Some Calculations of Failure Probabilities in Networks of Connected Detector Elements

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In this note, we would like to investigate how networks of detector elements which are connected together for power distribution may fail over time. In order to keep the analysis simple, we make some assumptions about the ways in which the elements fail and the consequences of the failures of elements. To motivate the discussion, consider Figure 1 at the back of this report. In this figure, there are a total of 8 detector elements receiving current from terminals called the tracker bulkhead. The figure shows a directed graph where the vertices represent detector elements and the edges represent current carrying wires or cables that deliver electrical power to the detector elements. The directions of the arrows (edges) illustrate the directions of current flow. As noted in Figure 1, if all of the detector elements are powered and operating, there is no loss in the detector. We account for loss in the detector with a variable Z. If Z = 0 as in Figure 1, then the entire detector is assumed to be functioning. If Z = N, no elements in the detector are functioning.

The case of Figure 2 is an example of a detector which has experienced failure in one detector element. In this particular case, the failure of the indicated detector element does not cause other elements to fail since they are powered independently of the failed element. In this case, the value of Z = 1 since 1 element is not working.

The example of Figure 3 illustrates another assumption that will be used in this analysis. In this case, one detector element has failed. However, we assume as a worst case assumption that when this happens, any other elements in series with this element also stop functioning. This assumption is equivalent to treating the failures as “open circuit” conditions, leading to a condition in which current flow to other elements in the power circuit branch is “cut off”. In the example of Figure 3, this leads to a loss of not 1 but 2 detector elements.

For arbitrary configurations of detectors and power cables connecting them, the possibilities are too numerous to analyze unless the number of elements and connections between them are relatively small. However, there are some configurations which can be examined, either using analytical techniques or numerical simulations. The value in undertaking such an analysis is that it can yield results which represent limiting values of some parameters which give us an idea of bounds within which the solutions to more general cases must be constrained.

For example, consider the completely parallel and statistically independent network of Figure 4. In this network, since all of the elements are powered independently, if we assume that the failure of any one element has no bearing on the probability of failure for any of the other elements, we may argue that the rate at which this network experiences detector element failures over time must be lower than any other configuration. This network also serves as a similar limiting case for networks with elements that fail in some other fashion than resulting in an open circuit. In those cases, failure of an element in series does not necessarily result in the cut off condition described earlier and therefore, actual failures may more closely approximate the behavior of the completely independent network. For this reason, we will spend a good deal of this report addressing computations for this valuable case.

To examine the behavior of the loss of detector elements as a function of time, we define a random process $\{Z_n\}$, where, at discrete points in time indexed by an integer, n, the value of the process is defined by a random variable, Z_n. That is,

Z_n = the number of failed detector elements at time n where n = 0, 1, 2, …

We assume that $Z_n = 0$ at time n= 0 (that is, all elements are functional when the experiment begins). This implies that

$Z_0 = 0$

Assume that there are a total of N elements in the detector. We assume for simplicity that the probability of failure of each element is given by q with the probability that an element is functioning given by p. Since these are the only possibilities,

$q = 1 - p$

We assume that all elements fail with the same probability and that the failure of any element is independent of the failure of any other elements. In other words, we assume that the failures are independent and identically distributed (more will be said about this later). We also assume that the failure probability, q, is independent of time.

At time n = 1, the detector may have anywhere from 0 to N failed elements. These possibilities correspond to the following values for the random variable, Z_1:

$Z_1 = 0$ (all elements are functioning)
$Z_1 = 1$ (all but 1 of the elements are functioning)
$Z_1 = 2$ (all but 2 of the elements are functioning)
.
.
.
$Z_1 = k$ (all but k of the elements are functioning)
.
.
.
$Z_1 = N$ (all N elements have failed)

We want to compute the probabilities associated with each of these values of the random variable, Z_1.

By combinatorial arguments, we can establish the following:

$P[Z_1 = 0] = B_N(0)q^0p^{N-0} = B_N(0)\, p^N = p^N$

The term "$P[Z_k = x]$" means "the probability that $Z_k = x$". This gives the probability that x elements are no longer working in the detector. In this expression, the term $B_N(0)$ is a binomial coefficient with

$B_N(k) = N!/(k!(N-k)!)$

Furthermore, the remaining values are given by:

$P[Z_1 = 1] = B_N(1)q^1p^{N-1}$

$P[Z_1 = 2] = B_N(2)q^2p^{N-2}$

.

.

.

$P[Z_1 = k] = B_N(k)q^kp^{N-k}$

.

.

.

$P[Z_1 = N] = B_N(N)q^Np^{N-N} = B_N(N)\, q^N$

This handles all of the cases possible for n = 1. By the law of total probability:

$P[Z_1 =0] + P[Z_1 = 1] + P[Z_1 = 2] + \ldots P[Z_1 = N-1] + P[Z_1 = N] = 1$

This can also be shown using the binomial expansion for $(p+q)^N = 1^N$.

Next, consider the cases for n = 2, 3, 4, … (that is, we want to examine the possibilities as time evolves). In general, we want to find a suitable formula for n where n is an arbitrary nonnegative integer. In other words, we want the probability distributions for the random variables, Z_n, which characterize the time evolution of the process, $\{Z_n\}$. We start by examining the case of n = 2.

First we ask, "what is

$P[Z_2 = 0]$?"

If $Z_2 = 0$, this can only occur if there have been no elements lost up to this point in time. This can only be possible if

$Z_1 = 0$ (no elements are lost up to time n = 1)

and

$Z_2 - Z_1 = 0$ (there are no elements lost in the time between n = 1 and n = 2).

These conditions can be expressed as

$P[Z_2 = 0] = P[Z_2 - Z_1 = 0, Z_1 = 0]$

where the expression on the right hand side of the equation is the joint probability of the two conditions that must be met in order that $Z_2 = 0$. We can express this joint probability in a more convenient form through the use of the laws of conditional probability [Reference 1]:

$P[Z_2 = 0] = P[Z_2 - Z_1 = 0 \mid Z_1 = 0]\, P[Z_1 = 0]$

We choose to approach the problem this way because we have already computed the probability, $P[Z_1 = 0]$. Now, we need an expression for $P[Z_2 - Z_1 = 0 \mid Z_1 = 0]$.

Given that $Z_1 = 0$ means that there were, at time n = 1, N functioning elements in the detector. This means that there are a pool of N elements from which combinatorial arguments can be drawn in examining detector failure probabilities at time n = 2. Then, the conditional probability, $P[Z_2 - Z_1 = 0 \mid Z_1 = 0]$ is equal to the probability that no elements are lost in the detector during the time between n = 1 and n = 2, when considerations are made based on a pool of N available elements. Therefore,

$P[Z_2 - Z_1 = 0 \mid Z_1 = 0] = B_N(0)q^0p^{N-0} = B_N(0)\, p^N = p^N$

This is the only way in which we can have the result that $Z_2 = 0$.

Next, we ask "what is

$P[Z_2 = 1]$?"

There are two ways in which this can happen. These are:

$Z_1 = 0$ and $Z_2 - Z_1 = 1$

or

$Z_1 = 1$ and $Z_2 - Z_1 = 0$

These two possibilities are mutually exclusive and therefore, their probabilities may be summed. In other words,

$$P[Z_2 = 1] = P[Z_2 - Z_1 = 1, Z_1 = 0] + P[Z_2 - Z_1 = 0, Z_1 = 1]$$

Again, invoking conditional probability, we can write this as:

$$P[Z_2 = 1] = P[Z_2 - Z_1 = 1 \mid Z_1 = 0]\, P[Z_1 = 0]$$

$$+ \; P[Z_2 - Z_1 = 0 \mid Z_1 = 1]\, P[Z_1 = 1]$$

Given that $Z_1 = 0$ means that at time n = 1, N detector elements were functioning leaving a pool of N elements for combinatorial arguments. Then,

$$P[Z_2 - Z_1 = 1 \mid Z_1 = 0] = B_N(1)q^1p^{N-1}$$

This is just the number of ways in which a single element failure can occur in a pool of N elements ($B_N(1)$) times the probability that any single element (and only 1 such element) fails (q^1p^{N-1}).

The second term is different because, given that $Z_1 = 1$ means that there are only N-1 elements for consideration in combinatorial arguments for this case. Therefore,

$$P[Z_2 - Z_1 = 0 \mid Z_1 = 1] = B_{N-1}(0)q^0p^{(N-1)-0} = p^{(N-1)}$$

This is just the number of ways in which no element failures can occur in a pool of N-1 elements ($B_{N-1}(0)$) times the probability that no element failures occur ($q^0p^{(N-1)-0}$).

Now we look at the next case by asking "what is

$P[Z_2 = 2]$?"

By extending the arguments made above, it is possible to write:

$$P[Z_2 = 2] = P[Z_2 - Z_1 = 2 \mid Z_1 = 0]\, P[Z_1 = 0]$$

$$+ \; P[Z_2 - Z_1 = 1 \mid Z_1 = 1]\, P[Z_1 = 1]$$

$$+ \; P[Z_2 - Z_1 = 0 \mid Z_1 = 2]\, P[Z_1 = 2]$$

Again, by careful consideration of the implications of the conditioning events (for example, $Z_1 = 1$ or $Z_1 = 2$) and what that means to the size of the pool of available candidates for combinatorial arguments, we can write,

$$P[Z_2 - Z_1 = 2 \mid Z_1 = 0] = B_N(2)q^2p^{N-2}$$

$$P[Z_2 - Z_1 = 1 \mid Z_1 = 1] = B_{N-1}(1)q^1p^{(N-1)-1}$$

$$P[Z_2 - Z_1 = 0 \mid Z_1 = 2] = B_{N-2}(0)q^0p^{(N-2)-0} = p^{(N-2)}$$

The terms $P[Z_1 = 0]$, $P[Z_1 = 1]$, $P[Z_1 = 2]$ were all determined earlier.

Treatment of the remaining cases for n = 2 follows these arguments exactly. A pattern is seen to emerge that allows us to write:

$$
\begin{aligned}
P[Z_2 = k] = \; & P[Z_2 - Z_1 = k \mid Z_1 = 0]\, P[Z_1 = 0] \\
+ \; & P[Z_2 - Z_1 = k\text{-}1 \mid Z_1 = 1]\, P[Z_1 = 1] \\
+ \; & P[Z_2 - Z_1 = k\text{-}2 \mid Z_1 = 2]\, P[Z_1 = 2] \\
+ \; & P[Z_2 - Z_1 = k\text{-}3 \mid Z_1 = 3]\, P[Z_1 = 3] \\
& \vdots \\
+ \; & P[Z_2 - Z_1 = k\text{-}j \mid Z_1 = j]\, P[Z_1 = j] \\
& \vdots \\
+ \; & P[Z_2 - Z_1 = k\text{-}(k\text{-}1) \mid Z_1 = k\text{-}1]\, P[Z_1 = k\text{-}1] \\
+ \; & P[Z_2 - Z_1 = k\text{-}k \mid Z_1 = k]\, P[Z_1 = k]
\end{aligned}
$$

The parameter, k, can take on all values from 0 to N (that is, at any point in time, the number of elements that may have failed can be within the range of 0 elements to every element in the detector). We gave explicit formulas for the cases where k = 0, 1, and 2. The remaining formulas may be obtained by using the above general expression.

