-1)/2}} \quad (D \text{ odd}) \quad (3.39)$$

Stripping off a factor of  $e$  (by convention) this yields the consistent anomaly coefficient,  $\kappa'$ , for a right-handed Weyl spinor in even  $\hat{D} = D - 1$ ,

$$\kappa' = \frac{e^{\hat{D}/2}}{[((\hat{D} + 2)/2)!] (2\pi)^{(\hat{D})/2}} \quad (\hat{D} \text{ even}). \quad (3.40)$$

### 3.3 Covariant Anomaly Coefficients

In  $\hat{D}$ -even space-time we consider Weyl spinor theories  $\bar{\psi}(i\hat{\not{\partial}} - A_L)\psi_L$  ( $\bar{\psi}(i\hat{\not{\partial}} - A_R)\psi_R$ ) with currents  $j_{\mu L} = \bar{\psi}\gamma_\mu\psi_L$  ( $j_{\mu R} = \bar{\psi}\gamma_\mu\psi_R$ ), and we have the consistent anomalies:

$$\partial_\mu j_R^\mu = \kappa' dA_R\dots dA_R \quad \partial_\mu j_L^\mu = -\kappa' dA_L\dots dA_L \quad (3.41)$$

Now define:

$$A_L = V - A \quad A_R = V + A \quad (3.42)$$



whence:

$$\begin{aligned}\partial_\mu j_R^\mu &= \kappa' (dV\dots dV + \frac{\hat{D}}{2}dAdV\dots dV + \dots + dA\dots dA) \\ \partial_\mu j_L^\mu &= -\kappa' (dV\dots dV - \frac{\hat{D}}{2}dAdV\dots dV + \dots + (-)^{\hat{D}/2}dA\dots dA)\end{aligned}\quad (3.43)$$

or, in terms of vector and axial vector currents ( $j = j_L + j_R$ ,  $j^5 = j_R - j_L$ ) we have:

$$\partial_\mu j^\mu = \hat{D}\kappa' (dAdV\dots dV + \dots) \quad \partial_\mu j_5^\mu = 2\kappa' (dV\dots dV + \dots) \quad (3.44)$$

We see that the consistent anomalies violate both  $j$  and  $j^5$  conservation. In the application to vector-like theories, such as QED, it is desirable to treat  $V$  as a fundamental gauge field coupled to the vector current,  $j$ , and to maintain the conservation of  $j$  in the presence of both  $V$  and  $A$ . Thus, we can generate current correlators of  $j$  and  $j^5$  in which  $j$  is always conserved, by introducing counterterms, which is how of Adler's original analysis arrives at the covariant anomaly [4].

Many counterterms are possible in higher  $\hat{D}$ . It is sufficient, to determine the covariant anomaly coefficient, to consider the leading terms on the rhs of eq.(3.44). Consider

$$\mathcal{O} = -f \int AVdV\dots dV \quad (3.45)$$

If  $\mathcal{O}$  is added to the action, the currents are modified by corrections:

$$\begin{aligned}\delta j &= -\frac{\delta}{\delta V}\mathcal{O} = -\frac{1}{2}f \left( \hat{D}AdV\dots dV - (\hat{D} - 2)VdAdV\dots dV \right) \\ \delta j_5 &= -\frac{\delta}{\delta A}\mathcal{O} = f VdV\dots dV\end{aligned}\quad (3.46)$$

whence the full currents now satisfy:

$$\begin{aligned}\partial_\mu (j + \delta j)^\mu &= (\hat{D}\kappa' - f) dAdV\dots dV + \dots \\ \partial_\mu (j_5 + \delta j_5)^\mu &= 2 \left( \kappa' + \frac{f}{2} \right) (dV\dots dV) + \dots\end{aligned}\quad (3.47)$$

We thus specify  $f$  by demanding that the vector current is conserved, whence  $f = \kappa'\hat{D}$ :

$$\partial_\mu (j + \delta j)^\mu = 0 \quad \partial_\mu (j_5 + \delta j_5)^\mu = 2\kappa' \left( 1 + \frac{\hat{D}}{2} \right) (dV\dots dV + \dots) \quad (3.48)$$

Hence, using eq.(3.40) for  $\kappa'$ , the covariant anomaly coefficient is:

$$\tilde{\kappa} = 2\kappa \left( 1 + \frac{\hat{D}}{2} \right) = \frac{2e^{\hat{D}/2}}{(2\pi)^{\hat{D}/2}(\hat{D}/2)!} \quad (3.49)$$

and the covariant axial current divergence is:

$$\partial_\mu \tilde{j}_5^\mu = \tilde{\kappa} (dV \dots dV + \dots) \quad (3.50)$$

As a check, we see that this reproduces Adler's original result:

$$\partial_\mu \tilde{j}_5^\mu = \frac{e^2}{4\pi^2} dV dV = \frac{e^2}{8\pi^2} F \tilde{F} \quad (3.51)$$

where  $F \tilde{F} = 2dV dV$ , where  $\tilde{F}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ . Our result is consistent with Frampton and Kephart [7] (*c.f.* their eqs.(5.16) and (5.17) for  $2^{\ell-1} X_\ell$ ; note their result is quoted in momentum space and imbeds the operator matrix element; once one determines the general  $1/(2\pi)^{\hat{D}/2}(\hat{D}/2)!$  behavior the result is determined from the  $\hat{D} = 2, 4$  results).

