

Full one-loop amplitudes from tree amplitudes

Walter T. Giele*

Fermilab, Batavia, IL 60510, USA

Zoltan Kunszt†

Institute for Theoretical Physics, ETH, CH-8093 Zürich, Switzerland

Kirill Melnikov‡

*Department of Physics and Astronomy, University of Hawaii,
2505 Correa Rd. Honolulu, HI 96822*

(Dated: January 16, 2008)

Abstract

We establish an efficient polynomial-complexity algorithm for one-loop calculations, based on generalized D -dimensional unitarity. It allows automated computations of both cut-constructible *and* rational parts of one-loop scattering amplitudes from on-shell tree amplitudes. We illustrate the method by (re)-computing all four-, five- and six-gluon scattering amplitudes in QCD at one-loop.

PACS numbers: 13.85.-t,13.85.Qk

*Electronic address: giele@fnal.gov

†Electronic address: kunszt@itp.phys.ethz.ch

‡Electronic address: kirill@phys.hawaii.edu

I. INTRODUCTION

Many complex processes that contain multi-jet final states will be observed at the Large Hadron Collider (LHC). Detailed studies of these processes will give us information about the mechanism of electroweak symmetry breaking and, hopefully, reveal physics beyond the Standard Model. To assess this information and interpret experimental data correctly, an accurate theoretical description of processes at the LHC is required. In principle, Monte-Carlo programs based on leading order (LO) computations do provide such a description [1, 2, 3, 4, 5]. However, as experience with the TEVATRON, LEP and HERA data has shown, LO predictions often give only rough estimates. To extract maximal information from data, more precise predictions based on next-to-leading order (NLO) calculations are required. NLO predictions are also instrumental for reliable estimates of theoretical uncertainties related to the truncation of the perturbative expansion.

For sufficiently complicated final states, computations required for the LHC are very challenging. In the standard approach one uses perturbative expansion of scattering amplitudes in terms of Feynman diagrams. This gives an algorithm suitable for numerical implementation. However, even for tree amplitudes, the number of Feynman diagrams grows faster than factorial with the number of external particles involved in a scattering process. As a consequence, computing time needed to evaluate a scattering amplitude at a single phase-space point, grows at least as fast. This means that computational algorithms based on expansion in Feynman diagrams are necessarily of exponential complexity, an undesirable feature.

In tree-level calculations exponential growth in complexity is avoided by employing recursion relations. These relations re-use recurring groups of off-shell Feynman graphs in an optimal manner [6, 7, 8, 9, 10]. The use of recursion relations for tree amplitudes leads to a computational algorithm of polynomial complexity so that computing time grows as some power of the number of external legs. Because of that, the problem of evaluating tree amplitudes is considered a solved problem by the high-energy physics community. As was pointed out in Ref. [11], a polynomial-complexity algorithm is not available for one-loop computations; constructing such an algorithm is the goal of the present paper.

When performing NLO computations in the Standard Model, many difficulties arise. We need to calculate both virtual one-loop corrections and real emission processes with one additional particle in the final state. Currently, the bottleneck in NLO calculations for multi-

particle processes is the computation of virtual corrections. The difficulty related to the factorial growth in the number of Feynman diagrams is further amplified by a large number of terms generated when tensor loop integrals are reduced to scalar integrals. Nevertheless, thanks to modern computational resources, standard methods based on Feynman diagrams and tensor integrals reduction [12] may be extended brute force to deal with multi-particle processes. Striking examples of the success of this approach are recent computations of electroweak corrections to $e^+e^- \rightarrow$ four fermions process [13] and one-loop six-gluon scattering amplitudes Ref. [14]. Note, however, that a single phase space point for six-gluon scattering is evaluated in about nine seconds, which is 10,000 times slower than the evaluation time for four-gluon scattering amplitudes generated using the same procedure. It is clear that further application of brute force approaches to yet higher multiplicity processes are becoming unfeasible.

Unitarity-based methods for multi-loop calculations were suggested as an alternative to the expansion in Feynman diagrams long ago. We review the status of these calculations in the next section. The goal of the present paper is to describe a polynomial-complexity computational algorithm for one-loop amplitudes that provides both cut-constructible *and* rational parts. We start with the idea of generalized unitarity in D -dimensions and develop it to an algorithm amenable to numerical implementation. Since one-loop amplitudes are built up from tree amplitudes, albeit in higher-dimensional space-time, the polynomial complexity of the algorithm is ensured. The method is flexible and can be applied to scattering amplitudes with arbitrary internal and external particles. In particular, dealing with massive particles is straightforward.

The outline of the paper is as follows. In section II we give an overview of the current status of unitarity techniques. The structure of one-loop scattering amplitudes in D -dimensions is discussed in section III. Section IV contains the discussion of the D -dimensional residues and algebraic extraction of the coefficients of master integrals. In section V we apply the formalism to gluon scattering amplitudes. Numerical results for four-, five- and six-gluon scattering amplitudes are reported in section VI. We conclude in section VII.

II. THE STATUS OF UNITARITY METHODS

Unitarity-based methods for loop calculations were suggested as an alternative to the Feynman-diagrammatic expansion long ago [15, 16, 17]. It was argued that for gauge theories these methods lead to higher computational efficiency than traditional methods [18, 19]. Within unitarity-based methods, computations employ tree scattering amplitudes, rather than Feynman diagrams, thereby avoiding many complications. Unitarity cuts factorize one-loop amplitudes into products of tree amplitudes. Therefore, in numerical implementations computing time grows with the number of unitarity cuts, rather than Feynman diagrams, and depends on the efficiency of algorithms employed for evaluating tree amplitudes.

The unitarity-based approach gives a complete description of one-loop scattering amplitudes if it is applied in D dimensions. Any dimensionally regulated multi-loop Feynman integral is fully reconstructible from unitarity cuts [17]. Clearly, in this case we have to associate D -dimensional momenta and polarization vectors with each cut on-shell line, to obtain one-loop amplitudes [20].

The first successful application of unitarity-based techniques for one-loop computations employed a four-dimensional variant of the unitarity-based approach, where four-dimensional states were associated with each cut line [21]. In analytic calculations, such a procedure has the advantage of allowing full use of the spinor-helicity formalism. In this way, however, we obtain only the so-called cut-constructible part of the full one-loop amplitude. The missing part is referred to as the rational part and has to be determined by other methods. In particular, in supersymmetric theories, rational parts are known to vanish. In other cases, rational parts can be fixed by using factorization properties of one-loop amplitudes in collinear limits [21].

An important step in developing unitarity-based methods was made in Ref. [22] where the idea of generalized unitarity was introduced. In particular, it was shown that coefficients of four-point scalar integrals for multi-gluon processes can be calculated using quadruple cuts. This approach is also suitable for direct numerical implementation. A quadruple cut factorizes an one-loop amplitude into a product of four tree amplitudes. From unitarity constraints, we derive two complex solutions for the loop-momentum. The product of tree amplitudes can be evaluated using these solutions. Coefficients of scalar four-point functions are obtained by taking averages.

Unfortunately, the simplicity of the above procedure does not generalize easily to the computation of full one-loop amplitudes. In that case, we also have to determine coefficients of one-, two- and three-point scalar integrals. For example, when a double cut is applied to determine coefficients of two-point functions, we have to account for the fact that parts of these contributions are already contained in quadruple and triple cuts. Analytic separation of these overlapping contributions proved to be complicated.

An efficient algebraic method to separate the overlapping contributions was outlined in Ref. [23]. Developing upon this idea, flexible unitarity-based computational techniques were suggested in Refs. [24, 25] where it was shown how generalized unitarity can be implemented numerically. Analytic techniques were further developed in Refs. [26, 27]. All these papers address primarily cut-constructible parts of scattering amplitudes.

These developments provide a polynomial-complexity computational algorithm for cut-constructible parts of scattering amplitudes. However, techniques for calculating rational parts are much less developed. Three methods for computing rational parts are currently available.

In Refs. [28, 29] it was suggested to determine rational parts of tensor one-loop integrals analytically and employ traditional diagrammatic methods to obtain scattering amplitudes. This leads to an exponential-, rather than polynomial-complexity algorithm, negating all the progress achieved with the determination of the cut-constructible part using numerical unitarity techniques.

The other two methods are so far analytic, but both should in principle be suitable for numerical implementation. The so-called bootstrap method sets up a recursive computation of rational parts [30, 31] similar to tree-level unitarity-based recursion relations [10]. However, in its current formulation the bootstrap approach is not directly suitable for numerical implementation since both the cut-constructible and rational parts contain spurious poles. When the two parts are added together, spurious poles cancel. Unitarity-based recursion relations for the rational part can only be constructed once spurious poles are removed from the rational part. The procedure to remove these poles from the rational part is called cut-completion; it requires analytic knowledge of the cut-constructible part. Clearly, for complicated multi-particle processes this is a serious disadvantage.