We can also exploit the pattern that emerged through the expression of probabilities based on conditioning events (the number of elements that had failed up until the current point in time). Doing so results in the following:

$$
\begin{aligned}
P[Z_2 - Z_1 = k \mid Z_1 = 0] &= B_N(k) q^k p^{N-k} &&= \alpha_{k,0} \\
P[Z_2 - Z_1 = k\text{-}1 \mid Z_1 = 1] &= B_{N-1}(k\text{-}1) q^{k-1} p^{(N-1)-(k-1)} &&= \alpha_{k,1} \\
P[Z_2 - Z_1 = k\text{-}2 \mid Z_1 = 2] &= B_{N-2}(k\text{-}2) q^{k-2} p^{(N-2)-(k-2)} &&= \alpha_{k,2} \\
P[Z_2 - Z_1 = k\text{-}3 \mid Z_1 = 3] &= B_{N-3}(k\text{-}3) q^{k-3} p^{(N-3)-(k-3)} &&= \alpha_{k,3} \\
&\vdots \\
P[Z_2 - Z_1 = k\text{-}j \mid Z_1 = j] &= B_{N-j}(k\text{-}j) q^{k-j} p^{(N-j)-(k-j)} &&= \alpha_{k,j} \\
&\vdots
\end{aligned}
$$

$$P[Z_2 - Z_1 = k\text{-}(k\text{-}1) \mid Z_1 = (k\text{-}1)] = B_{N-(k-1)}(k\text{-}(k\text{-}1))q^{k-(k-1)}p^{(N-(k-1))-(k-(k-1))} \quad = \alpha_{k,(k-1)}$$

$$P[Z_2 - Z_1 = k\text{-}k \mid Z_1 = k] = B_{N-k}(k\text{-}k)q^{k-k}p^{(N-k)-(k-k)} \quad = p^{(N-k)} \quad = \alpha_{k,k}$$

We have expressed the conditional probabilities as elements of a matrix, $\underline{A}$. That is, the term $\alpha_{i,j}$ is the element in the ith row and the jth column of an (N+1)x(N+1) matrix, $\underline{A}$ ("underlined" terms are to be interpreted as vectors or matrices). Therefore, each of the terms identified above represent the elements of the kth row of the matrix, $\underline{A}$. These elements correspond to conditional probabilities that are based on all possible conditioning events which can result in a loss of k elements at time n = 2. The use of this notation and the reasons behind it will be made clearer in a little while.

In the expressions above, we have chosen not to simplify many of the terms (such as (k-(k-1)) = 1) so as to explicitly demonstrate from where the results were obtained. Such an approach can also more effectively suggest how the computation of the necessary terms may be automated.

Remember that all of the results expressed above were obtained while considering the possibilities at time n = 2. However, there is really nothing special about the analysis presented up to this point in time. This is due to the fact that we have assumed that the probabilities of failure for the elements, q, are independent of time. This simplification should be reconsidered when attempting to more accurately model the behavior of a connected power distribution network. For example, one might expect that, as elements fail, other elements in the network might be more electrically stressed (dissipating more power than nominal conditions might dictate) resulting in higher probabilities of failures in these more stressed elements.

If we continue to ignore the effects of increased power dissipation and possible modification of element failure probabilities, we can find a convenient way to express the general time evolution of the network (that is, the actual evolution of the random process $\{Z_n\}$).

Let us define the N+1 dimensional column vector of probabilities (for n = 1) as:

$$\underline{P_Z(1)} = (P[Z_1 = 0] \quad P[Z_1 = 1] \quad P[Z_1 = 2] \quad \ldots \quad P[Z_1 = N\text{-}1] \quad P[Z_1 = N])^T$$

where the superscript "T" indicates the transpose of this N+1 dimensional row vector to form an N+1 dimensional column vector and the "underlined" terms are to be interpreted as vectors or matrices. The appearance of the "(1)" portion of $\underline{P_Z(1)}$ is meant to indicate that these are the probabilities at time n = 1.

Similarly, define the (N+1) dimensional column vector of probabilities (for n = 2):

$$\underline{P_Z(2)} = (P[Z_2 = 0] \quad P[Z_2 = 1] \quad P[Z_2 = 2] \quad \ldots \quad P[Z_2 = N\text{-}1] \quad P[Z_2 = N])^T$$

In general, we define (for arbitrary n where n = 1, 2, 3…) the (N+1) dimensional column vector of probabilities:

$$\underline{P_Z(n)} = (P[Z_n = 0] \quad P[Z_n = 1] \quad P[Z_n = 2] \quad \ldots \quad P[Z_n = N-1] \quad P[Z_n = N])^T$$

This is the vector of probabilities at the time n.

Then, we may write the results for n = 2 in terms of the results of n = 1 as follows:

$$\underline{P_Z(2)} = \underline{A}\ \underline{P_Z(1)}$$

where the matrix A is an (N+1) by (N+1) matrix made up of the elements $a_{k,j}$ the formulas for which were outlined earlier. The notation "(N+1) by (N+1)" means the matrix has N+1 rows and N+1 columns; we use the notation "(N+1)x(N+1)" to indicate the same thing. Note that

$$\alpha_{k,j} = 0 \qquad \text{(for all } j > k)$$

This means that there is zero probability of having fewer than k failed elements once k elements have already failed (no elements come back to life). This results in a matrix which is lower triangular.

The assumption that the failure probabilities are independent in time allows us to realize that the matrix, A, may be repeatedly applied in the time evolution of this random process. In other words, we may write, in general:

$$\underline{P_Z(n+1)} = \underline{A}\ \underline{P_Z(n)} \qquad \text{for } n = 1, 2, 3, \ldots$$

This may also be written in the following form:

$$\underline{P_Z(n+1)} = \underline{A}^n\ \underline{P_Z(1)}$$

where $\underline{P_Z(1)}$ is the (N+1) dimensional column vector of probabilities that we computed for the time n = 1. Since the matrix, $\underline{A}$, is lower triangular, so too will be all products of the form $\underline{A}^n$. This formula gives us a particularly simple way to determine the probabilities associated with the possible states of detector loss in the network as a function of time. From the formula just given for $\underline{P_Z(n+1)}$, it is possible to evolve the failure probabilities simply by n repeated matrix multiplications of the matrix $\underline{A}$. Such an approach is sometimes discussed in the literature on stochastic processes associated with Markov chains and the Chapman-Kolmogorov equations [Reference 2].

One application of the formulas just demonstrated is to compute the expected value of the detector loss function of the configuration. Such a computation will actually represent a lower bound and will give us, as a function of time, the level of lost elements below which (on average) we should not expect to experience. In other words, for a given number of elements (ignoring how they are connected) and a given elemental failure

probability, q, we can estimate a function beyond which (on average), we should expect to do no better. Now, we will show how to compute this.

The expected value of the discrete random variable Z_{n+1} is given by

$$E[Z_{n+1}] = (0 \quad 1 \quad 2 \quad \ldots \quad N\text{-}1 \quad N)^T \, \underline{P_Z(n+1)}$$

$$= \underline{J}^T \, \underline{P_Z(n+1)}$$

$$= \underline{J}^T \, \underline{A}^n \, \underline{P_Z(1)}$$

where

$$\underline{J}^T = (0 \quad 1 \quad 2 \quad \ldots \quad N\text{-}1 \quad N)^T$$

The vector $\underline{J}$ is just the set of values that Z_{n+1} can assume (from 0 to N elements can be lost).

A useful property to demonstrate is that the diagonal elements are all distinct. This follows from the previously derived formula for the diagonal matrix elements, $\alpha_{k,k}$:

$$\alpha_{k,k} = P[Z_2 - Z_1 = k\text{-}k \mid Z_1 = k] = B_{N-k}(k\text{-}k)q^{k-k}p^{(N-k)-(k-k)}$$

$$= B_{N-k}(0)q^0p^{(N-k)}$$

$$= p^{(N-k)}$$

Since

$B_{N-k}(0) = 1$ (for all N, k = 0, 1, 2, …, N)

and

$q^0 = 1$

we have

$\alpha_{k,k} = p^{(N-k)}$ (for all N, k = 0, 1, 2, …, N)

If j is not equal to k, then $\alpha_{j,j}$ is not equal to $\alpha_{k,k}$ for any two allowable values of j and k (as long as p is different from 1 and 0). Furthermore, all of the diagonal elements are positive if $p > 0$.

The terms along the diagonal of $\underline{A}$ are given by:

$\alpha_{0,0} = p^{(N-0)} = p^N$

$\alpha_{1,1} = p^{(N-1)}$

$\alpha_{2,2} = p^{(N-2)}$

.

.

.

$\alpha_{N-1,\,N-1} = p^{(N-(N-1))} = p^1$

Observe that:

$\alpha_{N,\,N} = p^{(N-N)} = p^0 = 1$ (for any value of N).

Therefore, with the exception of the $\alpha_{N,N}$ term, all of the diagonal elements are positive with a value less than 1 (if p is not equal to 1).

Consider the meaning of the $\alpha_{N,\,N}$ term. This term is equal to the conditional probability (at time n+1):

$\alpha_{N,N} = P[Z_{n+1} - Z_n = 0 \mid Z_n = N] = B_0(0)q^0p^0 = 1$

This seems correct since we are certain (the probability is 1) that, given that we have lost all elements, we are going to lose no more.

The fact that the matrix, $\underline{A}$, is lower triangular means that the eigenvalues of that matrix are the diagonal elements. Furthermore, the formula for $\alpha_{k,k}$ allows us to conclude that the diagonal elements are all nonzero provided $p > 0$. Since we have demonstrated that the diagonal elements are all distinct, that is, if $\alpha_{k,k}$ is not equal to $\alpha_{j,j}$ for j not equal to k, then we can be sure that a similarity transformation defined by a nonsingular matrix, $\underline{X}$, exists such that:

$\underline{\Lambda} = \underline{X}^{-1}\underline{A}\,\underline{X}$

where $\underline{\Lambda}$ is a diagonal matrix with elements $\lambda_{k,k}$. This assurance follows from the fact that there are no repeated eigenvalues which implies that there are N+1 linearly independent eigenvectors of $\underline{A}$ [Reference 3]. These eigenvectors can be used as the columns of the transformation matrix, $\underline{X}$. To see this, define:

$\underline{X} = [\underline{x}_0 \quad \underline{x}_1 \quad \underline{x}_2 \quad \underline{x}_3 \quad \ldots \quad \underline{x}_N]$

where $\underline{x}_i$ = the ith N+1 dimensional column vector of $\underline{X}$

= the ith eigenvector of $\underline{X}$ corresponding to the eigenvalue, λ_i.

