

The Dipion Mass Spectrum In e^+e^- Annihilation and τ Decay : A Dynamical (ρ , ω , ϕ) Mixing Approach

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Abstract

We readdress the problem of finding a simultaneous description of the pion form factor data in e^+e^- annihilations and in τ decays. For this purpose, we work in the framework of the Hidden Local Symmetry (HLS) Lagrangian and modify the vector meson mass term by including the pion and kaon loop contributions. This leads us to define the physical ρ , ω and ϕ fields as linear combinations of their ideal partners, with coefficients being meromorphic functions of s , the square of the 4-momentum flowing into the vector meson lines. This allows us to define a dynamical, *i.e.* s -dependent, vector meson mixing scheme. The model is overconstrained by extending the framework in order to include the description of all meson radiative ($VP\gamma$ and $P\gamma\gamma$ couplings) and leptonic (Ve^+e^- couplings) decays and also the isospin breaking ($\omega/\phi \rightarrow \pi^+\pi^-$) decay modes. The model provides a simultaneous, consistent and good description of the e^+e^- and τ dipion spectra. The expression for pion form factor in the latter case is derived from those in the former case by switching off the isospin breaking effects specific to e^+e^- and switching on those for τ decays. Besides, the model also provides a good account of all decay modes of the form $VP\gamma$, $P\gamma\gamma$ as well as the isospin breaking decay modes. It leads us to propose new reference values for the $\rho^0 \rightarrow e^+e^-$ and $\omega \rightarrow \pi^+\pi^-$ partial widths which are part of our description of the pion form factor. Other topics ($\phi \rightarrow K\bar{K}$, the ρ meson mass and width parameters) are briefly discussed. The most important consequence of this work is indirect and confirms the known 3.3σ discrepancy between the direct BNL measurement of the muon anomalous moment and its theoretical estimate relying on e^+e^- data.

1 Introduction

In order to study the phenomenology of light flavor mesons below 1 GeV, like partial decay widths or meson form factors, one needs a framework which includes in a well defined manner the lowest mass nonets of pseudoscalar (P) and vector (V) mesons. Such a framework is well represented by the hidden local symmetry (HLS) model [1, 2]. In this approach, vector mesons are gauge bosons of a spontaneously broken hidden local symmetry which generates their (Higgs–Kibble, HK) masses. Besides the non–anomalous sector, this model has an anomalous sector, hereafter called FKTUY Lagrangian [3], which aims at describing couplings of the form VVP , $VP\gamma$, $P\gamma\gamma$, $VPPP$ for light flavor mesons.

In its original form, the full (non–anomalous and anomalous) bare HLS Lagrangian fulfills the $U(3)$ symmetry, as it possesses both Nonet Symmetry and $SU(3)$ flavor symmetry. As such, the HLS model covers a limited phenomenological scope. In order to broaden this scope, especially in order to account for $VP\gamma$ and $P\gamma\gamma$ couplings as derived from measured partial widths, the original HLS model should be supplemented with symmetry breaking mechanisms.

Several $SU(3)$ symmetry breaking schemes have been proposed [4, 5, 6] following the original idea of Bando, Kugo and Yamawaki (BKY) [7]. It has been shown [8, 9] that the most successful variant is the so–called “new scheme” of Ref. [6] briefly recalled in Appendix C. However, breaking only the $SU(3)$ symmetry is insufficient [8, 9] in order to reach a satisfactory description of the data on $VP\gamma$ and $P\gamma\gamma$ couplings which requires one to also break Nonet Symmetry. This was first performed in an *ad hoc* manner in [8] with the aim of recovering the radiative decay couplings of O’Donnell [10] which is the most general set fulfilling only $SU(3)$ symmetry ; therefore, the model developed in [8] was indeed in agreement with general group theoretical considerations with additionally a mechanism smoothly breaking $SU(3)$ flavor symmetry.