The other approach is a variant of D -dimensional unitarity method [32, 33]. Since the cut lines are D -dimensional, tree scattering amplitudes have to be calculated in D -dimensions.

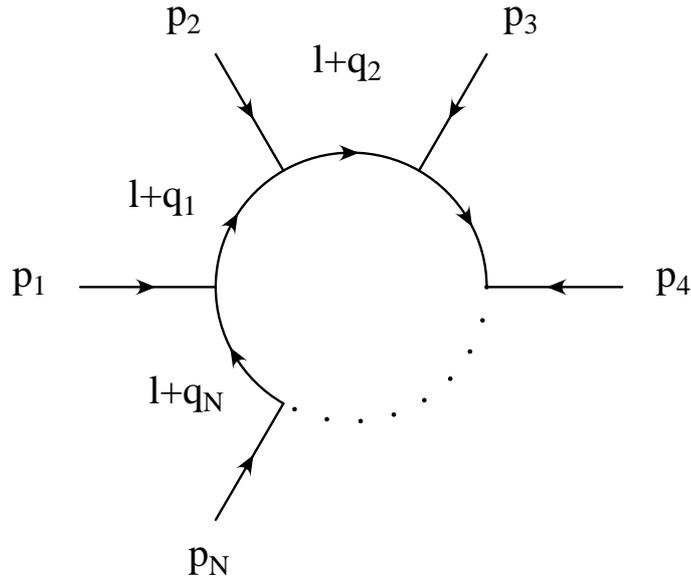


FIG. 1: The generic N -point loop amplitude.

Currently, this method is restricted to analytic applications for purely gluonic one-loop scattering amplitudes with a scalar particle propagating in the loop. In the following sections we describe a computational method based on D -dimensional unitarity that permits straightforward calculation of both cut-constructible and rational parts of arbitrary one-loop scattering amplitudes.

III. ONE-LOOP AMPLITUDES AND DIMENSIONALITY OF SPACE-TIME

Since one-loop calculations in quantum field theory lead to divergent expressions, we require regularization at intermediate stages of the calculation. Such regularization is accomplished by continuing momenta and polarization vectors of unobserved virtual particles to $D \neq 4$ dimensions [34]. On the other hand, it is convenient to keep momenta and polarization vectors of all external particles in four dimensions since this allows us to define one-loop scattering amplitudes through helicities of external particles. Once the dependence of a one-loop amplitude on the dimensionality of space-time is established, we interpolate to a non-integer number of dimensions $D = 4 - 2\epsilon$. The divergences of one-loop amplitudes are regularized by the parameter ϵ .

We wish to arrive at a numerical implementation of this procedure. To this end, it is

crucial to keep the number of dimensions in which virtual unobserved particles are allowed to propagate as integer since only in this case loop momenta and polarization sums of unobserved particles are fully defined. Therefore, we determine the dependence of one-loop amplitudes on the dimensionality of space-time treating the latter as integer and arrive at non-integer values (e.g. $D = 4 - 2\epsilon$) later on, by simple polynomial interpolation.

Any D -dimensional cyclic-ordered N -particle one-loop scattering amplitude (Fig. 1) can be written as

$$\mathcal{A}_{(D)}(\{p_i\}, \{J_i\}) = \int \frac{d^D l}{i(\pi)^{D/2}} \frac{\mathcal{N}(\{p_i\}, \{J_i\}; l)}{d_1 d_2 \cdots d_N}, \quad (1)$$

where $\{p_i\}$ and $\{J_i\}$ are the two sets that represent momenta and sources (polarization vectors, spinors, etc.) of external particles. The numerator structure $\mathcal{N}(\{p_i\}, \{J_i\}; l)$ depends on the particle content of the theory. The denominator is a product of inverse propagators

$$d_i = d_i(l) = (l + q_i)^2 - m_i^2 = \left(l - q_0 + \sum_{j=1}^i p_j \right)^2 - m_i^2, \quad (2)$$

where the four-vector q_0 represents the arbitrary parameterization choice of the loop momentum.

The one-loop amplitude can be written as a linear combination of master integrals

$$\begin{aligned} \mathcal{A}_{(D)} = & \sum_{[i_1|i_5]} e_{i_1 i_2 i_3 i_4 i_5} I_{i_1 i_2 i_3 i_4 i_5}^{(D)} + \sum_{[i_1|i_4]} d_{i_1 i_2 i_3 i_4} I_{i_1 i_2 i_3 i_4}^{(D)} \\ & + \sum_{[i_1|i_3]} c_{i_1 i_2 i_3} I_{i_1 i_2 i_3}^{(D)} + \sum_{[i_1|i_2]} b_{i_1 i_2} I_{i_1 i_2}^{(D)} + \sum_{[i_1|i_1]} a_{i_1} I_{i_1}^{(D)}, \end{aligned} \quad (3)$$

where we introduced the short-hand notation $[i_1|i_n] = 1 \leq i_1 < i_2 < \cdots < i_n \leq N$. The master integrals on the r.h.s. of Eq. (3) are defined as

$$I_{i_1 \cdots i_M}^{(D)} = \int \frac{d^D l}{i(\pi)^{D/2}} \frac{1}{d_{i_1} \cdots d_{i_M}}. \quad (4)$$

The coefficients of master integrals for this choice of basis depend on the number of dimensions D which, in practical calculations in dimensional regularization, needs to be taken as $D = 4 - 2\epsilon$. Since we aim at numerical implementation of D -dimensional unitarity, this is inconvenient. We explain below how to change the basis of master integrals to make coefficients D -independent.

The D -dependence of one-loop scattering amplitudes associated with virtual particles comes from two sources. When we continue loop momenta and polarization vectors to higher-

dimensional space time, the number of spin eigenstates changes. For example, massless spin-one particles in D_s dimensions have $D_s - 2$ spin eigenstates while spinors in D_s dimensions have $2^{(D_s-2)/2}$ spin eigenstates. In the latter case, D_s should be even.

The spin density matrix for a massless spin-one particle with momentum l and polarization vectors $e_\mu^{(i)}$ is given by

$$\sum_{i=1}^{D_s-2} e_\mu^{(i)}(l) e_\nu^{(i)}(l) = -g_{\mu\nu}^{(D_s)} + \frac{l_\mu b_\nu + b_\mu l_\nu}{l \cdot b}, \quad (5)$$

where b_μ is an arbitrary light-cone gauge vector associated with a particular choice of polarization vectors. Similarly, the spin density matrix for a fermion with momentum l and mass m is given by

$$\sum_{i=1}^{2^{(D_s-2)/2}} u^{(i)}(l) \bar{u}^{(i)}(l) = \not{l} + m = \sum_{\mu=1}^D l_\mu \gamma^\mu + m. \quad (6)$$

While, as we see from these examples, the number of spin eigenstates depends explicitly on the space-time dimensionality, the loop-momentum l itself has implicit D -dependence. We can define the loop momentum as a D -dimensional vector, with the requirement $D \leq D_s$ [35]. We now extend the notion of dimensional dependence of the one-loop scattering amplitude in Eq. (1) by taking the sources of all unobserved particles in D_s -dimensional space-time

$$\mathcal{A}_{(D,D_s)}(\{p_i\}, \{J_i\}) = \int \frac{d^D l}{i(\pi)^{D/2}} \frac{\mathcal{N}^{(D_s)}(\{p_i\}, \{J_i\}; l)}{d_1 d_2 \cdots d_N}. \quad (7)$$

The numerator function $\mathcal{N}^{(D_s)}(\{p_i\}, \{J_i\}; l)$ depends explicitly on D_s through the number of spin eigenstates of virtual particles. However, the dependence of the numerator function on the loop momentum dimensionality D emerges in a peculiar way. Since external particles are kept in four dimensions, the dependence of the numerator function on $D - 4$ components of the loop momentum l appears only through its dependence on l^2 . Specifically

$$l^2 = \bar{l}^2 - \tilde{l}^2 = l_1^2 - l_2^2 - l_3^2 - l_4^2 - \sum_{i=5}^D l_i^2, \quad (8)$$

where \bar{l} and \tilde{l} denote four- and $(D - 4)$ -dimensional components of the vector l . It is apparent from Eq. (8) that there is no preferred direction in the $(D - 4)$ -dimensional subspace of the D -dimensional loop momentum space.