Then,

$\underline{A}\ \underline{X} = [\lambda_0\ \underline{x}_0 \qquad \lambda_1\ \underline{x}_1 \qquad \lambda_2\ \underline{x}_2 \quad \ldots \qquad \lambda_N\ \underline{x}_N]$

But

$\underline{X}\ \underline{\Lambda} = [\lambda_0\ \underline{x}_0 \qquad \lambda_1\ \underline{x}_1 \qquad \lambda_2\ \underline{x}_2 \quad \ldots \qquad \lambda_N\ \underline{x}_N]$

as well (as can be seen by multiplying out the terms) and therefore,

$\underline{A}\ \underline{X} = \underline{X}\ \underline{\Lambda}$

Since the columns of $\underline{X}$ are linearly independent, the inverse $\underline{X}^{-1}$ exists and the result

$\underline{\Lambda} = \underline{X}^{-1}\underline{A}\ \underline{X}$

is valid. From this, we can write $\underline{A}$ as:

$\underline{A} = \underline{X}\ \underline{\Lambda}\ \underline{X}^{-1}$

We will have more to say about the similarity transformation and how to compute it in an appendix at the end of this note.

As a check on the correctness of the formulation, we should be able to demonstrate that the probabilities which make up the components of $\underline{P_Z(n+1)}$ all sum to 1 for any value of n. This computation is given by:

$P[Z_{n+1} =0] + P[Z_{n+1} = 1] + P[Z_{n+1} = 2] + \ldots\ P[Z_{n+1}= N-1] + P[Z_{n+1} = N] = 1$

In terms of the elements of the matrix, $\underline{A}$, we write this as:

$$
\begin{array}{l}
P[Z_n = 0]\ (\alpha_{0,0} + \alpha_{1,0} + \alpha_{2,0} + \alpha_{3,0} + \alpha_{4,0} + \ldots \alpha_{N-2,0} + \alpha_{N-1,0} + \alpha_{N,0}) + \\
P[Z_n = 1]\ (0 \quad + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1} + \ldots \alpha_{N-2,1} + \alpha_{N-1,1} + \alpha_{N,1}) + \\
P[Z_n = 2]\ (0 \quad + \ 0 \ + \alpha_{2,2} + \alpha_{3,2} + \alpha_{4,2} + \ldots \alpha_{N-2,2} + \alpha_{N-1,2} + \alpha_{N,2}) + \\
P[Z_n = 3]\ (0 \quad + \ 0 \ + \ 0 \ + \alpha_{3,3} + \alpha_{4,3} + \ldots \alpha_{N-2,3} + \alpha_{N-1,3} + \alpha_{N,3}) + \\
. \\
. \\
. \\
P[Z_n = N-2]\ (\ 0 \ + \ 0 \ + \ 0 \ + \ 0 \ + \ 0 \ + \ldots + \ 0 \ + \ \alpha_{N-2,N-2} + \alpha_{N-1,N-2} + \alpha_{N,N-2}) + \\
P[Z_n = N-1]\ (\ 0 \ + \ 0 \ + \ 0 \ + \ 0 \ + \ 0 \ + \ldots + \ 0 \ + \ 0 \qquad + \alpha_{N-1,N-1} + \alpha_{N,N-1}) + \\
P[Z_n = N]\ \ (\ 0 \ + \ 0 \ + \ 0 \ + \ 0 \ + \ 0 \ + \ldots + \ 0 \ + \ 0 \qquad + \ 0 \qquad + \alpha_{N,N}) = 1
\end{array}
$$

where, as before, we have the formula for the matrix elements:

$$\alpha_{k,j} = P[Z_2 - Z_1 = k\text{-}j \mid Z_1 = j] = B_{N-j}(k\text{-}j)q^{k-j}p^{(N-j)-(k-j)}$$

$$= B_{N-j}(k\text{-}j)q^{k-j}p^{(N-j)-(k-j)}$$

The fact that this expression is correct may be demonstrated by recognizing that each of the sums of terms in parentheses is equal to 1. Each of these sums represents the summation of the elements of a single column of $\underline{A}$. Therefore,

$$\alpha_{0,0} + \alpha_{1,0} + \alpha_{2,0} + \alpha_{3,0} + \alpha_{4,0} + \ldots \alpha_{N-2,0} + \alpha_{N-1,0} + \alpha_{N,0} = 1$$

$$\alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1} + \ldots \alpha_{N-2,1} + \alpha_{N-1,1} + \alpha_{N,1} = 1$$

$$\alpha_{2,2} + \alpha_{3,2} + \alpha_{4,2} + \ldots \alpha_{N-2,2} + \alpha_{N-1,2} + \alpha_{N,2} = 1$$

$$\alpha_{3,3} + \alpha_{4,3} + \ldots \alpha_{N-2,3} + \alpha_{N-1,3} + \alpha_{N,3} = 1$$

.

.

.

$$\alpha_{N-2,N-2} + \alpha_{N-1,N-2} + \alpha_{N,N-2} = 1$$

$$\alpha_{N-1,N-1} + \alpha_{N,N-1} = 1$$

$$\alpha_{N,N} = 1$$

This follows from the fact that each sum of terms for the columns of $\underline{A}$ is a sum of the binomial coefficients for pools of size N (0th column), N-1 (1st column), N-2 (2nd column), …, 1 ($N\text{-}1^{st}$ column), and 0 (N^{th} column). One way to see this is to write the elements of the matrix $\underline{A}$ in terms of the binomial coefficients:

$$\underline{A} = \begin{matrix}
B_N(0)p^N & 0 & 0 & 0 \ldots & 0 \\
B_N(1)p^{N-1}q^1 & B_{N-1}(0)p^{N-1} & 0 & 0 \ldots & 0 \\
B_N(2)p^{N-2}q^2 & B_{N-1}(1)p^{N-2}q^1 & B_{N-2}(0)p^{N-2} & 0 \ldots & 0 \\
B_N(3)p^{N-3}q^3 & B_{N-1}(2)p^{N-3}q^2 & B_{N-2}(1)p^{N-3}q^1 & B_{N-3}(0)p^{N-3} \ldots & 0 \\
\vdots & & & & \\
B_N(N-2)p^2q^{N-2} & B_{N-1}(N-3)p^2q^{N-3} & B_{N-2}(N-4)p^2q^{N-4} & B_{N-3}(N-5)p^2q^{N-5} \ldots & 0 \\
B_N(N-1)p^1q^{N-1} & B_{N-1}(N-2)p^1q^{N-2} & B_{N-2}(N-3)p^1q^{N-3} & B_{N-3}(N-4)p^1q^{N-4} \ldots & 0 \\
B_N(N)q^N & B_{N-1}(N-1)q^{N-1} & B_{N-2}(N-2)q^{N-2} & B_{N-3}(N-3)q^{N-3} \ldots & B_N(0)p^0
\end{matrix}$$

Since such sums of binomial coefficients always equal 1, we have demonstrated what was needed to establish the correctness of the formulation.

Another useful check is to consider the limiting probabilities. In other words, what is the behavior of the vector of probabilities, $\underline{P_Z(n)}$ as n goes to infinity? We might expect the following to be true:

$$\lim_n \underline{P_Z(n)} = \underline{e}_N$$

In other words, we might expect the detector to end up after a long time in the state where all of the elements have failed. We can prove that the limit exists and that $\lim_n \underline{P_Z(n)} = \underline{e}_N$ with the following argument by starting with the relation that:

$$\underline{P_Z(n+1)} = \underline{A}\ \underline{P_Z(n)}$$

Therefore, if the limits exist, then

$$\lim_n \underline{P_Z(n+1)} = \lim_n \underline{A}\ \underline{P_Z(n)} = \underline{A} \lim_n \underline{P_Z(n)}$$

First assume that the above limit exists and define

$$\underline{P_Z} = \lim_n \underline{P_Z(n)} = \lim_n \underline{P_Z(n+1)}$$

Then, we have:

$$\underline{P_Z} = \underline{A}\ \underline{P_Z}$$

If we guess that

$$\underline{P_Z} = \underline{e}_N$$

we can test the validity of

$$\underline{A}\ \underline{e}_N = \underline{e}_N$$

By direct multiplication, we see that this is true (refer to the form of $\underline{A}$ where the elements are expressed in terms of the binomial coefficients). The only requirement is that

$$B_N(0)p^0 = 1$$

which is true. In other words, the limiting value of the sequence of probabilities, $\underline{P_Z(n)}$, is an eigenvector of the matrix $\underline{A}$ corresponding to the eigenvalue, 1.

To prove that the limit above does exist, we simply consider the form produced under repeated applications of the similarity transformation:

$$\underline{P_Z(n+1)} = \underline{A}^n\ \underline{P_Z(n)}$$

$$= (\underline{X}\,\underline{\Lambda}\,\underline{X}^{-1})^n\,\underline{P_Z(1)}$$

$$= \underline{X}\,\underline{\Lambda}^n\,\underline{X}^{-1}\,\underline{P_Z(1)}$$

Therefore,

$$\lim_n \underline{P_Z(n+1)} = \lim_n \underline{X}\,\underline{\Lambda}^n\,\underline{X}^{-1}\,\underline{P_Z(1)}$$

$$= \underline{X} \lim_n \underline{\Lambda}^n\,\underline{X}^{-1}\,\underline{P_Z(1)}$$

The notation "$\lim_n f(n)$" means "the limit of f(n) as n approaches infinity". This leads to:

$$\lim_n \underline{\Lambda}^n = \lim_n (\mathrm{diag}((\lambda_{0,0})^n, (\lambda_{1,1})^n, (\lambda_{2,2})^n, (\lambda_{3,3})^n, \ldots (\lambda_{N,N})^n))$$

$$= \mathrm{diag}(\lim_n(\lambda_{0,0})^n, \lim_n(\lambda_{1,1})^n, \lim_n(\lambda_{2,2})^n, \lim_n(\lambda_{3,3})^n, \ldots \lim_n(\lambda_{N,N})^n)$$

$$= \mathrm{diag}(\lim_n(\alpha_{0,0})^n, \lim_n(\alpha_{1,1})^n, \lim_n(\alpha_{2,2})^n, \lim_n(\alpha_{3,3})^n, \ldots \lim_n(\alpha_{N,N})^n)$$

$$= \mathrm{diag}(0, 0, 0, \ldots, 0, 1)$$

$$= [\,\underline{0} \quad \underline{0} \quad \underline{0} \quad \underline{0} \ldots \quad \underline{0} \quad \underline{e}_N]$$

where

$\underline{0}$ = the N+1 dimensional column vector of all zero components

$\underline{e}_N$ = the N+1 dimensional standard basis column vector with all components equal to 0 except the N+1st component which is equal to 1

This result follows from the formulas derived for the terms $\alpha_{k,k}$ earlier in which all $\alpha_{k,k}$ except $\alpha_{N,N}$ are expressed as powers of the probability, p. Therefore, the limiting values in the expression above all tend towards 0 with the exception of $\alpha_{N,N}$ which is identically equal to 1. Therefore the limit of the product exists.