Slightly later, it was shown [9, 11] that an appropriate mechanism for Nonet Symmetry breaking can be produced by adding determinant terms [12] to the HLS Lagrangian ; the result was shown to meet all properties of Extended Chiral Perturbation Theory (EChPT) [13, 14, 15] at leading order in the breaking parameters. Additionally, it was also proved [11] that the Nonet Symmetry breaking mechanism proposed in [8] was indeed an appropriate approximation of the (rigorous) mechanism derived from adding the determinant terms to the bare HLS Lagrangian.

However, breaking $SU(3)$ and Nonet symmetries is still not enough to be in position of describing fully the whole set of radiative decays $VP\gamma$; indeed, a process like $\phi \rightarrow \pi^0\gamma$ requires including a $\omega - \phi$ mixing scheme ; additionally, any global fit of all available $VP\gamma$ transitions cannot be successful without introducing such a mixing. Traditionally, the $\omega - \phi$ mixing is described [8, 9] by rotating the fields ω_I and ϕ_I which are the entries of the bare vector field matrix, generally called ideal fields. An angle involving the mixing of the η and η' mesons is also required which has been shown [11] to vanish in the limit of exact $SU(3)$ symmetry of the HLS Lagrangian.

A short account of the HLS model in its anomalous (FKTUY) and non–anomalous sectors is given in Appendices A and B, mostly focused on the subject of this work. Appendix C describes shortly but fully the various symmetry breaking procedures and the field renormalization scheme, except for the $SU(2)$ breaking mechanism which is the main subject of this paper. All this illustrates that the model depends on a very few number of free parameters as

clear from Appendix D where they all appear. Among these free parameters, some are specific to our HLS based model, but some others are constrained. For instance, one of the two SU(3) breaking parameters is the ratio of decay constants f_K/f_π and can be constrained by the corresponding measured value.

On another hand, decay processes like $\omega/\phi \rightarrow \pi^+\pi^-$ can hardly be understood without some scheme for $\rho^0 - \omega$ and $\rho^0 - \phi$ mixings. Moreover, the decay behaviour of ρ^0 and ρ^\pm cannot differ at the coupling constant level. It happens that the HLS model at one loop order provides a mechanism which allows us to perform a full $\rho^0 - \omega - \phi$ mixing starting from the corresponding ideal (bare) fields. Within the non-anomalous HLS Lagrangian, the precise mechanism, kaon loop contributions, has been already described in [16]. However, we shall see that the (FKTUY) anomalous sector provides K^*K loops as additional mechanism.

Within the HLS model, as recalled below, $V_i \leftrightarrow V_j$ transitions among the ideal ρ_I^0, ω_I and ϕ_I fields are generated by loop effects ; these transitions generally have not a constant amplitude but rather depend on the squared 4-momentum flowing through the V_i (and V_j) line(s). We show in Section 4 that charged and neutral kaon and K^*K loops come through their sum in the $\omega_I \leftrightarrow \phi_I$ amplitude, while they come through their difference in $\rho_I^0 \leftrightarrow \phi_I$ and $\rho_I^0 \leftrightarrow \omega_I$ amplitudes. This also means that the $\omega - \phi$ mixing proceeds from quantum effects rather than symmetry breaking effects, in contrast with the $\rho^0 - \omega$ or $\rho^0 - \phi$ mixings. These transition amplitudes are given by Dispersion Relations which should be subtracted in order to make the integral convergent (see, for instance, Appendix A in [9]). This gives rise to polynomials with real coefficients to be fixed using external renormalization conditions.