A simple, but important observation is that in one-loop calculations, the dependence of scattering amplitudes on D_s is *linear*. This happens because, for such dependence to appear,

we need to have a closed loop of contracted metric tensors and/or Dirac matrices coming from vertices and propagators. Since only a single loop can appear in one-loop calculations, we find

$$\mathcal{N}^{(D_s)}(l) = \mathcal{N}_0(l) + (D_s - 4)\mathcal{N}_1(l). \quad (9)$$

We emphasize that there is no explicit dependence on either D_s or D in functions $\mathcal{N}_{0,1}$.

For numerical calculations we need to separate the two functions $\mathcal{N}_{0,1}$. To do so, we compute the left hand side of Eq. (9) for $D_s = D_1$ and $D_s = D_2$ and, after taking appropriate linear combinations, obtain

$$\begin{aligned} \mathcal{N}_0(l) &= \frac{(D_2 - 4)\mathcal{N}^{(D_1)}(l) - (D_1 - 4)\mathcal{N}^{(D_2)}(l)}{D_2 - D_1}, \\ \mathcal{N}_1(l) &= \frac{\mathcal{N}^{(D_1)}(l) - \mathcal{N}^{(D_2)}(l)}{D_2 - D_1}. \end{aligned} \quad (10)$$

Because both D_1 and D_2 are integers, amplitudes are numerically well-defined. We will comment more on possible choices of $D_{1,2}$ in the forthcoming sections; here suffice it to say that if fermions are present in the loop, we have to choose *even* D_1 and D_2 .

Having established the D_s -dependence of the amplitude, we discuss analytic continuation for sources of unobserved particles. We can interpolate D_s either to $D_s \rightarrow 4 - 2\epsilon$ (the t'Hooft-Veltman (HV) scheme) [34] or to $D_s \rightarrow 4$ (the four-dimensional helicity (FDH) scheme) [35]. The latter scheme is of particular interest in supersymmetric (SUSY) calculations since all SUSY Ward identities are preserved. We see from Eq. (9) that the difference between the two schemes is simply $-2\epsilon\mathcal{N}_1$.

We now substitute Eq.(10) into Eq. (7). Upon doing so, we obtain explicit expressions for one-loop amplitudes in HV and FDH schemes. We derive

$$\begin{aligned} \mathcal{A}^{\text{FDH}} &= \left(\frac{D_2 - 4}{D_2 - D_1} \right) \mathcal{A}_{(D, D_s=D_1)} - \left(\frac{D_1 - 4}{D_2 - D_1} \right) \mathcal{A}_{(D, D_s=D_2)}, \\ \mathcal{A}^{\text{HV}} &= \mathcal{A}^{\text{FDH}} - \left(\frac{2\epsilon}{D_2 - D_1} \right) (\mathcal{A}_{(D, D_s=D_1)} - \mathcal{A}_{(D, D_s=D_2)}). \end{aligned} \quad (11)$$

We emphasize that $D_s = D_{1,2}$ amplitudes on the r.h.s. of Eq. (11) are conventional one-loop scattering amplitudes whose numerator functions are computed in higher-dimensional space-time, i.e. all internal metric tensors and Dirac gamma matrices are in integer $D_s = D_{1,2}$ dimensions. The loop integration is in $D \leq D_s$ dimensions. It is important that explicit dependence on the regularization parameter $\epsilon = (4 - D)/2$ is not present in these amplitudes.

For this reason, Eq. (11) renders itself to straightforward numerical implementation. In particular, numerical implementation of D_s -dimensional unitarity cuts is now straightforward, as cut internal lines possess well-defined spin density matrices. The D_s -dimensional unitarity cuts, that we discuss in detail in the next section, decompose amplitudes into a linear combination of master integrals, see Eq. (3). We can choose the basis of master integrals in such a way that no explicit D -dependence in the coefficients appears. Only after the reduction to master integrals is established, we continue the space-time dimension associated with the loop momentum to $D \rightarrow 4 - 2\epsilon$, thereby regularizing the master integrals.

IV. D_s -DIMENSIONAL UNITARITY CUTS

The amplitudes on the r.h.s. of Eq. (11) are most efficiently calculated by using generalized unitarity applied in D_s dimensions. Since many aspects of the calculation in this case are the same as in four-dimensional generalized unitarity, we focus on new features that appear for $D_s > 4$. Our discussion follows Ref. [24] which details the $D_s = 4$ case.

The important issue is the set of master integrals that we have to deal with when applying D_s -dimensional unitarity cuts. Cutting a propagator i requires finding the loop momentum l such that equation $d_i(l) = 0$ is satisfied. Therefore, each cut imposes one constraint on the loop momentum. In four dimensions, where loop momentum has only four components, one can cut at most four propagators without over-constraining the system of equations. This leads to the conclusion that four-point integrand functions in four-dimensions are required whereas five-point and higher-point functions aren't.

When we compute an amplitude in $D_s > 4$ dimensions, we may also use the additional $D - 4$ components of the loop momenta to set more inverse propagators to zero. However, since all external momenta are four-dimensional, additional components of the loop momentum enter all propagators in a particular combination

$$s_e^2 = - \sum_{i=5}^D (l \cdot n_i)^2 = - \sum_{i=5}^D (\tilde{l} \cdot n_i)^2, \quad (12)$$

where $n_{i>4}$ are orthonormal basis vectors ($n_i \cdot n_j = \delta_{ij}$) of the $(D - 4)$ -dimensional sub-space embedded in a D -dimensional space. Therefore, when we move from four to D_s dimensions, at most *five* inverse propagators can be set to zero. Therefore, for $D_s > 4$, the highest-point master integral that should be included into the master integrals basis is a five-point

function. Hence the integrand of the N -particle amplitude in Eq. (1) can be parameterized as

$$\begin{aligned} \frac{\mathcal{N}^{(D_s)}(l)}{d_1 d_2 \cdots d_N} &= \sum_{[i_1|i_5]} \frac{\bar{e}_{i_1 i_2 i_3 i_4 i_5}^{(D_s)}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\bar{d}_{i_1 i_2 i_3 i_4}^{(D_s)}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} \\ &+ \sum_{[i_1|i_3]} \frac{\bar{c}_{i_1 i_2 i_3}^{(D_s)}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\bar{b}_{i_1 i_2}^{(D_s)}(l)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\bar{a}_{i_1}^{(D_s)}(l)}{d_{i_1}}. \end{aligned} \quad (13)$$

where the dependence on the external momenta and sources are suppressed. From four-dimensional unitarity we know that computation of each cut of the scattering amplitude is simplified if convenient parameterization of the residue is chosen. We now discuss how these parameterizations change when D_s -dimensional unitarity cuts are considered.

A. Pentuple residue

To calculate the pentuple residue, we choose momentum l such that five inverse propagators in Eq. (13) vanish. We define

$$\bar{e}_{ijkmn}^{(D_s)}(l_{ijkmn}) = \text{Res}_{ijkmn} \left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1 \cdots d_N} \right). \quad (14)$$

The momentum l_{ijkmn} satisfies the following set of equations $d_i(l_{ijkmn}) = \cdots = d_n(l_{ijkmn}) = 0$. The solution is given by

$$l_{ijkmn}^\mu = V_5^\mu + \sqrt{\frac{-V_5^2 + m_n^2}{\alpha_5^2 + \cdots + \alpha_D^2}} \left(\sum_{h=5}^D \alpha_h n_h^\mu \right), \quad (15)$$

where m_n is the mass in the propagator d_n which is chosen to be as $d_n = l^2 - m_n^2$ by adjusting the reference vector q_0 . The parameters α_h can be chosen freely. The four-dimensional vector V_5^μ depends only on external momenta and propagator masses. It is explicitly constructed using the Vermaseren-van Neerven basis as outlined in Ref. [24]. The $D - 4$ components of the vector l_{ijkmn} are necessarily non-vanishing; for simplicity we may choose l_{ijkmn} to be five-dimensional, independent of D_s . We will see below that this is sufficient to determine pentuple residue.