This demonstration also allows us to satisfy another intuitive consideration, namely that, over a long time, we expect all of the elements to eventually fail. This is equivalent to the assertion that:

$$\lim_n E[Z_{n+1}] = N$$

This follows immediately from the formula:

$$\lim_n E[Z_{n+1}] = \lim_n \underline{J}^T\,\underline{P_Z(n+1)}$$

$$= \underline{J}^T \lim_n \underline{P_Z(n+1)}$$

$$= \underline{J}^T \underline{e}_N$$

$$= N$$

Mathematically, the solution just demonstrated is equivalent to a detector constructed of completely independent, parallel elements all supplied by their own cables (see Figure 4 at the back of this note). If we assume that the probability of failure is the same for all elements and that these failure events are all independent, we will have the same solution. Such a detector represents the most reliable construction possible and confirms our assumption that the results presented here represent the best case scenario (for other designs, the expected value of the loss function will rise more rapidly). Figure 5 (see the back of this note) illustrates this behavior. The mathematical form of this plot will be demonstrated presently.

The time where the plot of the expected value crosses a threshold of acceptable performance (where that threshold is between 0 and N) is called the average survival time for the level of performance. From the plot, we should expect that survival time to be maximized for a fixed level of acceptable performance when the network is the independent parallel network.

The shape of the curve can be examined by computing the difference term:

$$E[Z_{n+1}] - E[Z_n] = \underline{J}^T (\underline{P}_Z(n+1) - \underline{P}_Z(n))$$

$$= \underline{J}^T (\underline{A}^n - \underline{A}^{n-1}) \underline{P}_Z(1)$$

$$= \underline{J}^T \underline{A}^{n-1}(\underline{A} - \underline{I}) \underline{P}_Z(1)$$

In the limit of large values of n, we have:

$$\lim_n (E[Z_{n+1}] - E[Z_n]) = \lim_n \underline{J}^T (\underline{A}^n - \underline{A}^{n-1}) \underline{P}_Z(1)$$

$$= \underline{J}^T \lim_n (\underline{A}^n - \underline{A}^{n-1}) \underline{P}_Z(1)$$

$$= \underline{J}^T (\lim_n \underline{A}^n - \lim_n \underline{A}^{n-1}) \underline{P}_Z(1)$$

$$= \underline{J}^T (\underline{0}) \underline{P}_Z(1)$$

$$= \underline{0}^T \underline{P}_Z(1)$$

$$= 0$$

since

$$\lim_n \underline{A}^n = \lim_n \underline{A}^{n-1}$$

$\lim_n \underline{A}^n - \lim_n \underline{A}^{n-1} = \underline{0}$ ($\underline{0}$ is an N+1 by N+1 dimensional matrix in this last equation)

Therefore, the slope of the curve for $E[Z_n]$ approaches 0 as n approaches infinity.

Alternative Analytical Approach Based on Single Element Probabilities:
Suppose that we consider a single detector element with a probability of failure given by:

$q = 1 - p$

We define a random process, $\{X_n, n = 0, 1, 2, \ldots\}$ such that:

$X_n = 0$ if the element is functioning at time n
$X_n = 1$ if the element is functioning at time n

$X_{n+1} = 1$ if $X_n = 1$ for n = 1, 2, …

$X_0 = 0$

Then we can ask, what is $P[X_n = k]$ where k = 0 or k = 1.

Consider n = 1:

$P[X_1 = 0] = p$

$P[X_1 = 1] = 1 - p = q$

Consider n = 2:

$$P[X_2 = 0] = P[X_2 - X_1 = 0, X_1 = 0]$$

$$= P[X_2 - X_1 = 0 \mid X_1 = 0]\, P[X_1 = 0]$$

$$= P[X_2 - X_1 = 0]\, P[X_1 = 0]$$

$$= p\, P[X_1 = 0]$$

$$\begin{aligned} P[X_2 = 1] &= P[X_2 - X_1 = 1, X_1 = 0] \\ &+ P[X_2 - X_1 = 0, X_1 = 1] \\ &= P[X_2 - X_1 = 1 \mid X_1 = 0]\, P[X_1 = 0] \\ &+ P[X_2 - X_1 = 0 \mid X_1 = 1]\, P[X_1 = 1] \end{aligned}$$

$$= P[X_2 - X_1 = 1]\, P[X_1 = 0]$$

$$+ P[X_2 - X_1 = 0 \mid X_1 = 1]\, P[X_1 = 1]$$

Given that $X_1 = 1$, we must have:

$P[X_2 - X_1 = 0 \mid X_1 = 1] = 1$ (it must be so if there is only 1 element)

Therefore:

$$P[X_2 = 1] = (1 - p)\, P[X_1 = 0] + 1\,(1-p)$$

We can summarize the possibilities for n = 2 in the matrix formulation:

$$\underline{P_X(2)} = \underline{A}\ \underline{P_X(1)}$$

where:

$$\underline{P_X(2)} = \begin{matrix} P[X_2 = 0] \\ P[X_2 = 1] \end{matrix}$$

$$\underline{P_X(1)} = \begin{matrix} P[X_1 = 0] \\ P[X_1 = 1] \end{matrix}$$

$$\underline{A} = \begin{matrix} p & 0 \\ 1 - p & 1 \end{matrix}$$

In general, as for the case of N detector elements, we may write:

$$\underline{P_X(n+1)} = \underline{A}^n\ \underline{P_X(\ (1)}$$

We may also diagonalize $\underline{A}$ to produce the matrix, $\underline{\Lambda}$, using the similarity matrix, $\underline{X}$, where:

$$\underline{\Lambda} = \underline{X}^{-1}\ \underline{A}\ \underline{X}$$

$$\underline{\Lambda} = \begin{matrix} p & 0 \\ 0 & 1 \end{matrix}$$

Using the formulas derived earlier for an N+1 by N+1 matrix, A, we find that:

$$\underline{X} = \begin{matrix} 1 & 0 \\ -1 & 1 \end{matrix}$$

$$\underline{X}^{-1} = \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}$$

Writing

$$\underline{A}^n = \underline{X}\,\underline{\Lambda}^n\,\underline{X}^{-1}$$

we can compute:

$$\underline{P_X(n+1)} = \underline{X}^{-1}\,\underline{\Lambda}^n\,\underline{X}\,\underline{P_X(1)}$$

First, compute:

$$\underline{X}^{-1}\,\underline{\Lambda}^n\,\underline{X} =$$

$$\begin{matrix} 1 & 0 & p^n & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & -1 & 1 \end{matrix}$$

$$= \begin{matrix} p^n & 0 \\ 1-p^n & 1 \end{matrix}$$

Therefore:

$$\underline{P_X(n+1)} = \begin{matrix} P[X_{n+1} = 0] \\ P[X_{n+1} = 1] \end{matrix} = \begin{matrix} p^n P[X_1 = 0] \\ (1- p^n)P[X_1 = 0] + P[X_1 = 1] \end{matrix} = \begin{matrix} p^{n+1} \\ (1- p^n)p + (1-p) \end{matrix}$$

$$= \begin{matrix} p^{n+1} \\ 1 - p^{n+1} \end{matrix}$$

This leads us to a simple formula for $E[X_{n+1}]$:

$$E[X_{n+1}] = 0\ P[X_{n+1} = 0] + 1\ P[X_{n+1} = 1]$$

$$= P[X_{n+1} = 1]$$

$$= 1 - p^{n+1}$$

If we realize that a collection of N such detector elements has an average value of N times this expectation for X_{n+1}, we conclude that:

$$E[Z_{n+1}] = N(1 - p^{n+1})$$

Using this formulation, it is much easier to examine properties of the expectation curve, $E[Z_{n+1}]$, as a function of time (index n). For example, the monotonically increasing nature of the curve is demonstrated as follows:

$$E[Z_{n+1}] - E[Z_n] = N(1 - p^{n+1}) - N(1 - p^n)$$

$$= N - N p^{n+1} – N + N p^n$$

$$= N (p^n - p^{n+1})$$

$$= N p^n (1 – p)$$

$$> 0 \qquad \text{(since } p < 1\text{)}$$

Taking some limits for increasing n yields:

$$\lim_n E[Z_{n+1}] = \lim_n N (1 – p^{n+1})$$

$$= N$$

$$\lim_n (E[Z_{n+1}] - E[Z_n]) = \lim_n N p^n (1 – p)$$

$$= 0$$

Therefore, the curve for $E[Z_{n+1}]$ asymptotically approaches N in a monotonically increasing fashion as time grows larger and larger. This is consistent with the plots shown in Figure 2 for the average behavior of the loss function. An interesting result from the analysis of this process is that the variance of the random variable Z_n actually decreases with time. This is due to the fact that elements are being lost and the possible states of the detector are becoming more and more limited as time increases.

Another moment that we can compute using this formulation is the variance of the process, $\{Z_n\}$, as a function of time. This is defined as:

$$Var(Z_n) = E[(Z_n - E[Z_n])^2]$$

$$= E[(Z_n)^2] - (E[Z_n])^2$$

Since the parallel elements fail independently of each other, we can use the property that, for independent random variables $X_1, X_2, X_3, \ldots X_m$

$$Var(X_1 + X_2 + X_3 + \ldots + X_m) = Var(X_1) + Var(X_2) + Var(X_3) + \ldots Var(X_m)$$

This will allow us to express $Var(Z_n)$ as:

$$Var(Z_n) = N\, Var(X_n)$$

$$= N(E[(X_n)^2] - (E[X_n])^2)$$

Then, we write:

$$E[(X_n)^2] = 0^2\, P[X_{n+1} = 0] + 1^2\, P[X_{n+1} = 1]$$

$$= E[X_n]$$

$$= 1 - p^n$$

Then we compute:

$$(E[X_n])^2 = (1 - p^n)^2$$

$$= 1 - 2p^n + p^{2n}$$

Taking the difference of these two terms yields:

$$Var(X_n) = 1 - p^n - (1 - 2p^n + p^{2n})$$

$$= p^n (1 - p^n)$$

Another way to express this is:

$Var(X_n) = (1 - p^n) - (1 - p^n)^2$

Therefore, for the loss in all of the detector elements, we have:

$Var(Z_n) = N\ Var(X_n) = N\ p^n\ (1 - p^n)$

We can demonstrate two properties of $Var(Z_n)$ from this formulation. First, we take the lim as n tends to infinity:

$\lim_n Var(Z_n) = 0$

This means that the uncertainty of the loss of the number of elements eventually disappears (this makes sense; eventually we know that all of the elements will be lost).