In the exact SU(3) symmetry limit, charged and neutral kaons carry the same mass and one can expect (or require) the polynomials associated with the charged and neutral kaon loops to coincide. Likewise, the charged and neutral K^*K loop functions can be made equal. In this case, the $\omega_I \leftrightarrow \phi_I$ amplitude survives with its renormalization polynomial, while the $\rho_I^0 \leftrightarrow \phi_I$ and $\rho_I^0 \leftrightarrow \omega_I$ transition amplitudes exactly vanish. If one breaks the SU(3) flavor symmetry leaving conserved the (u, d) quark sector, the same conclusion holds. However, if one introduces a breaking of SU(2) flavor symmetry, the mass difference between u and d quarks generates a mass difference between the charged and neutral kaons (and K^* 's). Then, the three possible transition amplitudes do not identically vanish any longer and, moreover, they depend on the invariant mass associated by the 4-momentum flowing through the vector meson lines. Stated otherwise : the ρ_I^0, ω_I and ϕ_I mixing into the physical ρ^0, ω and ϕ fields should also be invariant mass dependent. As already noted in [9, 16] this implies that the vector squared mass matrix, which has to be diagonalized in order to define the physical fields, is also invariant mass dependent and that the notion of mass for the physical vector fields becomes unclear as soon as one goes beyond tree level.

The property that $\rho^0 - \omega$ mixing should be invariant mass dependent has been the subject of several studies in the framework of general local effective field theories [17], then in Vector Dominance Models [18, 19] where it was pointed out that the mixing amplitude should vanish at $s = 0$, as the ρ^0 and ω self-masses, in order to preserve gauge invariance. One may also quote other studies going in the same direction [20, 21] attributing the mixing to finite width effects, or quark loops (and pion loops). Using only the pion form factor data available at that time, Ref. [22] derived an approximate expression for the $\rho^0 - \omega$ mixing amplitude ; however, limiting that much the kind of data used, one cannot really observe a clear mass dependence effect. On the other hand, one should also note that Ref. [23] proved that isospin violation

effects describing the $\rho^0 - \omega$ mixing are not accounted for by the low energy constants (LEC) of Chiral Perturbation Theory (ChPT) but are generated near threshold by the difference between charged and neutral kaon loops ; it was also shown that these effects are tiny at the two-pion threshold (10^{-4}) while they are known to be at the percent level in resonance peak region. This illustrates the invariant mass dependence of the $\rho^0 - \omega$ mixing. However, one should note that the absence of the LECs in this calculation suggests a one-loop result may be unreliable.

Therefore, the question we address is to consider the effects of loops on ρ^0 , ω , ϕ mixing. However, in order to have some chance to single out the mass dependent behaviour, one clearly has to treat simultaneously the pion form factor data in the spacelike and timelike regions together with the largest possible set of light meson decays (radiative, leptonic and isospin violating strong decays) within a single framework. As argued above, the HLS model, equipped with suitable symmetry breaking mechanisms, seems able to provide such a framework. We shall not try to include Isospin Symmetry breaking effects into the coupling constants which would help by providing more parameter freedom in the fit procedures. If really needed, it can certainly be done along the lines of the BKY breaking scheme [7, 6] as illustrated by [24].

The question of having a unified description of the largest possible set of low energy data is by itself interesting. However, this also addresses the puzzling and long-standing problem of the difference between the (estimated) isospin 1 part of the pion form factor in e^+e^- annihilations and in τ decays which are related through the Conserved Vector Current assumption (CVC). The importance of the problem is enhanced by its implication for the predicted value of the muon anomalous magnetic moment a_μ to be compared with the direct BNL measurement [25] ; referring to the latest account by M. Davier [26] the estimate of the hadronic vacuum polarization (which enters the theoretical estimate) derived from e^+e^- data provides a 3.3σ disagreement between the theoretical estimation of a_μ and the BNL direct measurement [25] ; moreover, the τ data estimate of the hadronic vacuum polarization provides a value of a_μ very close to its direct measurement [26].

Except for an experimental problem with e^+e^- annihilation data (which seems by now unlikely) in the data recently collected at Novosibirsk [27, 28, 29, 30], or some new (or unidentified) physics effect, the disagreement between e^+e^- and τ data [31, 32, 33] is hard to explain. Indeed, *a priori* the single difference between these two channels, should be due to Isospin Symmetry breaking (IB) effects. However, the comparison has been already performed with IB effects accounted for in both e^+e^- and τ data. This includes [34, 35, 36] pion mass values in kinematical factors, (a parametrization of the) $\rho - \omega$ mixing, charged and neutral ρ mass and width differences, short range [37] and long range [38, 39, 40, 41, 42, 43] IB effects in the τ partial decay width to two pions.