## 4 Conclusions and Discussion

### 4.1 Summary of Results

We can always introduce an integer ‘‘index’’  $N$  into the anomaly and CS term coefficients which counts the ‘‘number of (spectator) colors’’ for fermion loops. In odd  $D$  the Chern-Simons term for a  $U(1)$  gauge theory is:

$$-\kappa \epsilon_{ABC\dots CD} A^A \partial^B A^C \dots \partial^C A^D \quad (4.52)$$

where:

$$\kappa = \frac{N e^{(D+1)/2}}{[((D+1)/2)!] (2\pi)^{(D-1)/2}} \quad (4.53)$$

In even  $\hat{D}$  the right-handed Weyl spinor current coupled to a  $U(1)$  gauge field has the anomaly:

$$\partial_A \bar{\psi} \gamma^A \psi_R = \frac{N e^{\hat{D}/2}}{[(\hat{D}+2)/2)!] (2\pi)^{(\hat{D})/2}} \epsilon_{AB\dots CD} \partial^A A^B \dots \partial^C A^D \quad (4.54)$$

where  $\psi_R = (1 + \gamma^{\hat{D}+1})\psi/2$  (and correspondingly for  $\bar{\psi} \gamma^A \psi_L$  with a minus sign).

In even  $\hat{D}$  the covariant anomaly for the axial current current coupled to a  $U(1)$  gauge vector-field  $V$  has the anomaly:

$$\partial_A \bar{\psi} \gamma^A \gamma^{\hat{D}+1} \psi = \frac{2e^{\hat{D}/2}}{[(\hat{D})/2)!] (2\pi)^{(\hat{D})/2}} \epsilon_{AB\dots CD} \partial^A V^B \dots \partial^C V^D \quad (4.55)$$

(this has additional terms if there is also an axial photon as in [5]).

For Yang-Mills theories the CS term has the structure in odd  $D$ :  $\kappa \text{Tr}(AdA\dots dA + \dots)$  where  $A = A_A^a T^a$ . It is the  $D + 1$ -th component of a current,  $K$ , whose divergence is  $\partial K = \text{Tr}(F \wedge F \wedge \dots F)$ , with  $(D + 1)/2$  field strength factors  $F$ . The normalization of  $T^a$  is irrelevant since it can be absorbed into the definition of  $e$ . Note that for an extra dimensional theory the CS term coefficient steps through  $\kappa$  as we cross a brane with  $N$  fermion species, *i.e.*,  $\kappa_R - \kappa_L = \kappa$ . When compactifying onto a circle,  $S_1$ ,  $\kappa$  is in principle arbitrary because there is no net anomaly on a boundary. This is the analogue of the  $\theta$  term in  $D = 5$ . The  $D = 5$  nonabelian CS term under suitable compactification becomes the Wess-Zumino-Witten term and the coefficient carries over [24].

## 4.2 Discussion and Summary

Axial anomalies are traditionally viewed as intrinsically fermionic in origin, [2, 3, 4, 5, 6]. However, axial anomalies are often present in purely bosonic effective theories and their coefficients can be fixed modulo an integer by self-consistency arguments. A familiar example is a low energy effective lagrangian of the  $\pi^0$  and the photon, including a term of the form  $(N_c \alpha / 2\pi f_\pi) \pi^0 F_{\mu\nu} \tilde{F}^{\mu\nu}$ . This is a term in the overall Wess-Zumino-Witten term [18, 19] of low energy QCD chiral dynamics. Its coefficient can be determined by the underlying anomalous quark loop structure [20], and the WZW term then gives a complete bosonic description of the anomaly structure.

However, we can determine the WZW term coefficient, hence the anomaly coefficient, from a purely bosonic argument, without resort to fermion loops. The full WZW term is a necessary part of the effective action in the IR theory required to generate a complete physical description of the skyrmion, the low energy description of the baryon [19]. The skyrmion is topologically stable and has conserved topological currents, *e.g.*, the singlet Goldstone-Wilczek [21] current. Noether variation of the WZW term *wrt* the  $\omega$  meson generates the gauge invariant form of the Goldstone-Wilczek current (this requires care in the Standard Model [22]). We can therefore determine the coefficient of the WZW term by choosing the Goldstone-Wilczek charge, the baryon number of the skyrmion, to be an integer. The WZW term also has been argued to control the spin and statistics of the skyrmion, confirming its interpretation as the low energy description of the baryon in QCD when the index  $N \equiv N_c = 3$  [19, 23]. Remarkably, this determines the anomaly coefficient and  $\pi^0$  decay without resort to a fermion loop calculation.