To restrict the functional form of the pentuple residue $\bar{e}_{ijkmn}(l)$ we apply the same reasoning as in four-dimensional unitarity case, supplemented with the requirement that $\bar{e}_{ijkmn}^{(D_s)}(l)$

depends only on even powers of s_e ; this requirement is a necessary consequence of the discussion around Eq. (8). These considerations lead to the conclusion that the pentuple residue is independent of the loop momentum

$$\bar{e}_{ijkmn}^{(D_s)}(l) = e_{ijkmn}^{(D_s, (0))}. \quad (16)$$

To calculate $e^{(0)}$ in the FDH scheme, we employ Eq. (10) and obtain

$$e_{ijkmn}^{(0), \text{FDH}} = \left(\frac{D_2 - 4}{D_2 - D_1} \right) \text{Res}_{ijkmn} \left(\frac{\mathcal{N}^{(D_1)}(l)}{d_1 \cdots d_N} \right) - \left(\frac{D_1 - 4}{D_2 - D_1} \right) \text{Res}_{ijkmn} \left(\frac{\mathcal{N}^{(D_2)}(l)}{d_1 \cdots d_N} \right). \quad (17)$$

The calculation of the residues of the amplitude on the r.h.s. of Eq. (17), is simplified by their factorization into products of tree amplitudes

$$\begin{aligned} \text{Res}_{ijkmn} \left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1 \cdots d_N} \right) &= \sum \mathcal{M}(l_i; p_{i+1}, \dots, p_j, -l_j) \times \mathcal{M}(l_j; p_{j+1}, \dots, p_k, -l_k) \\ &\times \mathcal{M}(l_k; p_{k+1}, \dots, p_m, -l_m) \times \mathcal{M}(l_m; p_{m+1}, \dots, p_n, -l_n) \times \mathcal{M}(l_n; p_{n+1}, \dots, p_i, -l_i). \end{aligned} \quad (18)$$

Here, the summation is over all different quantum numbers of the cut lines. In particular, we have to sum over polarization vectors of the cut lines. This generates explicit D_s dependence of the residue, as described in the previous section. Note that the complex momenta $l_h^\mu = l^\mu + q_h^\mu$ are on-shell due to the unitarity constraint $d_h = 0$.

B. Quadrupole residue

The construction of the quadrupole residue follows the discussion of the previous subsection and generalizes the four-dimensional case studied in [23, 24]. We define

$$\bar{d}_{ijkn}^{(D_s)}(l) = \text{Res}_{ijkn} \left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1 \cdots d_N} - \sum_{[i_1]i_5} \frac{e_{i_1 i_2 i_3 i_4 i_5}^{(D_s, (0))}}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} \right), \quad (19)$$

where the last term in the r.h.s. is the necessary subtraction of the pentuple cut contribution. We now specialize to the FDH scheme. In this case, the most general parameterization of the quadrupole cut is given by

$$\bar{d}_{ijkn}^{\text{FDH}}(l) = d_{ijkn}^{(0)} + d_{ijkn}^{(1)} s_1 + (d_{ijkn}^{(2)} + d_{ijkn}^{(3)} s_1) s_e^2 + d_{ijkn}^{(4)} s_e^4, \quad (20)$$

where $s_1 = l \cdot n_1$. We used the fact that, in renormalizable quantum field theories, the highest rank of a tensor integral that may contribute to a quadrupole residue is four and

that only even powers of s_e can appear on the r.h.s of Eq. (20). The solution of the unitarity constraint is given by

$$l_{ijkn}^\mu = V_4^\mu + \sqrt{\frac{-V_4^2 + m_n^2}{\alpha_1^2 + \alpha_5^2 + \dots + \alpha_D^2}} \left(\alpha_1 n_1^\mu + \sum_{h=5}^D \alpha_h n_h^\mu \right). \quad (21)$$

The vector V_4 is defined in the space spanned by the three independent inflow momenta $\{k_i, k_j, k_k\}$ and n_1 is the unit vector that describes the one-dimensional ‘‘transverse space’’, i.e. $k_i \cdot n_1 = 0$, $k_j \cdot n_1 = 0$, $k_k \cdot n_1 = 0$, $n_i \cdot n_j = \delta_{ij}$ [24].

To determine the momentum-independent coefficients $d_{ijkn}^{(0,1,\dots,4)}$ in Eq. (20), we compute the quadrupole residue of the one-loop scattering amplitude for different values of the loop momentum l that satisfies the unitarity constraint. This entails choosing different values for parameters α_i in Eq. (21). As a first step, we may choose l to be a four-dimensional vector embedded in D_s -dimensional space ($\alpha_{i \geq 5} = 0$). Then $s_e = 0$ and the parameterization of the residue in Eq. (20) becomes identical to a four-dimensional case. Standard manipulations described in Refs. [23, 24] then allow us to find $d_{ijkn}^{(0)}$ and $d_{ijkn}^{(1)}$. To determine the remaining coefficients, we consider loop momenta in dimensions $D > 4$ such that $d_i = d_j = d_k = d_m = 0$ but $s_e \neq 0$. We accomplish this by adjusting the value of α_5 . By choosing appropriate loop momenta, we determine all the remaining three coefficients of the quadrupole residue by solving a linear system of equations.

C. Triple , double and single-line cuts

Calculation of triple, double and single cuts proceeds in full analogy with what has been described for pentuple and quadrupole cuts. The only modification concerns parameterization of residues.

The general parameterization of a triple cut in the FDH scheme is given by

$$\begin{aligned} \bar{c}_{ijk}^{\text{FDH}}(l) = & c_{ijk}^{(0)} + c_{ijk}^{(1)} s_1 + c_{ijk}^{(2)} s_2 + c_{ijk}^{(3)} (s_1^2 - s_2^2) + s_1 s_2 (c_{ijk}^{(4)} + c_{ijk}^{(5)} s_1 + c_{ijk}^{(6)} s_2) \\ & + c_{ijk}^{(7)} s_1 s_e^2 + c_{ijk}^{(8)} s_2 s_e^2 + c_{ijk}^{(9)} s_e^2, \end{aligned} \quad (22)$$

where $s_1 = l \cdot n_1$ and $s_2 = l \cdot n_2$. The solution of the unitarity constraint is given by

$$l_{ijk}^\mu = V_3^\mu + \sqrt{\frac{-V_3^2 + m_k^2}{\alpha_1^2 + \alpha_2^2 + \alpha_5^2 + \dots + \alpha_D^2}} \left(\alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \sum_{h=5}^D \alpha_h n_h^\mu \right). \quad (23)$$

The vector V_3 is defined in the space spanned by the two independent inflow momenta $\{k_i, k_j\}$ and $n_{1,2}$ are orthonormal vectors that describe the two-dimensional “transverse space”, i.e. $k_i \cdot n_{1,2} = 0$, $k_j \cdot n_{1,2} = 0$, $n_i \cdot n_j = \delta_{ij}$ [24].

The general parameterization of a double cut is given by

$$\begin{aligned} \overline{b}_{ij}^{\text{FDH}}(l) = & b_{ij}^{(0)} + b_{ij}^{(1)} s_1 + b_{ij}^{(2)} s_2 + b_{ij}^{(3)} s_3 + b_{ij}^{(4)} (s_1^2 - s_3^2) + b_{ij}^{(5)} (s_2^2 - s_3^2) + b_{ij}^{(8)} s_2 s_3 \\ & + b_{ij}^{(6)} s_1 s_2 + b_{ij}^{(7)} s_1 s_3 + b_{ij}^{(9)} s_e^2, \end{aligned} \quad (24)$$

where $s_1 = l \cdot n_1$, $s_2 = l \cdot n_2$ and $s_3 = l \cdot n_3$. The solution of the unitarity constraint is given by

$$l_{ij}^\mu = V_2^\mu + \sqrt{\frac{-V_2^2 + m_j^2}{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_5^2 + \dots + \alpha_D^2}} \left(\alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \alpha_3 n_3^\mu + \sum_{h=5}^D \alpha_h n_h^\mu \right). \quad (25)$$

The vector V_2 is proportional to the inflow momentum $\{k_i\}$ and $n_{1,2,3}$ are orthonormal vectors that describe the three-dimensional “transverse space”, i.e. $k_i \cdot n_{1,2,3} = 0$, $n_i \cdot n_j = \delta_{ij}$ [24]. Finally, we note that the parameterization of a single-line residue is the same as in the four-dimensional case described in Ref. [24].

D. One-loop amplitudes and master integrals

Following the strategy outlined in previous subsections, we obtain pentuple, quadrupole, triple, double and single-line residues. These residues give us coefficients of master integrals through which the one-loop amplitude is expressed.

Because we have more coefficients in our parameterization of residues, than in the four-dimensional case, we end up with a larger number of master integrals. The new master integrals include the five-point function and various s_e -dependent terms that appear in quadrupole, triple and double residues. We will comment on the fate of the five-point function shortly.