This persistent disagreement may point towards new physics effects [44] ; however, one should also note that the way some IB effects are accounted for has been questioned several times. For instance, effects due to the charged and neutral ρ pole positions [45, 46] were considered, but have not been found sufficient in order to solve the observed discrepancy [35, 36] ; $\rho - \omega$ mixing effects may also have been poorly estimated [47]. However, based on sum rules derived using an OPE input, K. Maltman [48, 49] concluded there is inconsistency of the (presently) estimated isospin 1 part of e^+e^- data with expectation, while τ data provide a nice agreement.

We plan to address this question once again by building an effective model relying on the

(symmetry broken) HLS model. In this approach, we plan to have a framework giving simultaneously an account of the partial decay widths of light mesons decays (radiative, leptonic, isospin breaking decay modes), of the pion form factor in e^+e^- data (both timelike and spacelike) and in τ decay. By construction, the corresponding expressions of the pion form factor will be such that they will solely differ from each other by Isospin Symmetry breaking effects, mostly located in the (ρ, ω, ϕ) mixing scheme which, of course, has no counterpart in τ decay. Stated otherwise, our model is built in such a way that going from the pion form factor expression in e^+e^- annihilations to the expression valid for τ decays is performed by switching off IB effects specific to e^+e^- and switching on those specific to τ decays.

At start, the model we built is rendered complicated by the large number of possible loops involved. Fortunately, it can be somewhat simplified without loosing too much physics insight. Of course, this model depends on some parameters to be fixed in a fitting procedure ; we should define a fitting procedure flexible enough that τ decay data can be removed or kept. Stated otherwise, the light mesons decays and the pion form factor in e^+e^- data are expected to fix practically the U(3)/SU(3)/SU(2) breaking model. In this approach, we may get a prediction of the τ decay 2-pion spectrum which can be compared with the existing measurements. Including τ decay data should only refine the values of the fitted parameters.

The paper is organized as follows : In Section 2, we derive the Lagrangian pieces of relevance in order to deal with the pion form factor in e^+e^- annihilations and τ decays, while in Section 3 we give the pion form factor expressions without loop corrections and symmetry breaking effects, mostly for illustration. In Section 4 we discuss the loop corrections which modify the vector meson mass matrix and perform already some simplifications. The modified vector meson squared mass matrix we propose is given in Section 5 with the diagonalization procedure and the relation between physical and ideal vector meson fields. The method used in order to renormalize the loop functions defining the self-energies and the transition amplitudes is sketched in Section 6. The form factor functions used for τ decays and e^+e^- annihilations are given in resp. Sections 7 and 8. A necessary ingredient affecting the pion form factor in e^+e^- annihilations is the photon vacuum polarization (VP), which is discussed in Section 9. Fitting with the partial width expressions is briefly discussed in Section 10 ; more details can be found in [16], where a practically identical method is used with constant mixing functions, however. The way to deal with the various kinds of data used in the fit procedures is described in Section 11, with a special emphasis on our dealing with some correlation phenomena present in the existing data. In Section 12, we fully describe the fit procedures we worked out under various conditions and the results and conclusions we reach ; we also comment on the numerical and physical properties of our model. Finally, Section 13 is devoted to a summary of our conclusions.

As already commented upon in course of the Introduction, several Appendices gather results of ours or others already published. They are given in an attempt to be as self contained as possible ; they are placed outside the main text in the interest of clarity and ease of reading.

2 The HLS Lagrangian Model

We outline in the Appendices the main features of the HLS Model in both the non-anomalous (Appendix A) and anomalous (Appendix B) sectors. This allows us to derive the leading terms of the non-anomalous Lagrangian of interest for the present paper by expanding the exponentials defining the ξ fields. The breaking of flavour symmetries, SU(3) and the Nonet Symmetry is sketched in Appendix C.