We can determine the CS term coefficient for a nonabelian Yang-Mills theory in  $D = 5$  in a similar fashion. In fact, a  $D = 5$  Yang-Mills theory of flavor, suitably compactified, becomes a chiral lagrangian of pions (the  $A^5$  zero modes) coupled to flavored gauge fields in  $\hat{D} = 4$ , and the topological aspects of QCD emerge from the  $D = 5$  CS term [24]. In the  $D = 5$  Yang-Mills theory there exists an “instantonic vortex,” which is a stable soliton solution, and under compactification to  $\hat{D} = 4$  this object holographically maps onto the skyrmion, [25]; naturally, the  $D = 5$  CS currents, in turn, match onto the skyrme currents, [26]. Again, demanding that the instantonic vortex carry an integer charge dictates the CS term coefficient, hence the “consistent anomaly” coefficient in  $\hat{D} = 4$ . Via fermion loops, this involves the computation of triangle and box diagrams [5], but the form of the consistent anomaly can be immediately obtained from the boundary variation of the properly normalized CS term under a gauge transformation [27].

In the present paper we have illustrated a general construction of a field configuration that carries the  $U(1)$  CS charge in any  $U(1)$  theory in any  $D$ -odd dimension. This begins with Dirac’s solenoid construction of the magnetic monopole in  $D = 3$  for a  $U(1)$  gauge theory. The solenoidal flux is chosen so that the phase shift of an encircling “electron” of charge  $e$  is  $2\pi$ , thus “cloaking” the solenoid. This quantizes the monopole’s magnetic charge as  $g_m = \hbar/2e$ .

However, when we turn on the CS term,  $\kappa AdA = \kappa \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho$ , three things happen: (1) the theory becomes that of a massive photon [12, 13] and we can easily see that monopole solutions *cannot exist* when  $\kappa \neq 0$ ; (2) the solenoid must therefore be either a closed loop, or an infinite line in  $D = 1 + 2$ ; (3) with  $\kappa \neq 0$ , the Dirac solenoid becomes physical: an electric current flows in a closed solenoid loop; alternatively, in  $D = 1 + 2$ , an infinite timelike solenoid becomes the world-line of a charged particle that can emit or absorb photons. The Dirac solenoid has become a bosonic string with  $\kappa \neq 0$ .

We must then impose a “consistency condition,” *i.e.*, that the induced charge carrier in the solenoid current loop (*i.e.*, the electric charge of the  $1 + 2$  particle solenoid world-line) have the same value as the defining electric charge,  $e$ , employed in the original Dirac quantization condition. With the consistency condition the coefficient of the CS term is determined and found to be  $e^2/4\pi$ . This, in turn, dictates the axial anomaly coefficient in  $\hat{D} = 2$ .

We generalize the solenoid construction to Dirac branes, first introduced by Teitelboim [8]. In higher odd dimensionality  $D$ , we consider Dirac branes that are extended objects that can always be encircled by a charged particle world-line loop. The branes

are therefore  $D - 2$  hypersurfaces. With the encircling loop we can impose the Dirac flux quantization condition on a Dirac brane, analogous to the solenoid construction in  $D = 3$ . In *e.g.*,  $D = 5$ , this becomes a three dimensional extended surface carrying a flux  $F_{\mu\nu\rho}^*$ .

The simultaneous intersection of  $(D - 1)/2$  of Dirac branes defines a world-line in  $D$  dimensions. This intersection becomes the world-line of an electrically charged particle when the  $D$  dimensional CS term,  $AdAdA\dots dA$ , is switched on. We must impose the consistency condition that this world-line carry the defining charge used to quantize the Dirac brane flux. In this manner we thus obtain the CS term coefficient in any odd  $D$ .

Compactification of the  $D$ -odd theory to an even  $\hat{D} = D - 1$  theory generates the consistent anomaly coefficient on the boundary branes from the CS term coefficient. One can introduce fermions on the  $\hat{D}$  boundaries to cancel these anomalies if one wants an anomaly free theory [14]. In this picture, fermions are “spectators” to the bosonic theory that can remedy the uncanceled bosonic anomalies. This is a “yin-yang” view of the role of bosons vs. fermions in axial anomalies.