Consider now the s_e -dependent master integrals. Some of them contain scalar products $l \cdot n_{1,2}$ and vanish upon angular integration over l . Neglecting these spurious terms, we have to deal with four additional master integrals. We can rewrite those integrals through conventional four-, three- and two-point functions in higher-dimensional space-time. We

find

$$\begin{aligned}
\int \frac{d^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} &= -\frac{D-4}{2} I_{i_1 i_2 i_3 i_4}^{D+2}, \\
\int \frac{d^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} &= \frac{(D-2)(D-4)}{4} I_{i_1 i_2 i_3 i_4}^{D+4}, \\
\int \frac{d^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} &= -\frac{(D-4)}{2} I_{i_1 i_2 i_3}^{D+2}, \\
\int \frac{d^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} &= -\frac{(D-4)}{2} I_{i_1 i_2}^{D+2}.
\end{aligned} \tag{26}$$

Using Eq. (26), we arrive at the following representation of the scattering amplitude

$$\begin{aligned}
\mathcal{A}_{(D)} &= \sum_{[i_1|i_5]} e_{i_1 i_2 i_3 i_4 i_5}^{(0)} I_{i_1 i_2 i_3 i_4 i_5}^{(D)} \\
&+ \sum_{[i_1|i_4]} \left(d_{i_1 i_2 i_3 i_4}^{(0)} I_{i_1 i_2 i_3 i_4}^{(D)} - \frac{D-4}{2} d_{i_1 i_2 i_3 i_4}^{(2)} I_{i_1 i_2 i_3 i_4}^{(D+2)} + \frac{(D-4)(D-2)}{4} d_{i_1 i_2 i_3 i_4}^{(4)} I_{i_1 i_2 i_3 i_4}^{(D+4)} \right) \\
&+ \sum_{[i_1|i_3]} \left(c_{i_1 i_2 i_3}^{(0)} I_{i_1 i_2 i_3}^{(D)} - \frac{D-4}{2} c_{i_1 i_2 i_3}^{(9)} I_{i_1 i_2 i_3}^{(D+2)} \right) \\
&+ \sum_{[i_1|i_2]} \left(b_{i_1 i_2}^{(0)} I_{i_1 i_2}^{(D)} - \frac{D-4}{2} b_{i_1 i_2}^{(9)} I_{i_1 i_2}^{(D+2)} \right) + \sum_{i_1=1}^N a_{i_1}^{(0)} I_{i_1}^{(D)}.
\end{aligned} \tag{27}$$

We emphasize that the explicit D -dependence on the r.h.s. of Eq. (27) is the consequence of our choice of the basis for master integrals in Eq. (26).

We note that the above decomposition is valid for any value of D . We can now interpolate the loop integration dimension D to $D \rightarrow 4 - 2\epsilon$. The extended basis of master integrals that we employ provides a clear separation between cut-constructible and rational parts of the amplitude. The cut-constructible part is given by the integrals in D -dimensions in Eq. (27), while the rational part is given by the integrals in $D+2$ and $D+4$ dimensions. However, it is possible to use smaller basis of master integrals by rewriting integrals $\{I_{i_1 i_2 i_3 i_4}^{(D+4)}, I_{i_1 i_2 i_3 i_4}^{(D+2)}, I_{i_1 i_2 i_3}^{(D+2)}, I_{i_1 i_2}^{(D+2)}\}$ in terms of $\{I_{i_1 i_2 i_3 i_4}^{(D)}, I_{i_1 i_2 i_3}^{(D)}, I_{i_1 i_2}^{(D)}\}$ using the integration-by-parts techniques.

Since we are interested in NLO computations, we only need to consider the limit $\epsilon \rightarrow 0$ in Eq. (27) and neglect contributions of order ϵ . This leads to certain simplifications. First, in this limit, we can re-write the scalar 5-point master integral as a linear combination of four-point master integrals up to $\mathcal{O}(\epsilon)$ terms. If we employ this fact in Eq. (27), we obtain

$$\lim_{D \rightarrow 4-2\epsilon} \left(\sum_{[i_1|i_5]} e_{i_1 \dots i_5}^{(0)} I_{i_1 \dots i_5}^{(D)} + \sum_{[i_1|i_4]} d_{i_1 \dots i_4}^{(0)} I_{i_1 \dots i_4}^{(D)} \right) = \sum_{[i_1|i_4]} \tilde{d}_{i_1 \dots i_4}^{(0)} I_{i_1 \dots i_4}^{(4-2\epsilon)} + \mathcal{O}(\epsilon). \tag{28}$$

Note that since *scalar* five-point function is cut-constructible, it does not contribute to the rational part. Second, since $\lim_{D \rightarrow 4} (D - 4) \times I_{i_1 i_2 i_3 i_4}^{(D+2)} = 0$, the 6-dimensional 4-point integral can be neglected. Finally, the 6-dimensional 2-point, 6-dimensional 3-point and 8-dimensional 4-point integrals are all ultraviolet divergent and produce contributions of order ϵ^{-1} . Therefore, factors $(D - 4)$ that multiply these integrals in Eq. (27) pick up divergent terms and produce ϵ -independent, finite contributions

$$\begin{aligned} \lim_{D \rightarrow 4} \frac{(D - 4)(D - 2)}{4} I_{i_1 i_2 i_3 i_4}^{(D+4)} &= -\frac{1}{3}, \\ \lim_{D \rightarrow 4} \frac{(D - 4)}{2} I_{i_1 i_2 i_3}^{(D+2)} &= \frac{1}{2}, \\ \lim_{D \rightarrow 4} \frac{(D - 4)}{2} I_{i_1 i_2}^{(D+2)} &= -\frac{m_{i_1}^2 + m_{i_2}^2}{2} + \frac{1}{6} (q_{i_1} - q_{i_2})^2. \end{aligned} \quad (29)$$

Combining everything together, we can write any one-loop amplitude up to terms of order ϵ as a linear combination of cut-constructible and rational parts.

$$\mathcal{A}_N = \mathcal{A}_N^{CC} + R_N. \quad (30)$$

The expression for the cut-constructible part reads

$$\mathcal{A}_N^{CC} = \sum_{[i_1|i_4]} \tilde{d}_{i_1 i_2 i_3 i_4}^{(0)} I_{i_1 i_2 i_3 i_4}^{(4-2\epsilon)} + \sum_{[i_1|i_3]} c_{i_1 i_2 i_3}^{(0)} I_{i_1 i_2 i_3}^{(4-2\epsilon)} + \sum_{[i_1|i_2]} b_{i_1 i_2}^{(0)} I_{i_1 i_2}^{(4-2\epsilon)} + \sum_{i_1=1}^N a_{i_1}^{(0)} I_{i_1}^{(4-2\epsilon)}, \quad (31)$$

where coefficients $\tilde{d}_{i_1 \dots i_4}^{(0)}$ are implicitly defined in Eq. (28). For the rational part, we obtain

$$R_N = - \sum_{[i_1|i_4]} \frac{d_{i_1 i_2 i_3 i_4}^{(4)}}{3} - \sum_{[i_1|i_3]} \frac{c_{i_1 i_2 i_3}^{(9)}}{2} - \sum_{[i_1|i_2]} \left(\frac{(q_{i_1} - q_{i_2})^2}{6} - \frac{m_{i_1}^2 + m_{i_2}^2}{2} \right) b_{i_1 i_2}^{(9)}. \quad (32)$$

Finally, we point out that various master integrals required for one-loop computations can be found in Ref. [36].

V. GLUON SCATTERING AMPLITUDES IN QCD

We applied the method described in the previous sections to compute gluon scattering amplitudes in QCD. It is well-known that these amplitudes can be represented as linear combinations of simpler objects, called color-ordered sub-amplitudes. For example, the tree amplitude for n -gluon scattering reads [37, 38]

$$\mathcal{A}_n^{\text{tree}}(\{p_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in \mathcal{S}_n / \mathbb{Z}_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) \mathcal{A}_n^{\text{tree}}\left(p_{\sigma(1)}^{\lambda_{\sigma(1)}}, \dots, p_{\sigma(n)}^{\lambda_{\sigma(n)}}\right), \quad (33)$$

where p_i, λ_i, a_i stand for momenta, helicities and color indices of external gluons, g is the coupling constant and T^a are generators of the $SU(N_c)$ color algebra normalized as $\text{Tr}(T^a T^b) = \delta^{ab}$. The sum in Eq. (33) runs over $(n-1)!$ non-cyclic permutations of the set $\{1, \dots, n\}$. Amplitudes $\mathcal{A}_n^{\text{tree}}(p_{\sigma(1)}^{\lambda_{\sigma(1)}}, \dots, p_{\sigma(n)}^{\lambda_{\sigma(n)}})$ are color-ordered tree sub-amplitudes.