Several pieces of the HLS Lagrangian of relevance for our problem will be given explicitly in the main text ; first, the part describing the photon sector (traditional VMD) is :

$$\begin{aligned}
\mathcal{L}_{VMD} = & ie\left(1 - \frac{a}{2}\right)A \cdot \pi^- \overleftrightarrow{\partial} \pi^+ + i\frac{e}{z_A}\left(z_A - \frac{a}{2} - b\right)A \cdot K^- \overleftrightarrow{\partial} K^+ + i\frac{e}{z_A}bA \cdot K^0 \overleftrightarrow{\partial} \overline{K}^0 \\
& + \frac{ia g}{2}\rho_I^0 \cdot \pi^- \overleftrightarrow{\partial} \pi^+ + \frac{ia g}{4z_A}(\rho_I^0 + \omega_I - \sqrt{2}z_V\phi_I)K^- \overleftrightarrow{\partial} K^+ + \frac{ia g}{4z_A}(\rho_I^0 - \omega_I + \sqrt{2}z_V\phi_I)K^0 \overleftrightarrow{\partial} \overline{K}^0 \\
& - eagf_\pi^2 \left[\rho_I^0 + \frac{1}{3}\omega_I - \frac{\sqrt{2}}{3}z_V\phi_I \right] \cdot A + \frac{1}{9}af_\pi^2e^2(5 + z_V)A^2 + \frac{af_\pi^2g^2}{2} \left[(\rho_I^0)^2 + \omega_I^2 + z_V\phi_I^2 \right]
\end{aligned} \tag{1}$$

limiting oneself to vector mesons, pion and kaon fields. Flavor symmetries have been broken and, as noted in [7, 6], this implies a pseudoscalar field renormalization. The pseudoscalar field renormalization has been performed (following the prescription given by Eqs. (80) or, rather, by Eqs. (84) which include Nonet Symmetry breaking). The free Lagrangian of the vector meson fields is standard [1, 2], as well as the (canonical) pseudoscalar kinetic energy piece, the leptonic (see Eq.(71)) and photonic free Lagrangian pieces.

The parameter g is the traditional universal vector meson coupling constant. On the other hand, the parameter a is specific of the HLS model and fulfills $a = 2$ in standard VMD approaches ; however such a stringent condition is not mandatory and several phenomenological studies involving pion form factor data on the one hand [50, 51] and light meson decays on the other hand [8, 9] concluded that a much better favored value is $a \simeq 2.4 \div 2.5$. This opens a way to a direct coupling of photons to pseudoscalar pairs within VMD-like approaches. One should remark the presence of a photon mass term of order e^2 which is traditionally removed by field redefinition [2, 52] (see also [53] and the discussion concerning the photon pole position). It can also be removed by renormalization conditions at one loop order.

The parameter b in Eq. (1) is $b = a(z_V - 1)/6$ where z_V is the SU(3) breaking parameter of the \mathcal{L}_V part of the HLS Lagrangian, while $z_A = [f_K/f_\pi]^2 = 1.495 \pm 0.031$ [54] is the SU(3) breaking parameter of its \mathcal{L}_A part [1, 2]. z_A is almost fixed numerically, while z_V is the major origin of the HK mass difference between the ϕ meson and the (ω, ρ^0) system ; it has to be fitted as the relation between vector meson masses determined experimentally and the (Higgs–Kibble) masses occurring in Lagrangians is unclear [52, 9]. On the other hand, the value for f_K/f_π provided by the Review of Particle Properties (RPP) [54] can be added to the set of measurements to be fit.

A subscript I on the fields, standing for “ideal”, affects the neutral vector meson fields. It indicates that the corresponding fields occurring in the Lagrangian are not the physical fields.