We can also demonstrate that the uncertainty is a monotonically decreasing function of the time index, n. We see this by observing that:

$Var(Z_n) / Var(Z_{n-1}) = [N\ p^n\ (1 - p^n)] / [N\ p^{n-1}\ (1 - p^{n-1})]$

$= p\ ((1 - p^n)/ (1 - p^{n-1}))$

Since $p < 1$,

$(1 - p^n) > (1 - p^{n-1})$

so

$(1 - p^n)/ (1 - p^{n-1}) > 1$

Notice that, in the limit as n tends towards infinity,

$\lim_n (1 - p^n)/ (1 - p^{n-1}) = 1$

This means that we can find an index, n_0, such that, $(1 - p^n)/ (1 - p^{n-1})$ is as close to 1 as we want it to be. Then, we can find an n_0 such that $p\ ((1 - p^n)/ (1 - p^{n-1}))$ is less than 1. For all $n > n_0$,

$Var(Z_n) / Var(Z_{n-1}) < 1$

or

$Var(Z_n) < Var(Z_{n-1})$

This means that for sufficiently large n, the variance monotonically decreases with increasing n.

Note that we can not conclude that, for all values of p and n, $Var(Z_n)$ tends monotonically towards 0. As an example, consider the case where n = 2 and p = 3/4. In this case,

$$Var(Z_n) / Var(Z_{n-1}) = p\,((1 - p^n)/ (1 - p^{n-1}))$$

$$= p\,((1 - p^2)/ (1 - p^1))$$

$$= ¾\,((1 - (¾)^2)/ (1 - ¾))$$

$$= ¾\,(7/16)/(1/4)$$

$$= 21/16$$

Since this is greater than 1, we conclude that, for some range of increasing n, the variance is actually growing. But, eventually, it will begin to decrease and once it does, it decreases monotonically for all increasing n.

Now we can relate the probabilities for a single detector element to the probabilities for a collection of N independent, identically distributed such elements. Define

$$\underline{P_Z(n)} = (\ P[Z_n = 0] \quad P[Z_n = 1] \quad P[Z_n = 2] \quad \ldots \quad P[Z_n = N-1] \quad P[Z_n = N]\ )^T$$

$$\underline{P_X(n)} = (\ P[X_n = 0] \quad P[X_n = 1]\ )^T$$

Then, by considering the combination of N of the single detectors with their two state probabilities, we may express the components of $\underline{P_Z(n+1)}$ as:

$$P[Z_{n+1} = 0] = B_N(0)\ \ (P[X_{n+1} = 0])^N\,(P[X_{n+1} = 1])^0$$

$$= B_N(0)\,(p^{n+1})^N$$

$$P[Z_{n+1} = 1] = B_N(1)\,(P[X_{n+1} = 0])^{N-1}\,(P[X_{n+1} = 1])^1$$

$$= B_N(1)\,(p^{n+1})^{N-1}\,(1 - p^{n+1})^1$$

$$P[Z_{n+1} = 2] = B_N(2)\,(P[X_{n+1} = 0])^{N-2}\,(P[X_{n+1} = 1])^2$$

$$= B_N(2)\,(p^{n+1})^{N-2}\,(1 - p^{n+1})^2$$

.

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.

$$P[Z_{n+1} = k] = B_N(k)\,(P[X_{n+1} = 0])^{N-k}\,(P[X_{n+1} = 1])^k$$

$$= B_N(k)\,(p^{n+1})^{N-k}\,(1 - p^{n+1})^k$$

.

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$$P[Z_{n+1} = N\text{-}2] = B_N(N\text{-}2)\,(P[X_{n+1} = 0])^2\,(P[X_{n+1} = 1])^{N-2}$$

$$= B_N(N\text{-}2)\,(p^{n+1})^2\,(1 - p^{n+1})^{N-2}$$

$$P[Z_{n+1} = N\text{-}1] = B_N(N\text{-}1)\,(P[X_{n+1} = 0])^1\,(P[X_{n+1} = 1])^{N-1}$$

$$= B_N(N\text{-}1)\,(p^{n+1})^1\,(1 - p^{n+1})^{N-1}$$

$$P[Z_{n+1} = N] = B_N(N)\,(P[X_{n+1} = 0])^0\,(P[X_{n+1} = 1])^N$$

$$= B_N(N)\,(1 - p^{n+1})^N$$

These formulas offer another way to express the probabilities that were derived in terms of the Chapman-Kolmogorov equations (using the product of $\underline{A}^n$).

Survival Time Computation:

Assume that we are given a design target in the form of a requirement that some percentage of the detector still be functioning (on average) after the passage of some time period. Then, given the level of function (call it E, the number of elements still functioning out of N) and the probability, p, we may compute (from the expression for the average loss in the detector), $E[Z_n] = N(1 - p^n)$,

$$E < N(1 - p^n)$$

or

$$E/N < (1 - p^n)$$

This leads to

$$p^n < 1 – E/N$$

yielding

$$n \ln(p) < \ln (1 – E/N)$$

or (since $\ln(p) < 0$ for $0 < p < 1$)

$$n > [\ln (1 – E/N)]/[\ln(p)]$$

In this formulation, a higher level of E corresponds to a higher level of acceptable loss in the detector. For a given level of E, we call this the survival time of the detector.

Extensions to More Complicated Architectures:
The results just presented can be extended to architectures more complicated than a simple arrangement of parallel (independent) detectors. Such architectures include:

1. Parallel arrangements of serially connected simple detector elements
2. Series connections of parallel arrangements of simple detector elements
3. Parallel arrangements of serially connected elements with periodically spaced cross connections

We briefly discuss these three cases.

Extension to Parallel Arrangements of Series Connected Simple Detector Elements:
The first arrangement above results in a system design in which the number of cables from the tracker bulkhead to the detector elements is reduced (see Figure 6). Since the independence of all of the elements is reduced, we expect the reliability to suffer but the material budget to be reduced. The analysis is a straightforward generalization of the above results with modifications to the failure probabilities for each independent element and to the possible values which are part of the expected value calculation.

We may approach the extension to this architecture by considering each series connection of M elements as a single entity. The probability that this entity is functioning is given by p_M with:

$$p_M = p^M$$

where p is the probability that each independent element in the series connection is functioning. If we assume that the number of elements is still equal to N and that we have arranged these N elements in L independent strings of M elements each where:

$$N = LM$$

we may follow a path similar to the approach that led to the matrix formulation of the vector of probabilities, $\underline{P_Z(n)}$ to produce a new vector of probabilities. This new vector however differs in that the possibilities for the number of non-functioning elements is not 0, 1, 2, … but rather 0, M, 2M, 3M, … according to which of the series connected strings are not functioning. Rather than repeat the analysis here, we can make use of the fact that the problem of a single string of series connections is much the same as that for a single detector element of weight M with $p = p^M$. Furthermore, for such an arrangement, the only possibilities for the operating state (the number of failed detectors) are 0 (all of the elements in the string are functioning and powered) or M (at least one element has failed rendering the string of M elements non-functional).

This leads to a modified expectation for a single string of series connected detectors of:

$E[X_n] = 0\ P[X_n = 0] + M\ P[X_n = M]$

$= M\ P[X_{n+1} = M]$

$= M\ (1 - p^{Mn})$

For L parallel arrangements of these strings of M detectors in series, we have:

$E[W_n] = LM(1 – (p^M)^n)$

$= N(1 – (p^M)^n)$

where $\{W_n\}$ is a new random process with a spectrum of values drawn from the set 0, M, 2M, 3M, … and defined for integer n = 0, 1, 2, … with the restriction that $W_0 = 0$. As expected, this expected value rises more rapidly than does the similar computation for $E[Z_n]$ reflecting the fact that series connections lose more elements when an element fails. The local and asymptotic behavior of this function is in agreement with what we should expect for such an arrangement.

We can examine the consequences of this configuration on the survival time of the network, previously calculated for the independent parallel case.

From the relation,

$E[W_n] = N(1 – (p^M)^n)$

we have (for a given threshold of acceptable loss)

$n > [\ln (1 – E/N)]/[M \ln(p)]$

(found by replacing p with p^M in the previous expression)

As before, a higher level of E corresponds to a higher level of acceptable loss in the detector. We see that the move to series connected strings of length M for the architecture reduces the survival time of the resulting network of detector elements by a factor of 1/M.

An obvious deficiency of this arrangement is that the loss of a single detector element eliminates an entire string of elements.

Extension to Series Arrangements of Parallel Connected Simple Detector Elements:
The second arrangement above results in a system design in which the number of cables from the tracker bulkhead to the detector elements is also reduced (see Figure 7). This option avoids the deficiency of the first option in that the loss of multiple detector elements must be suffered before the series path is broken. However, once broken, the entire detector is lost. Again, since the independence of all of the elements is reduced, we

expect the reliability to suffer but the material budget to be reduced. The analysis is not however a straightforward generalization of the results for either the fully parallel topology or the parallel connection of series detectors. In particular, the probabilities of loss for the entire detector are considerably more complicated due to the dependent nature of the failure events introduced by the series connections. It is, however, possible to consider an approach to the calculation of the expected value of the detector function by some modifications to the probability distributions of the parallel blocks and the use of discrete time convolution applied to the specific calculation of the expected value. These ideas will be presented in another note.

Extension to Parallel Arrangements of Series Connected Simple Detector Elements with Periodically Spaced Cross Connections:

The third arrangement is an attempt to restore some reliability to the detector with a small penalty to the material budget by introducing local cables to reduce the consequences of individual detector element failures (see Figure 8). The analysis of this arrangement requires us to consider the lack of independence resulting from the cross connections introduced. A way to approach this analysis is to consider the output probabilities of a single unit of the detector made up of elements with the cross connections. By designing the detector as serially connected arrays of regularly space elements of this type, we will attempt to formulate the probabilities for a single such configuration. If this is possible, then the analytical techniques derived above will apply to independent arrays of this type.