One-loop amplitudes can be decomposed in a similar, but slightly more complicated fashion [39]. Considering one-loop amplitudes for n -gluon scattering, where all internal particles are also gluons, we write

$$\mathcal{A}_n(\{p_i, \lambda_i, a_i\}) = g^n \sum_{c=1}^{[n/2]+1} \sum_{\sigma \in S_n/S_{n,c}} \text{Gr}_{n;c}(\sigma) \mathcal{A}_{n,c}(\sigma), \quad (34)$$

where

$$\text{Gr}_{n;1}(\sigma) = N_c \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}), \quad (35)$$

and

$$\text{Gr}_{n;c}(\sigma) = \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(c-1)}}) \text{Tr}(T^{a_{\sigma(c)}} \dots T^{a_{\sigma(n)}}). \quad (36)$$

In Eq. (34) $[x]$ is the largest integer number smaller than or equal to x and $S_{n,c}$ are subsets of S_n that leave the double trace in Eq. (36) invariant.

Since sub-amplitudes possess a number of symmetries under cyclic permutations of external particles and parity, not all $\mathcal{A}_{n,1}$ amplitudes are independent. Moreover, all $\mathcal{A}_{n,c>1}$ amplitudes can be written as linear combinations of $\mathcal{A}_{n,1}$ amplitudes. Because of that, there are four independent helicity sub-amplitudes for four- and five-gluon scattering and eight independent helicity sub-amplitudes for six-gluon scattering. Scattering amplitudes for four, five and six gluons are available in the literature; this allows us to check the validity of our method.

The method described in the previous section is amenable to straightforward numerical implementation. We construct a list of all possible cuts of a given sub-amplitude. Those cuts that correspond to two-point functions for light-like incoming momenta are discarded since corresponding master integrals vanish in dimensional regularization.

Each cut in the list is computed as a product of tree amplitudes which are obtained using Berends-Giele recurrence relations [6]. The recurrence relations themselves do not change when we construct gluon scattering amplitudes in higher-dimensional space-time. However, polarization vectors for cut gluon lines have to be extended. To discuss this extension explicitly, we now choose specific values for space-time dimensionalities $D_{1,2}$ in Eq. (11).

Since we deal with pure gluonic amplitudes, we may consider $D_1 = 5$ and $D_2 = 6$. We obtain

$$\mathcal{A}^{\text{FDH}} = 2\mathcal{A}_{(D,D_s=5)} - \mathcal{A}_{(D,D_s=6)}. \quad (37)$$

Computation of residues discussed in the previous section requires $s_e \neq 0$. This can be easily accomplished by allowing the projection of the loop momentum on the fifth direction to be non-vanishing, while always keeping $l \cdot n_6 = 0$, even for $D_s = 6$. Hence, for computing residues of gluon amplitudes we have to consider a few cases of how four- and five-dimensional loop momentum can be embedded into five- and six-dimensional space-time. For the sake of clarity, we describe those cases separately.

For $D_s = 5$ and four-dimensional loop momentum, we have $l \cdot n_5 = 0$. This allows us to write

$$\sum_{i=1}^3 e_i^\mu e_i^\nu = \begin{cases} \rho^{\mu\nu}(l, \eta), & \mu, \nu \in 4 \text{ dim}; \\ -n_5^\mu n_5^\nu, & \mu = 5, \nu = 5; \\ 0, & \text{otherwise,} \end{cases} \quad (38)$$

where

$$\rho^{\mu\nu}(l, \eta) = -g^{\mu\nu} + \frac{(l^\mu \eta^\nu + l^\nu \eta^\mu)}{l \cdot \eta}, \quad (39)$$

and η is a four-dimensional light-cone vector such that $l \cdot \eta \neq 0$.

For $D_s = 6$ and four-dimensional loop momentum, we have $l \cdot n_5 = l \cdot n_6 = 0$. We then choose the following expression for gluon density matrix

$$\sum_{i=1}^4 e_i^\mu e_i^\nu = \begin{cases} \rho^{\mu\nu}(l, \eta), & \mu, \nu \in 4 \text{ dim}; \\ -n_5^\mu n_5^\nu, & \mu = 5, \nu = 5; \\ -n_6^\mu n_6^\nu, & \mu = 6, \nu = 6; \\ 0, & \text{otherwise.} \end{cases} \quad (40)$$

For $D_s = 5$ and five-dimensional loop momentum, we write $l^\mu = \bar{l}^\mu + \beta n_5^\mu$, where $\bar{l} \cdot n_5 = 0$. Also, $l^2 = 0$, but $\bar{l}^2 = -\beta^2 \neq 0$. Then, we construct polarization vectors by taking them to be in the three-dimensional subspace of a five-dimensional space, which is orthogonal to \bar{l} and n_5 . Specifically, we have

$$\sum_{i=1}^3 e_i^\mu e_i^\nu = \begin{cases} -\omega^{\mu\nu}(\bar{l}); & \mu, \nu \in 4 \text{ dim}; \\ 0, & \text{otherwise,} \end{cases} \quad (41)$$

where $\omega^{\mu\nu} = -g^{\mu\nu} + \bar{l}^\mu \bar{l}^\nu / \bar{l}^2$, with all indices restricted to four-dimensions. From a four-dimensional viewpoint, Eq. (41) is a density matrix of a spin-one particle with the mass \bar{l}^2 .

For $D_s = 6$ and five-dimensional loop momentum, we take $l = \bar{l}^\mu + \beta n_5^\mu$. In this case $l \cdot n_6 = 0$. With this choice, all we need to do is to add one more polarization direction to Eq. (41). We obtain

$$\sum_{i=1}^4 e_i^\mu e_i^\nu = \begin{cases} -\omega^{\mu\nu}(\bar{l}); & \mu, \nu \in 4\text{dim}; \\ -n_6^\mu n_6^\nu, & \mu = 6, \nu = 6, \\ 0, & \text{otherwise.} \end{cases} \quad (42)$$

We now make the following observation. For a chosen loop momentum l , be it four- or five-dimensional, the calculation in $D_s = 6$ differs from the calculation in $D_s = 5$ by a single polarization component of the gluon, denoted n_6 in Eqs. (40,42). All external momenta and polarizations are four-dimensional, and the cut loop momentum l that satisfies the unitarity constraint is at most five-dimensional. Because of this the n_6 polarization gives non-vanishing contribution when it is contracted with itself through a metric tensor. From this point of view, its contribution is equivalent to that of an additional (real) scalar particle in the loop. To see this more explicitly, consider a three-gluon vertex with two gluons polarized along the sixth dimension in six-dimensional space and the third gluon with the four-dimensional polarization vector. Since none of the gluon momenta have a component along the sixth dimension, we obtain

$$V_{\mu_1\mu_2\mu_3}^{(3)}(k_1, k_2, k_3) e_1^{\mu_1} n_6^{\mu_2} n_6^{\mu_3} \sim e_1^{\mu_1} (k_2 - k_3)_{\mu_1}. \quad (43)$$

The object on the r.h.s. of the above equation is the scalar-scalar-gluon vertex. Similar considerations applied to a four-gluon vertex with two gluons polarized along the n_6 direction immediately lead us to conclude that it becomes a scalar-scalar-gluon-gluon vertex. Hence, any tree n -gluon amplitude with two gluons whose polarization vectors are $e^\mu = n_6^\mu$ is equivalent to a tree scattering amplitude of $n - 2$ gluons and two scalars. This allows us to write

$$\mathcal{A}_{(D, D_s=6)} = \mathcal{A}_{(D, D_s=5)} + \mathcal{A}_{(D)}^S, \quad (44)$$

where the amplitude \mathcal{A}_D^S is computed with a particular propagator for all internal gluon lines $-in_6^\mu n_6^\nu / l^2$ which, as we argued above, is equivalent to scalar contribution.

If we use Eq. (44) in Eq. (37), we derive the following result

$$\mathcal{A}^{\text{FDH}} = \mathcal{A}_{(D,D_s=5)} - \mathcal{A}_{(D)}^S. \quad (45)$$

It tells us that, from the point of view of generalized unitarity, computations of gluon scattering amplitudes in the four-dimensional helicity scheme are equivalent to calculations of gluon scattering amplitudes in five-dimensional space-time up to a term that can be associated with a scalar contribution.