One should note [6] that the SU(3) breaking of the HLS Lagrangian generates a non-resonant coupling of the photon to neutral kaon pairs ; this is a property common to all breaking

procedures of the HLS Lagrangian proposed so far [7, 6, 4, 5].

On the other hand, still limiting oneself to pions and kaon terms, the Lagrangian piece of relevance for τ decay after symmetry breaking and field renormalization is given by :

$$\begin{aligned} \mathcal{L}_\tau = & -\frac{ig_2}{2}V_{ud}W^+ \cdot \left[\left(1 - \frac{a}{2}\right)\pi^- \overleftrightarrow{\partial} \pi^0 + \left(z_A - \frac{a}{2}\right)\frac{1}{z_A\sqrt{2}}K^0 \overleftrightarrow{\partial} K^- \right] \\ & -\frac{af_\pi^2gg_2}{2}V_{ud}W^+ \cdot \rho^- - \frac{ia g}{2}\rho^- \left[\pi^0 \overleftrightarrow{\partial} \pi^+ - \frac{1}{z_A\sqrt{2}}\overline{K}^0 \overleftrightarrow{\partial} K^+ \right] \\ & + f_\pi^2g_2^2 \left\{ \frac{1+a}{4} [z_A|V_{us}|^2 + |V_{ud}|^2] + \frac{a}{4}[\sqrt{z_V} - z_A]|V_{us}|^2 \right\} W^+ \cdot W^- + af_\pi^2g^2\rho^+\rho^- \end{aligned} \quad (2)$$

plus the conjugate of the interaction term (the W^- term, not displayed). This Lagrangian piece depends on g_2 (which is fixed by its relation with the Fermi constant (see Eq. (72))), on the CKM matrix element $V_{ud} = 0.97377 \pm 0.00027$ [54], on the universal coupling g and on the breaking parameters z_A and z_V already defined. One should note, balancing the photon mass term in \mathcal{L}_{VMD} , a small mass term complementing the W mass of the Standard Model which could be removed by appropriate field redefinitions.

Finally, the effective Lagrangian of the Model we use in order to describe low energy physics is :

$$\mathcal{L} = \mathcal{L}_{VMD} + \mathcal{L}_\tau + \mathcal{L}_{anomalous} \quad (3)$$

where $\mathcal{L}_{anomalous}$ is given by Eq. (87). Its VVP part is not given in the Appendices, but can be found fully expanded in the Appendices of [8]. The first two terms in Eq. (3) allow us to build up the pion form factor in e^+e^- interactions and τ decay and the leptonic widths of neutral vector mesons, while the anomalous decays will be dealt with starting from the third piece. Except for the single fit parameter z_T which is important only for K^* radiative decays, all breaking parameters are common to all pieces of our \mathcal{L} ; more precisely, all parameters of our model, except those of the vector meson mixing, could be fixed from only $\mathcal{L}_{anomalous}$ and the leptonic decays of vector mesons. This was proved in [8, 9, 16] by adding various (ρ, ω, ϕ) mixing schemes including the most traditional (ω, ϕ) mixing in isolation [8].

3 The Pion Form Factor Without Symmetry Breaking

The Lagrangian given in Eq. (3) allows us to construct the pion form factor in e^+e^- annihilation and in τ decay. One has ($m_\pi \equiv m_{\pi^\pm}$) :

$$F_\pi(s) = \left(1 - \frac{a}{2}\right) - \frac{a^2g^2f_\pi^2}{2} \frac{1}{D_V(s)} \quad (4)$$

for both processes involving intermediate photon and W boson. We also have :

$$\left\{ \begin{array}{l} \sigma(e^+e^- \rightarrow \pi^+\pi^-) = \frac{8\pi\alpha^2}{3s^{5/2}} |F_\pi(s)|^2 q_\pi^3 \\ \frac{d\Gamma}{ds}(s) = \frac{|V_{ud}|^2 G_F^2}{64\pi^3 m_\tau^3} |F_\pi(s)|^2 [G_0(s) + \epsilon^2 G_2(s)] \end{array} \right. \quad (5)$$