Consistent anomalies, equivalent to “left-right symmetric anomalies” [5], occur in both the vector and axial-vector currents, are generated by Weyl spinor loops. For covariant anomalies the gauged currents (*e.g.*, vector current in QED) are conserved for any background fields upon including counterterms. Adler’s result for QED was a covariant anomaly (see section 3). In Yang-Mills theories, covariant anomalies are gauge invariant, while consistent anomalies are not. To construct the covariant anomaly in vectorlike theories, in which the gauged vector currents are conserved, requires including the “Bardeen counterterm” into the action [5]. We can easily construct the counterterm in any even  $\hat{D} = D - 1$  and demand vector current conservation, to obtain the covariant anomaly coefficient. We thus arrive at a final result for the consistent and covariant anomaly coefficients in any even  $\hat{D}$ . Our results confirm the fermion loop calculations of Frampton and Kephart in any  $\hat{D}$  [7].<sup>2</sup>

Our main result, in summary, is that there always exists, by construction, a gauge field configuration (albeit singular) that inherits an electric charge in the presence of the CS term in any odd  $D$ , for any gauge theory, since any gauge group  $G$  contains  $U(1)$ . This field configuration can be taken to be a collection of  $(D - 1)/2$  intersecting Dirac branes in a  $U(1)$  submanifold of  $G$ . The field configuration on a spacelike hypersurface,

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<sup>2</sup>Note that there are important instances in which we do not want to enforce vector current conservation with the Bardeen counterterm, such as the case of the Standard Model in which the gauged left-handed currents must be anomaly free. This requires the Standard Model counterterm of ref.[22] and leads to some interesting modifications in the low energy effective theory via the WZW term.

cutting through the brane intersection is an instanton with unit Pontryagin index. Hence, this brane intersection is related to the notion of an instantonic vortex. The consistency of the induced electric charge on the intersection with the Dirac quantization condition determines the CS term coefficient and hence the consistent anomaly. The covariant anomaly follows from the judicious choice of counterterm.

Axial anomalies can be viewed as purely bosonic in origin. They are intrinsically topological and arise holographically from Chern-Simons terms in odd  $D$ . We've shown that axial anomalies trace directly from the Dirac monopole construction. A Chern-Simons term destroys monopoles, and thus destroys the duality of electrodynamics, essentially enforcing a vector potential description rather than allowing, *e.g.*, an axial vector potential. This leads to the violation of axial current conservation.

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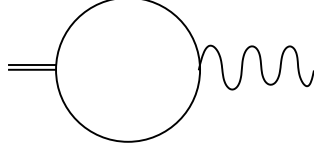


Figure 7: Anomaly in  $\hat{D} = 2$ .

### 4.3 Weyl Spinor Anomaly in $\hat{D} = 2$

We can readily demonstrate the consistent anomaly in the theory for a single Weyl fermion coupled to a photon in  $\hat{D} = 2$ ,

$$S = \int d^2x \bar{\psi}(i\cancel{\partial} - \cancel{A}_L)\psi_L \quad (\text{A.1})$$

where,  $\psi_L = (1 - \gamma^3)\psi/2$ . A Euclidean calculation suffices. We choose the  $\gamma$ -matrices to be the Pauli matrices,  $\gamma_0 = \tau^x$ ,  $\gamma_1 = \tau^y$ ,  $\gamma^3 = \tau^z$ . Then,  $\cancel{p} = p_0\tau^x + p_1\tau^y$ ,  $\text{Tr}(\tau^z\cancel{a}\cancel{b}) = 2i\epsilon_{ij}a^ib^j$ . The divergence of the current  $j_{\mu L} = \bar{\psi}\gamma_\mu\psi_L$  has a matrix element to a single photon of polarization  $\epsilon_\mu$  given by:

$$\begin{aligned} \langle 0 | \partial_\mu j_L^\mu | \epsilon \rangle &= - \int \frac{d^2\ell}{(2\pi)^2} \text{Tr} \left[ \frac{\ell \cancel{\epsilon} (\ell + \cancel{q}) \cancel{q} (1 - \tau^z)}{2(\ell + q)^2 \ell^2} \right] \\ &= \int_0^1 dx q^2 x(1-x) \int \frac{d^2\bar{\ell}}{(2\pi)^2} \text{Tr} \left[ \frac{\tau^z \cancel{\epsilon} \cancel{q}}{2(\bar{\ell}^2 + x(1-x)q^2)^2} \right] \\ &= -\frac{1}{4\pi} \epsilon_{\mu\nu} q^\mu \epsilon^\nu \end{aligned} \quad (\text{A.2})$$

where  $\bar{\ell} = \ell + xq$ , and the divergent part of the  $\bar{\ell}$  integral is zero owing to the identity  $\gamma_\mu \cancel{\ell} \gamma^\mu = 0$ . This is the ‘‘consistent’’ anomaly:

$$\partial^\mu \bar{\psi} \gamma_\mu \psi_L = -\frac{1}{4\pi} \epsilon_{\mu\nu} \partial^\mu A_L^\nu. \quad (\text{A.3})$$

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