Finally, we may now write explicitly the relation between amplitudes computed in FDH and HV schemes. Using Eq. (11) and Eq. (44), we derive

$$\mathcal{A}^{\text{HV}} = \mathcal{A}_{(D,D_s=5)} - (1 + 2\epsilon)\mathcal{A}_{(D)}^S. \quad (46)$$

We see that the difference between the HV and FDH schemes is entirely due to the additional “scalar” degree of freedom that contributes to one-loop amplitudes. Note that we re-derive the results of [19] without resorting to SUSY arguments. The relation between unrenormalized amplitudes in two schemes becomes

$$\mathcal{A}^{\text{FDH}} - \mathcal{A}^{\text{HV}} = 2\epsilon\mathcal{A}^S. \quad (47)$$

The contribution of a real scalar to one-loop gluon scattering amplitudes reads

$$\mathcal{A}^S = \frac{c_\Gamma}{6\epsilon}\mathcal{A}_{\text{tree}} + \text{finite} \quad , \quad (48)$$

where

$$c_\Gamma = \frac{\Gamma(1 + \epsilon)\Gamma(1 - \epsilon)^2\mu^{2\epsilon}}{(4\pi)^{2-\epsilon}} \quad , \quad (49)$$

is the usual normalization factor and μ is the scale that maintains the dimensionality of loop integrals after the loop momentum is continued to $D = 4 - 2\epsilon$ dimensions. Neglecting terms of order ϵ , we obtain the well-known relation between gluon amplitudes computed in FDH and HV schemes [40]

$$\mathcal{A}^{\text{HV}} = \mathcal{A}^{\text{FDH}} - \frac{c_\Gamma}{3}\mathcal{A}_{\text{tree}}. \quad (50)$$

VI. RESULTS

We now present the results of the calculation of four-, five- and six-gluon scattering amplitudes in QCD. If, for a given choice of gluon helicities, the corresponding tree amplitude

does not vanish, the n -gluon one-loop sub-amplitude is written in the following way

$$\mathcal{A}_{n,1}(1^{\lambda_1}, \dots, n^{\lambda_n}) = c_\Gamma \left(-\frac{n}{\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{11}{3} + \sum_{i=1}^n \ln(-s_{i,i+1}) \right) + \Delta_{\lambda_1, \dots, \lambda_n} \right) \mathcal{A}_{\text{tree}}(1^{\lambda_1}, \dots, n^{\lambda_n}), \quad (51)$$

where $s_{i,i+1} = 2p_i \cdot p_{i+1} + i\delta$, $p_{n+1} = p_1$. The convenience of representing scattering amplitudes as in Eq. (51) is that phase conventions for gluon polarization vectors drop out when functions $\Delta_{\lambda_1, \dots, \lambda_n}$ are computed; this feature allows for direct comparison with the literature.

We have verified that our calculations correctly reproduce the divergent parts of Eq. (51). In Tables I, II and III we give the results for finite parts of color-ordered sub-amplitudes $\Delta_{\lambda_1, \dots, \lambda_n}$. The finite parts are split into cut-constructible and rational parts

$$\Delta_{\lambda_1, \dots, \lambda_n} = \Delta_{\lambda_1, \dots, \lambda_n}^{\text{cut}} + \Delta_{\lambda_1, \dots, \lambda_n}^{\text{rat}}$$

for kinematic points specified below.

On the other hand, some color-ordered sub-amplitudes for n -gluon scattering are finite; the corresponding tree amplitudes vanish making representation Eq. (51) senseless. For those finite amplitudes explicit results are presented below. Note however that in this case results depend on phase conventions adopted for gluon polarization vectors. To get rid of this dependence, for finite amplitudes we compare absolute values $|\mathcal{A}_{n,1}|$ to the results available in the literature. All results for scattering amplitudes reported below are given in the FDH scheme; results in the HV scheme can be obtained using Eq. (50).

Finally, we note that QCD sub-amplitudes are often calculated using supersymmetric decomposition since in this way analytic computations are simplified. In particular, the only part of the gluon scattering amplitude that cannot be obtained from four-dimensional unitarity is the part where gluons scatter through a loop with virtual scalars. Dealing with virtual scalars is easier than with virtual gluons since the number of degrees of freedom is smaller. However, since our goal is to demonstrate the vitality of the method, we do not employ this simplification and compute sub-amplitudes, without using the supersymmetric decomposition.

A. Four gluon scattering

We consider color-ordered sub-amplitude $\mathcal{A}_{4,1}(1^{\lambda_1}, \dots, 4^{\lambda_4})$ for the following choice of external momenta ($p = (E, p_x, p_y, p_z)$)

$$\begin{aligned} p_1 &= (1, 0, 0, 1), & p_2 &= (1, 0, 0, -1), \\ p_3 &= (-1, \sin \theta, 0, \cos \theta), & p_4 &= (-1, -\sin \theta, 0, -\cos \theta). \end{aligned} \quad (52)$$

with $\theta = \pi/3$.

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	Δ^{cut}	Δ^{rat}	Δ
--++	2.53627	0.2222	2.75849
-+-+	1.90292-3.29626 i	0.66667	2.56959-3.29626 i

TABLE I: Finite parts of singular four-gluon scattering amplitudes for various gluon helicities. Cut-constructible and rational parts are shown separately.

There are four sub-amplitudes that we have to consider + + + +, - + + +, - - + + and - + - +. The amplitudes + + + + and - + + + are finite, i.e. the cut-constructible parts of these amplitudes vanish identically and the entire results are due to rational parts. We find

$$\begin{aligned} \mathcal{A}_{4,1}(1^+, 2^+, 3^+, 4^+) &= i c_{\Gamma} \times 0.33333, \\ \mathcal{A}_{4,1}(1^-, 2^+, 3^+, 4^+) &= i c_{\Gamma} \times 0.7500. \end{aligned} \quad (53)$$

Finite parts for the two divergent amplitudes - - + + and - + - + are given in Table I. The results for scattering amplitudes in Eqs. (51,53) and Table I are in complete agreement with Ref. [41].

B. Five-gluon scattering

We consider sub-amplitude $\mathcal{A}_{5,1}(1^{\lambda_1}, \dots, 5^{\lambda_5})$ and choose external momenta to be

$$\begin{aligned} p_1 &= (1, 0, 0, 1), & p_2 &= (1, 0, 0, -1), \\ p_3 &= \xi (-1, 1, 0, 0), & p_4 &= \xi (-\sqrt{2}, 0, 1, 1), \\ p_5 &= -p_1 - p_2 - p_3 - p_4, \end{aligned} \quad (54)$$

with $\xi = 2/(1 + \sqrt{2} + \sqrt{3}) = 0.4823619098$. We have to consider four sub-amplitudes $+++++$, $-++++$, $--+++$ and $-+-++$. The first two amplitudes are finite and, hence, are entirely due to rational parts.

$\lambda_1, \lambda_2, \dots, \lambda_5$	Δ^{cut}	Δ^{rat}	Δ
$--+++$	7.3382230-2.1860355 i	0.24488559-1.4089423 i	7.58310859-3.5949778 i
$-+-++$	12.059206+1.7853279 i	-7.5603579-8.4763597 i	4.4988481-6.6910318 i

TABLE II: Finite parts of singular five-gluon scattering amplitudes for various gluon helicities. The cut-constructible and rational parts are shown separately.

For the two finite amplitudes we obtain

$$\begin{aligned}\mathcal{A}_{5,1}(1^+, 2^+, 3^+, 4^+, 5^+) &= i c_{\Gamma} \times (0.01056 - 0.6614 i), \\ \mathcal{A}_{5,1}(1^-, 2^+, 3^+, 4^+, 5^+) &= i c_{\Gamma} \times (-0.6773 - 0.4976 i).\end{aligned}\tag{55}$$

The results for the two divergent amplitudes are given in Table II. For all scattering amplitudes, we find complete agreement with the results reported in Ref. [40].

C. Six gluon scattering

We now present the results for the six-gluon scattering. We consider color-ordered sub-amplitude $\mathcal{A}_{6,1}(1, 2, 3, 4, 5)$ for the same external momenta as considered in Ref. [14]¹. Specifically, we have

$$\begin{aligned}p_1 &= 3(-1, \cos \theta \sin \phi, \sin \theta, \cos \theta \cos \phi), \quad p_2 = -3(1, \cos \theta \sin \phi, \sin \theta, \cos \theta, \cos \phi); \\ p_3 &= 2(1, 0, 1, 0), \quad p_4 = \frac{6}{7}(1, \sin \beta, \cos \beta, 0), \\ p_5 &= (1, \cos \alpha \sin \beta, \cos \alpha \cos \beta, \sin \alpha), \quad p_6 = -p_1 - p_2 - p_3 - p_4 - p_5,\end{aligned}\tag{56}$$

where $\theta = \pi/4$, $\phi = \pi/6$, $\alpha = \pi/3$, $\cos \beta = -7/19$.

We compute sub-amplitude $\mathcal{A}_{6,1}(1, 2, 3, 4, 5, 6)$ for the following helicity configurations $+++++$, $-++++$, $--+++$, $-+-++$, $-++-++$, $---+++$, $--+-++$,

¹ Our conventions differ from Ref. [14] in that x and y axes are interchanged.