with¹ :

$$\begin{cases} G_0(s) = \frac{4(m_\tau^2 - s)^2(m_\tau^2 + 2s)}{s^{3/2}} Q_\pi^3 \\ G_2(s) = -\frac{(m_\tau^2 - s)^2(4s - m_\tau^2)}{s^{5/2}} m_\pi^4 Q_\pi \end{cases} \quad (6)$$

and :

$$\begin{cases} \epsilon = \frac{m_{\pi^0}^2 - m_{\pi^+}^2}{m_{\pi^+}^2} \simeq -0.06455, & (m_\pi \equiv m_{\pi^+}) \\ q_\pi = \frac{1}{2} \sqrt{s - 4m_\pi^2} \\ Q_\pi = \frac{\sqrt{[s - (m_{\pi^0} + m_{\pi^+})^2][s - (m_{\pi^0} - m_{\pi^+})^2]}}{2\sqrt{s}} \end{cases} \quad (7)$$

where one accounts for the pion mass difference. The $G_2(s)$ term gives a completely negligible contribution to the form factor and will be cancelled out from now on. The bare inverse propagator $D_V = s - m_\rho^2$ has to be modified for self-mass effects which fortunately shift the ρ meson pole off the physical region by giving it an imaginary part. At this stage, there is also no inclusion of loop effects in $\gamma\rho$ or $W\rho$ transition amplitudes in the expression for the pion form factor itself.

Additionally, there is clearly no interplay of the ω or ϕ mesons as can be seen from inspecting the various pieces of the full Lagrangian in Eq. (3) ; this should come from Isospin Symmetry breaking.

Including self-mass effects for the ρ (and adding the ω/ϕ meson contributions for e^+e^- annihilation), these expressions provide the usual HLS based framework for pion form factor fitting of e^+e^- data [51, 50]. Even if never done, in principle, Eq. (4) applies to τ data, again after shifting the ρ^\pm singularity off the physical region by means of a varying width Breit-Wigner amplitude, for instance.

4 Including One-Loop Effects In The HLS Lagrangian

From the expressions given in the previous Section, the ρ meson occurs as a pole on the physical region ; this is moved off the real axis by self-mass (loop) effects which essentially turn out to provide a width to the ρ through the imaginary part of the pion loop. However, besides this effect, a closer look at our \mathcal{L} allows us to see that loop effects contribute to generate self-masses to all vector mesons, *and* transition amplitudes among all neutral vector mesons. Assuming from now on SU(3) and SU(2) breaking effects, the charged and neutral pion and kaon masses become different. One can see that kaon loops and the anomalous VVP piece of the Lagrangian give the following transition amplitudes for neutral vector mesons (ideal fields

¹Of course, in the SU(2) symmetry limit, we have $\epsilon = 0$ and $q_\pi = Q_\pi$.

are understood, *i.e.* $\Pi_{\rho\rho}(s)$ should be understood as $\Pi_{\rho_I\rho_I}(s)$, for instance) :