Unfortunately, the analysis is complicated by the need to consider what we call the "feedback failures" on upstream elements induced by the failure of downstream elements (this is the "cut off" failure illustrated in Figure 3). This occurs when a downstream element failure causes the output current path for an otherwise functioning upstream element to be broken. This results in the loss of the upstream element as well. Analysis of the possibilities is complicated as the introduction of cross connections introduces many dependencies in the equations defining failure events and their resulting probabilities. This gives us an opportunity to consider other methods for analyzing the situation. One of these methods is to configure a model of the architecture in a Field Programmable Gate Array device and to perform repeated trials with random variable inputs to drive the gates of the synthesized model to evaluate the configuration. This approach which may be thought of as a type of "hardware assisted" Monte Carlo simulation will be discussed in another note.

Generalization of the Time Evolution Process:

A simplifying assumption that was made for all of the analysis above was that the probability of failure of detector elements is independent of time (and therefore, the state of the detector as a function of time). This approach can be generalized if we know how failure probabilities for individual detector elements may change as time passes and the distribution of power in the network changes as elements fail. The approach would be to modify the form of the matrix $\underline{A}$ in such cases. If that could be done, we would expect the form of the computation to become:

$$E[Z_{n+1}] = \underline{J}^T \, \underline{A(1)} \, \underline{A(2)} \, \underline{A(3)} \ldots \underline{A(n)} \, \underline{P_Z(1)}$$

where now, the matrices $\underline{A(k)}$ may all be different in general. If we can figure out a method to modify the matrix $\underline{A}$ (given failures in the network) that accounts for the effects of increased stress, we might be able to evaluate a topology for the dynamic case without having to resort to statistical simulation. It could also be profitable to identify sensitive parts of topologies and evaluate them for the impact of the loss at those parts.

Appendix: Some Properties of the Similarity Transformation of a Lower Triangular Matrix with Distinct, Nonzero Eigenvalues

Proposition: Let $\underline{A}$ be an MxM lower triangular matrix with distinct nonzero eigenvalues. In other words, the diagonal elements of $\underline{A}$ are such that $\alpha_{i,i}$ is not equal to $\alpha_{j,j}$ if i is not equal to j and $\alpha_{i,i}$ is not equal to 0 for all i. Then, there exists an MxM unit lower triangular matrix, $\underline{X}$, (that is, $\underline{X}$ is a lower triangular matrix with 1's at all diagonal elements) such that

$$\underline{A}\ \underline{X} = \underline{X}\ \underline{\Lambda}$$

where $\underline{\Lambda}$ is the diagonal matrix:

$$\underline{\Lambda} = \mathrm{diag}(\alpha_{1,1}\ ,\ \alpha_{2,2}\ ,\ \alpha_{3,3}\ ,\ .\ .\ .\ \alpha_{M,M}\ )$$

Proof: We follow the method of proof frequently used to establish the existence of the Schur decomposition of an arbitrary square matrix [Reference 4]. This method is based on induction on the dimension, M, of the matrix, $\underline{A}$.

Let M = 2 to establish the basis for induction. Then, we have:

$$\underline{A} = \begin{matrix} \alpha_{1,1} & 0 \\ \alpha_{2,1} & \alpha_{2,2} \end{matrix}$$

$$\underline{\Lambda} = \begin{matrix} \alpha_{1,1} & 0 \\ 0 & \alpha_{2,2} \end{matrix}$$

We seek a matrix, $\underline{X}$, such that

$$\underline{X} = \begin{matrix} \xi_{1,1} & 0 \\ \xi_{2,1} & \xi_{2,2} \end{matrix}$$

where

$$\xi_{1,1} = \xi_{2,2} = 1$$

and

$$\underline{A}\ \underline{X} = \underline{X}\ \underline{\Lambda}$$

By multiplying out the products represented by $\underline{A}\ \underline{X}$ and $\underline{X}\ \underline{\Lambda}$, and equating term by term, we have:

$\xi_{1,1}\ \alpha_{1,1} + \xi_{2,1}\ (0) = \xi_{1,1}\ \alpha_{1,1} + (0)\ 0$ (1, 1 elements)

$(0)\ \alpha_{1,1} + \xi_{2,2}\ (0) = \xi_{1,1}\ (0) + (0)\ \alpha_{2,2}$ (1, 2 elements)

$\xi_{1,1}\ \alpha_{2,1} + \alpha_{2,2}\ \xi_{2,1} = \xi_{2,1}\ \alpha_{1,1} + (0)\ \xi_{2,2}$ (2, 1 elements)

$(0)\ \alpha_{2,1} + \xi_{2,2}\ \alpha_{2,2} = \xi_{2,1}\ (0) + \xi_{2,2}\ \alpha_{2,2}$ (2, 2 elements)

From the equations for the 1, 1 elements and the 2, 2 elements, we are free to choose any nonzero values for $\xi_{1,1}$ and $\xi_{2,2}$. Therefore, we choose:

$$\xi_{1,1} = \xi_{2,2} = 1$$

Then, using these in the equation for the 2, 1 elements, we have:

$$\alpha_{2,1} + \alpha_{2,2}\ \xi_{2,1} = \xi_{2,1}\ \alpha_{1,1}$$

or

$$\xi_{2,1} = \alpha_{2,1}/(\ \alpha_{1,1} - \alpha_{2,2}\ )$$

Since $\alpha_{2,2}$ is not equal to $\alpha_{1,1}$ (by hypothesis), division in this formula is well defined. Therefore, since we have established the values of $\underline{X}$ consistent with the required formulas, we have proven that such a matrix exists for the case of M = 2.

Now, for the induction step, assume that the assertion is true for a value M = k-1. That is, assume that for every (k-1)x(k-1) matrix $\underline{A}$ which meets the requirements of the proposition, a (k-1)x(k-1) matrix, $\underline{X}$, which also meets the requirements always exists. Then, what does this imply for a kxk matrix, $\underline{A}$?

To examine this case, we partition the matrices $\underline{A}$ and $\underline{\Lambda}$ as:

$$\underline{A} = \begin{matrix} \underline{A}_0 & \underline{0} \\ \underline{a}^T & \alpha_{k,k} \end{matrix}$$

$$\underline{\Lambda} = \begin{matrix} \underline{\Lambda}_0 & \underline{0} \\ \underline{0} & \alpha_{k,k} \end{matrix}$$

Then, we seek a kxk unit lower triangular matrix, X such that

$\underline{A}\ \underline{X} = \underline{X}\ \underline{\Lambda}$

Let us partition $\underline{X}$ as:

$$\underline{X} = \begin{matrix} \underline{X}_0 & \underline{0} \\ \underline{x}^T & \xi_{k,k} \end{matrix}$$

In these partitions,

$\underline{A}_0$ is a (k-1)x(k-1) matrix
$\underline{\Lambda}_0$ is a (k-1)x(k-1) matrix
$\underline{X}_0$ is a (k-1)x(k-1) is a yet to be determined unit lower triangular matrix

$\underline{0}$ is a (k-1) column vector of zeroes
$\underline{a}^T$ is a (k-1) row vector
$\underline{x}^T$ is a (k-1) row vector which is yet to be determined

The value $\xi_{k,k}$ is also to be determined (in fact, we seek to set it to 1).

Then multiplying out the partitioned matrices on each side of $\underline{A}\ \underline{X} = \underline{X}\ \underline{\Lambda}$, we have the following requirements:

$\underline{A}_0\ \underline{X}_0 = \underline{X}_0\ \underline{\Lambda}_0$ (for the 1, 1 partition)

$\underline{A}_0\ \underline{0} + \xi_{k,k}\ \underline{0} = \underline{X}_0\ \underline{0} + \alpha_{k,k}\ \underline{0}$ (for the 1, 2 partition)

$\underline{a}^T\ \underline{X}_0 + \alpha_{k,k}\ \underline{x}^T = \underline{x}^T\ \underline{\Lambda}_0$ (for the 2, 1 partition)

$\underline{a}^T\ \underline{0} + \xi_{k,k}\ \alpha_{k,k} = \underline{x}^T\ \underline{0} + \alpha_{k,k}\ \xi_{k,k}$ (for the 2, 2 partition)

From the equations for the 1, 2 and 2, 2 partitions, we are free to choose

$\xi_{k,k} = 1$

The equation for the 1, 1 partition is valid by the hypothesis that has been made for systems of dimension (k-1) meeting the requirements. In other words, we assume as part

of the induction hypothesis that such a matrix $\underline{X}_0$ exists. The equation for the 1, 2 partition does not add any information. We are left to deal with the equation for the 2, 1 partition. Taking the transpose of this equation yields:

$$\underline{\Lambda}_0 \, \underline{x} = \underline{X}_0^T \, \underline{a} + \alpha_{k,k} \, \underline{x}$$

We may write this as:

$$(\underline{\Lambda}_0 - \alpha_{k,k} \, \underline{I}) \, \underline{x} = \underline{X}_0^T \, \underline{a}$$

where I is the (k-1)x(k-1) identity matrix.

The right hand side of this equation is just a constant vector. There will be a solution, $\underline{x}$, to this equation for any value of this vector, $\underline{X}_0^T \, \underline{a}$, if and only if the matrix

$$\underline{\Lambda}_0 - \alpha_{k,k} \, \underline{I}$$

is nonsingular. This is the case because none of the diagonal elements of $\underline{\Lambda}_0$ are equal to $\alpha_{k,k}$ since the diagonal elements of $\underline{A}$ have been assumed to be distinct. Therefore, this is a diagonal matrix with all nonzero diagonal elements and a solution does exist. Our proof by induction is complete.

The derivation just given suggests an algorithm for computing the transformation matrix, $\underline{X}$. This algorithm starts in the upper left corner of $\underline{A}$ and uses the equations on the 2x2 principal submatrix $\underline{A}_0$ to compute the 2x2 principal submatrix, $\underline{X}_0$. Computation of the 1x1 case is just to assign $\xi_{1,1} = 1$. A key step is the solution to the linear system of equations given by

$$(\underline{\Lambda}_0 - \alpha_{k,k} \, \underline{I}) \, \underline{x} = \underline{X}_0^T \, \underline{a}$$

However, the solution to this system is particularly simple given the diagonal nature of the matrix of coefficients, $(\underline{\Lambda}_0 - \alpha_{k,k} \, \underline{I})$. Then, the technique is expanded to the 3x3 principal submatrix, $\underline{A}_0$ to compute the 3x3 principal submatrix, $\underline{X}_0$ and so on. This special direct technique prevents us from having to use a general iterative technique for computing eigenvectors to formulate the transformation matrix, $\underline{X}$. Since the technique is direct, it has the benefit of avoiding questions of convergence. However, the efficiency of this direct approach should be compared with that of available methods (such as implemented in MATLAB).