$\lambda_1, \lambda_2, \dots, \lambda_6$	Δ^{cut}	Δ^{rat}	Δ
-- + + + +	-19.481065+78.147162 <i>i</i>	28.508591-74.507275 <i>i</i>	9.027526+3.639887 <i>i</i>
- + - + + +	-241.10930+27.176200 <i>i</i>	250.27357-25.695269 <i>i</i>	9.164272+1.480930 <i>i</i>
- + + - + +	5.4801516-12.433657 <i>i</i>	0.19703574+0.25452928 <i>i</i>	5.677187-12.179127 <i>i</i>
- - - + + +	15.478408-2.7380153 <i>i</i>	2.2486654+1.0766607 <i>i</i>	17.727073-1.661354 <i>i</i>
- - + - + +	-339.15056-328.58047 <i>i</i>	348.65907+336.44983 <i>i</i>	9.508509+7.869351 <i>i</i>
- + - + - +	31.947346+507.44665 <i>i</i>	-17.430910-510.42171 <i>i</i>	14.516436-2.975062 <i>i</i>

TABLE III: Finite parts of singular six-gluon scattering amplitudes for various gluon helicities. The cut-constructible and rational parts are shown separately.

- + - + - + -. Among eight amplitudes, two amplitudes + + + + + + and - + + + + + are finite; the corresponding tree amplitudes vanish. For these amplitudes we find

$$\begin{aligned}
\mathcal{A}_{6,1}(1^+, 2^+, 3^+, 4^+, 5^+, 6^+) &= i c_\Gamma \times (0.4968861 - 0.1838453 i), \\
\mathcal{A}_{6,1}(1^-, 2^+, 3^+, 4^+, 5^+, 6^+) &= i c_\Gamma \times (1.0182879 + 3.0968492 i).
\end{aligned} \tag{57}$$

The results for the finite parts of remaining six gluon scattering amplitudes are given in Table III, where cut-constructible and rational parts are shown separately. For all six-gluon amplitudes, we find an agreement², at least through seven digits, with the results reported in Ref. [14].

VII. CONCLUSIONS

In this paper we describe a novel method for calculating one-loop scattering amplitudes, including their rational parts. It is based on unitarity cuts in higher-dimensional space-time. Similar to four-dimensional unitarity, one-loop amplitudes are obtained from tree amplitudes. These tree amplitudes can be efficiently calculated using recursive algorithms of polynomial complexity leading to an efficient method for semi-numerical evaluation of one-loop amplitudes. Because the method is built around integer-dimensional unitarity cuts, we do not foresee any difficulty with its application to chiral gauge theories.

² To obtain numerical results reported Ref. [14] from our results, one should expand $\mu^{2\epsilon}$ included in the normalization factor c_Γ , Eq. (49), in powers of ϵ and substitute $\mu = 6$.

Our method solves the outstanding problem of an efficient semi-numerical evaluation of the so-called rational part of one-loop amplitudes, an important step towards automated computation of NLO cross sections. The generality of the method allows a straightforward calculation of NLO corrections to multi-particle processes that involve virtual particles of arbitrary spins and masses. We hope that further development of this method will finally bring within reach NLO computations for such complicated processes as $PP \rightarrow t\bar{t} + 2$ jets and $PP \rightarrow V + 3, 4$ jets.

Acknowledgments

K.M. is supported in part by the DOE grant DE-FG03-94ER-40833 and Outstanding Junior Investigator Award. We are indebted to G. Zanderighi for her help with comparison of our numerical results to that of Ref. [14]. Useful discussions with R.K. Ellis are gratefully acknowledged.

-
- [1] M. L. Mangano, M. Moretti, F. Piccinini, R. Pittau and A. D. Polosa, JHEP 0307, 001 (2003) [hep-ph/0206293].
 - [2] T. Stelzer and W. F. Long, Comput. Phys. Commun. 81, 357 (1994) [hep-ph/9401258].
 - [3] A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. 132, 306 (2000) [hep-ph/0002082].
 - [4] A. Pukhov *et al.*, hep-ph/9908288.
 - [5] F. Krauss, R. Kuhn and G. Soff, JHEP 0202, 044 (2002) [hep-ph/0109036].
 - [6] F. A. Berends and W. T. Giele, Nucl. Phys. B 306 (1988) 759.
 - [7] F. Caravaglios and M. Moretti, Phys. Lett. B 358, 332 (1995) [arXiv:hep-ph/9507237].
 - [8] P. Draggiotis, R. H. P. Kleiss and C. G. Papadopoulos, Phys. Lett. B 439, 157 (1998) [arXiv:hep-ph/9807207].
 - [9] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 715, 499 (2005) [hep-th/0412308].
 - [10] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. 94, 181602 (2005) [hep-th/0501052].
 - [11] R. K. Ellis, W. T. Giele and Z. Kunszt, Contribution to the workshop “ Physics at TeV

colliders”, Les Houches, 11-29 June 2007.

- [12] G. Passarino and M. J. G. Veltman, Nucl. Phys. B 160, 151 (1979).
- [13] A. Denner, S. Dittmaier, M. Roth and L. H. Wieders, Phys. Lett. B 612, 223 (2005) [arXiv:hep-ph/0502063]; A. Denner, S. Dittmaier, M. Roth and L. H. Wieders, Nucl. Phys. B 724, 247 (2005) [arXiv:hep-ph/0505042].
- [14] R. K. Ellis, W. T. Giele, and G. Zanderighi, JHEP 05, 027 (2006) [hep-ph/0602185].
- [15] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).
- [16] G. 't Hooft and M. Veltman, Diagrammar, CERN Report 73-9, Geneva (1973)
- [17] W. L. van Neerven, Nucl. Phys. B 268, 453 (1986).
- [18] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 425, 217 (1994) [hep-ph/9403226].
- [19] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 435, 59 (1995) [hep-ph/9409265].
- [20] Z. Bern, A.G. Morgan, Nucl. Phys. B 467 (1996), 479; Z. Bern, L. Dixon, D.C. Dunbar and D.A. Kosower, Phys. Lett. B 394, 105 (1997).
- [21] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B 513, 3 (1998) [hep-ph/9708239].
- [22] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 725, 275 (2005) [hep-th/0412103].
- [23] G. Ossola, C. G. Papadopoulos, and R. Pittau, Nucl. Phys. B 763, 147 (2007) [hep-ph/0609007].
- [24] R. K. Ellis, W. T. Giele and Z. Kunszt, hep-ph/07082398.
- [25] G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP 0707, 085 (2007) [arXiv:0704.1271 [hep-ph]]; G. Ossola, C. G. Papadopoulos and R. Pittau, arXiv:0711.3596 [hep-ph].
- [26] D. Forde, Phys. Rev. D 75, 125019 (2007) [arXiv:0704.1835 [hep-ph]].
- [27] W. B. Kilgore, arXiv:0711.5015 [hep-ph].
- [28] Z. Xiao, G. Yang and G. J. Zhu, Nucl. Phys. B 758, 53 (2006).
- [29] T. Binoth, J. P. Guillet and G. Heinrich, JHEP 0702, 013 (2007).
- [30] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D 73, 065013 (2006) [arXiv:hep-ph/0507005].
- [31] C.F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, Phys. Rev. D 75, 016006 (2007).
- [32] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, Phys. Lett. B 645, 213 (2007);

JHEP 0703, 111 (2007).

- [33] R. Britto and B. Feng, hep-ph/07114284.
- [34] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B 44, 189 (1972).
- [35] Z. Bern, A. De Freitas, L. J. Dixon and H. L. Wong, Phys. Rev. D 66, 085002 (2002) [arXiv:hep-ph/0202271].
- [36] A repository of one-loop master integrals can be found at <http://qcdloop.fnal.gov/> . See also R.K. Ellis and G. Zanderighi, arXiv:0712.1851; J.R. Andersen, T. Binoth, G. Heinrich and J.M. Smillie, arXiv:0709.3513.
- [37] F. A. Berends and W. Giele, Nucl. Phys. B 294, 700 (1987).
- [38] M. L. Mangano, S. J. Parke and Z. Xu, Nucl. Phys. B 298, 653 (1988).
- [39] Z. Bern and D. A. Kosower, Nucl. Phys. B 362, 389 (1991).
- [40] Z. Bern, L.J. Dixon and D.A. Kosower, Phys. Rev. Lett. 70, 2677 (1993).
- [41] Z. Kunszt, A. Signer and Z. Trocsanyi, Nucl. Phys. B 411, 397 (1994).