References:

[1] “Probability and Random Processes”, Wilbur B. Davenport, Jr, McGraw-Hill, 1970, pp. 69-84.

[2] “Stochastic Processes”, Sheldon Ross, John Wiley and Sons, 1996, pp. 167-168.

[3] “Introduction to Matrix Computations”, G.W. Stewart, Academic Press, 1973, pp. 276-277.

[4] “Matrix Analysis and Applied Linear Algebra”, Carl D. Meyer, Society for Industrial and Applied Mathematics, 2000, pp. 508-509.

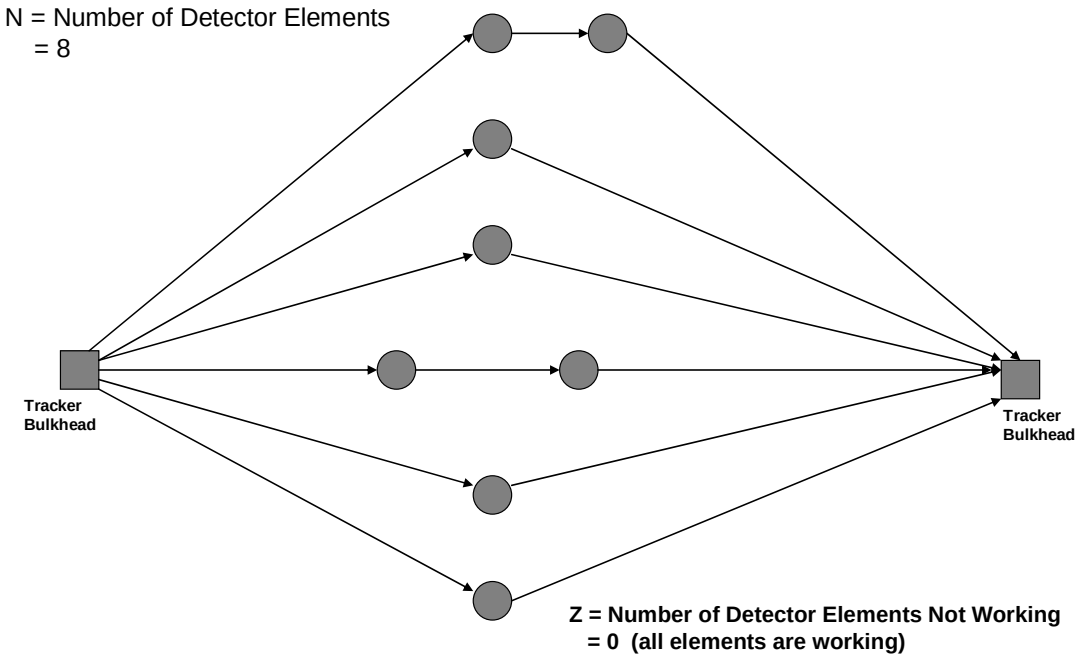

Figure 1:
A Distribution Network of Detector Elements

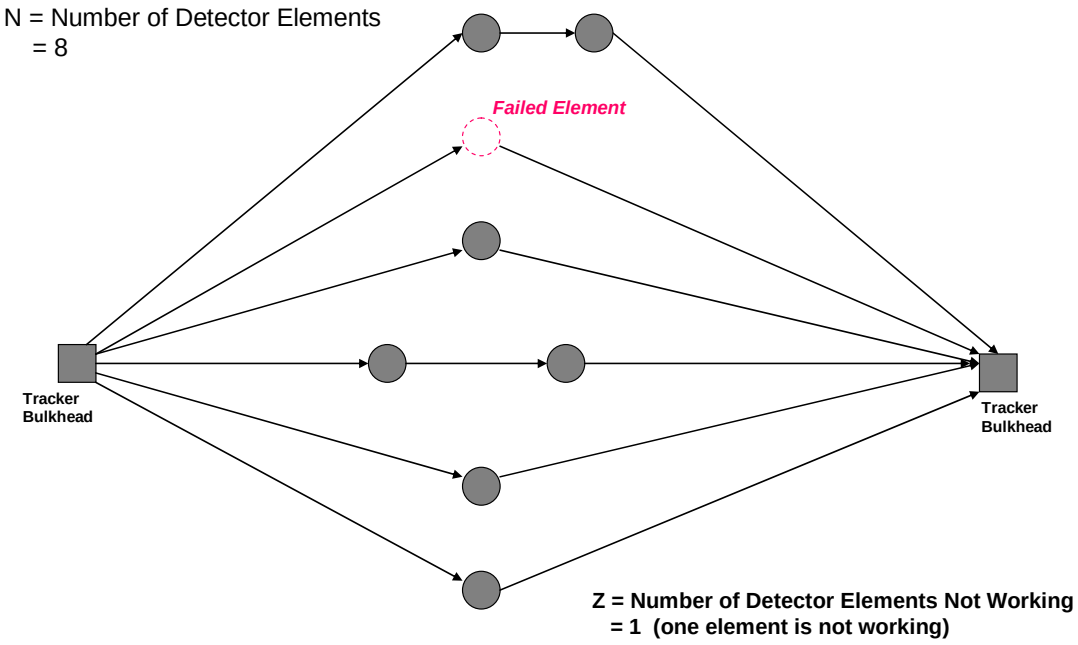

Figure 2:
Failure of an Isolated Detector Element

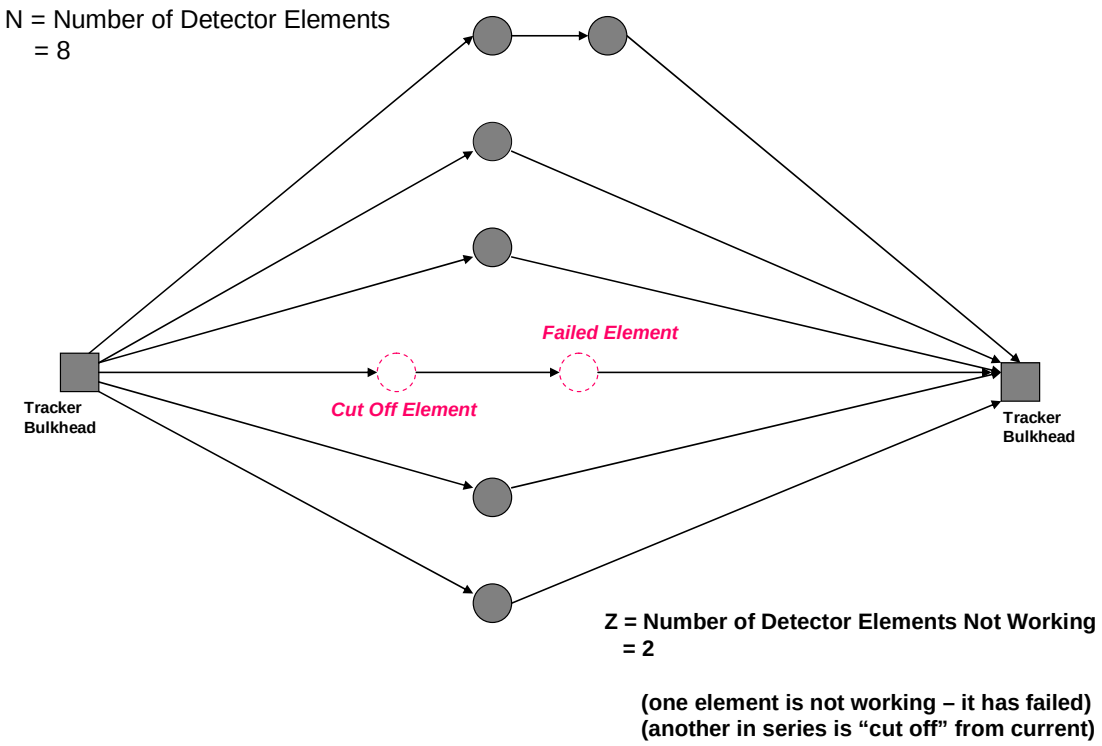

Figure 3:
Failure of Two Dependent Detector Elements

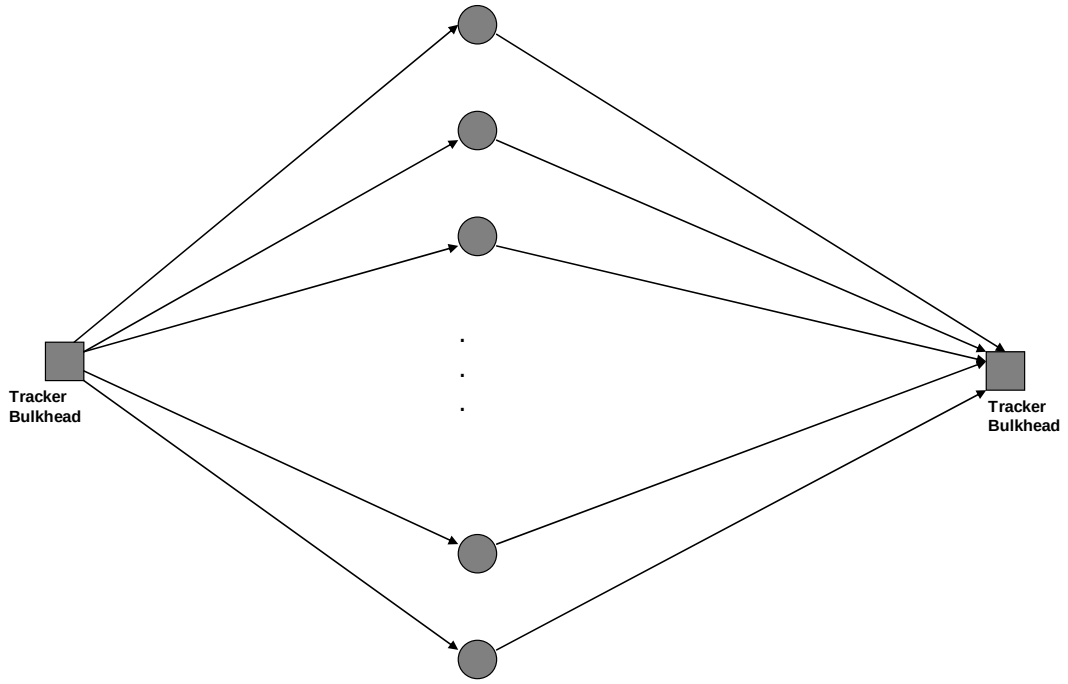

Figure 4:
A Parallel Network of Independent Detector Elements

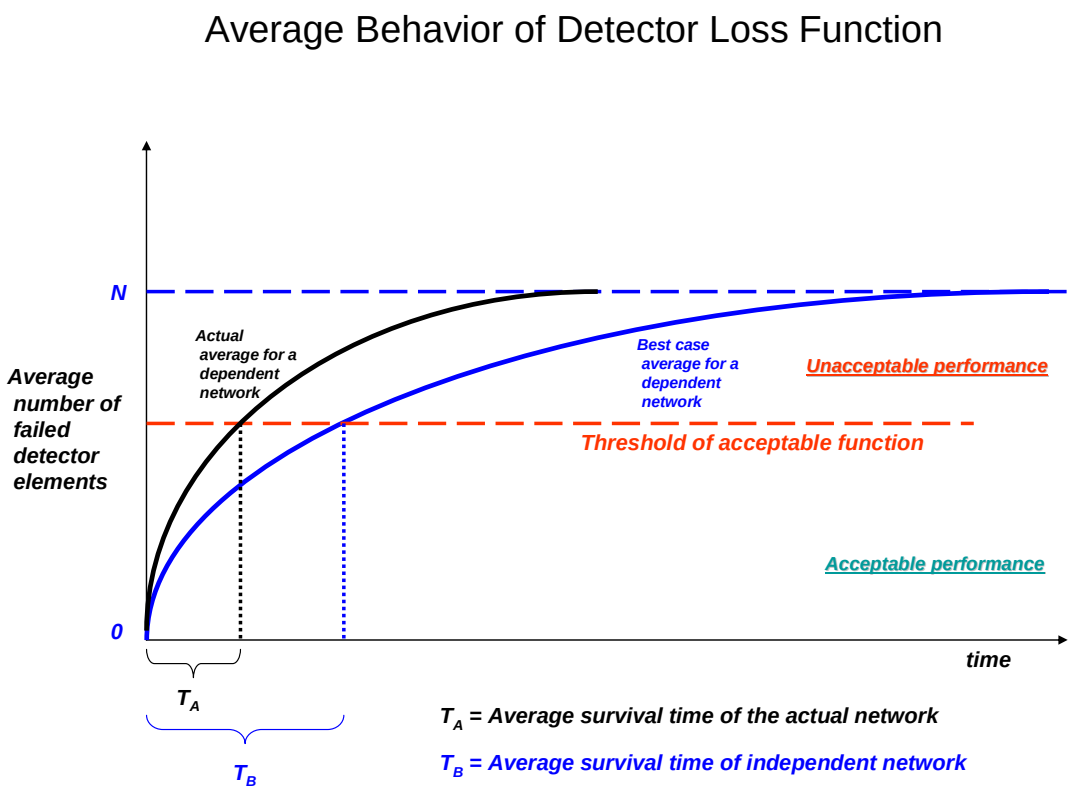

Figure 5:
Average Behavior of Loss of Elements in the Parallel Detector Architecture

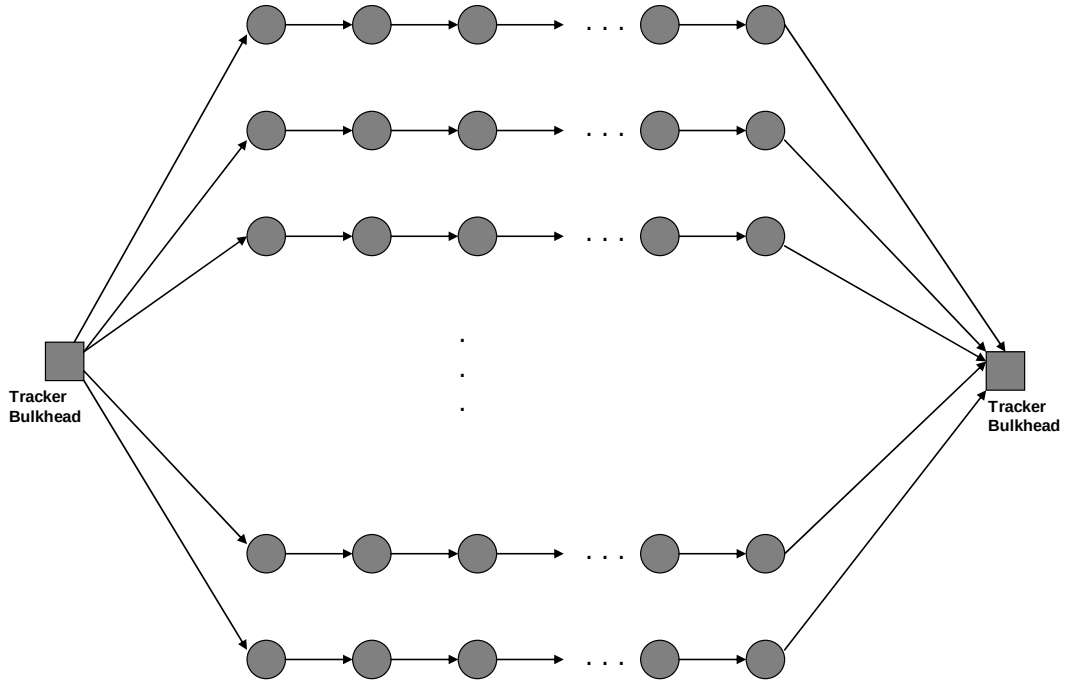

Figure 6:
A Parallel Network of Serially Connected Detector Elements

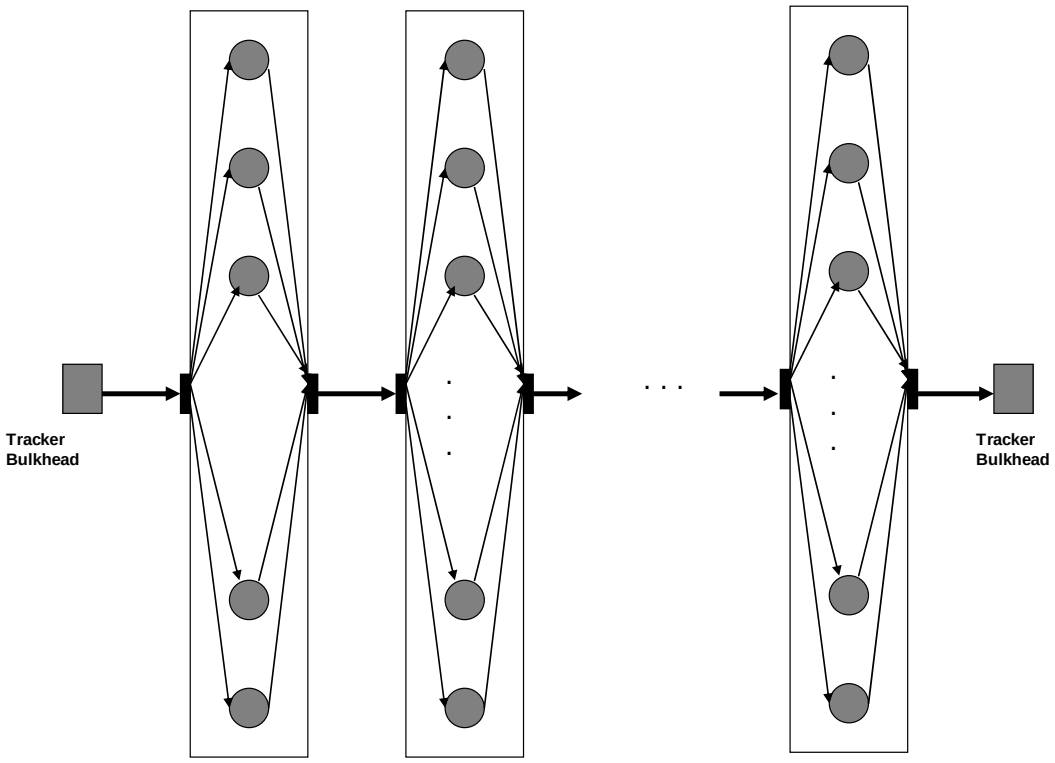

Figure 7:
A Serial Network of Parallel Connected Detector Elements

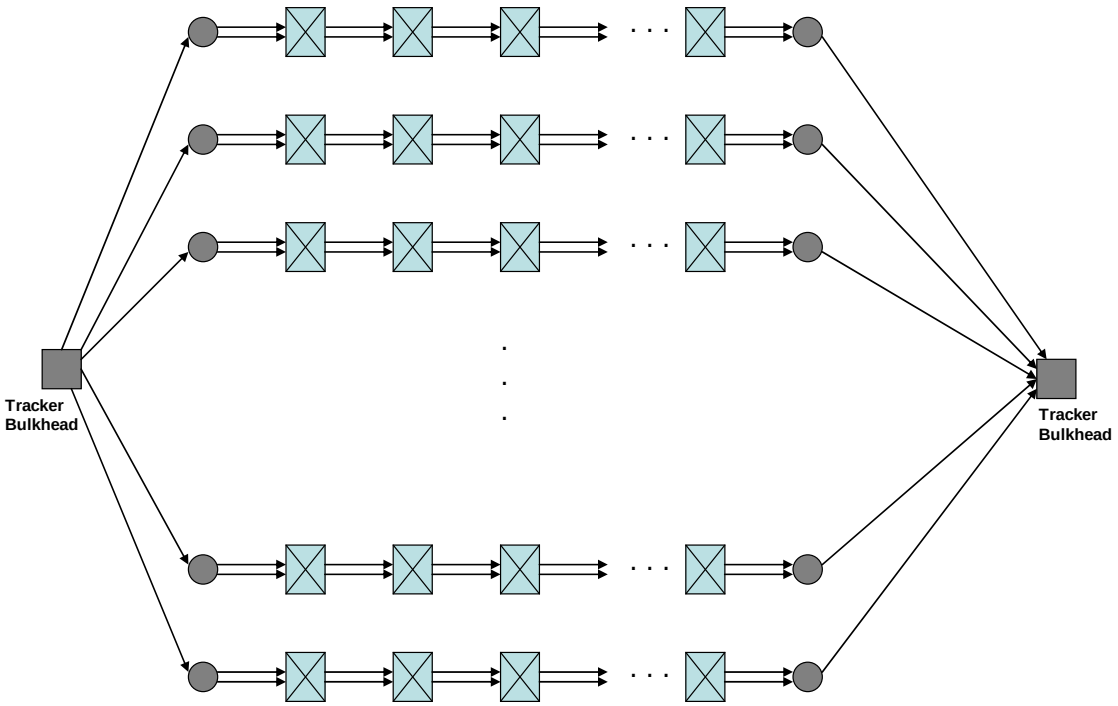

Figure 8:
A Parallel Network of Serially Connected Detector Elements
With
Cross Connections Between Serial Chains