$$\left\{ \begin{array}{ll} \Pi_{\rho\rho}(s) = g_{\rho\pi\pi}^2 \Pi'(s) + g_{\rho KK}^2 (\Pi_+(s) + \Pi_0(s)) & + [g_{\rho\omega\pi}^2 \Pi_{\omega\pi}(s) + \dots] \\ \Pi_{\omega\omega}(s) = g_{\omega KK}^2 (\Pi_+(s) + \Pi_0(s)) & + [g_{\rho\omega\pi}^2 \Pi_{\rho\pi}(s) + \dots] \\ \Pi_{\phi\phi}(s) = g_{\phi KK}^2 (\Pi_+(s) + \Pi_0(s)) & + [g_{\phi K^*K}^2 \Pi_{K^*+K^-}(s) + \dots] \\ \Pi_{\omega\phi}(s) = -g_{\omega KK} g_{\phi KK} (\Pi_+(s) + \Pi_0(s)) & + [2 g_{\phi K^*K} g_{\omega K^*K} (\Pi_{K^*\pm K^\mp}(s) + \Pi_{K^*0K^0}(s))] \\ \Pi_{\rho\omega}(s) = g_{\rho KK} g_{\omega KK} (\Pi_+(s) - \Pi_0(s)) & + [2 g_{\omega K^*K} g_{\rho K^*K} (\Pi_{K^*\pm K^\mp}(s) - \Pi_{K^*0K^0}(s))] \\ \Pi_{\rho\phi}(s) = -g_{\rho KK} g_{\phi KK} (\Pi_+(s) - \Pi_0(s)) & + [2 g_{\phi K^*K} g_{\rho K^*K} (\Pi_{K^*\pm K^\mp}(s) - \Pi_{K^*0K^0}(s))] \end{array} \right. \quad (8)$$

where we have defined $g_{\rho\pi\pi} = ag/2$, $g_{\rho KK} = g_{\omega KK} = ag/(4z_A)$ and $g_{\phi KK} = \sqrt{2}agz_V/(4z_A)$. $\Pi'(s)$ is the charged pion loop, while $\Pi_+(s)$ and $\Pi_0(s)$ are the charged and neutral kaon loops, each amputated from their couplings to vector mesons (*i.e.* loops carrying unit coupling constants).

The contributions of the anomalous loops have been displayed between square brackets. The anomalous FKTUY Lagrangian gives several terms contributing to the self-masses $\Pi_{\rho\rho}(s)$, $\Pi_{\omega\omega}(s)$ and $\Pi_{\phi\phi}(s)$. They can easily be constructed from the VVP Lagrangian given in Appendix 4 of [8] ; for these, we have displayed in Eqs. (8) only one representative of the full list which includes always $K^{*\pm}K^\mp$, $K^{*0}\bar{K}^0, \bar{K}^{*0}K^0$ and, depending on the particular self-mass considered, contributions from $\omega\pi^0$, $\rho\pi^0$, $\omega\eta$, $\rho\eta$, $\phi\eta$, $\omega\eta'$, $\rho\eta'$ or $\phi\eta'$ loops.

The anomalous parts of all transition amplitudes have been entirely displayed, as they exhibit an interesting correspondence with the non-anomalous contributions. We have identified to each other both $K^{*\pm}K^\mp$ loops on the one hand, and $K^{*0}\bar{K}^0$ with $\bar{K}^{*0}K^0$ on the other hand. Using the present set of notations, we have defined $g_{\rho K^*K} = g_{\omega K^*K} = \sqrt{z_T/z_A} G_{anom}/2$ and $g_{\phi K^*K} = G_{anom}/\sqrt{2z_A z_T}$ with [8] $G_{anom} = -3g^2/(8\pi^2 f_\pi)$. We have also denoted by $\Pi_{K^*\pm K^\mp}(s)$ and $\Pi_{K^*0K^0}(s)$ the amputated $K^{*\pm}K^\mp$ and $K^{*0}K^0$ loop functions.

In the exact Isospin Symmetry limit, one has $\Pi_+(s) = \Pi_0(s)$ on the one hand, and $\Pi_{K^*\pm K^\mp}(s) = \Pi_{K^*0K^0}(s)$ on the other hand. Then, all transition amplitudes vanish except for $\omega\phi$.

Therefore, within the HLS model, the $\omega\phi$ mixing is a natural feature generated by loop effects and not from some breaking mechanism. In contrast, the $\rho\phi$ mixing and the prominent $\rho\omega$ mixing are fully due to Isospin Symmetry breaking. Including the anomalous sector does not change the picture.

If one denotes by $\Pi_{+/0}(s)$ the amputated K^+K^0 loop and by $\Pi''(s)$ the amputated $\pi^\pm\pi^0$ loop, the charged ρ self-mass reads :

$$\Pi'_{\rho\rho}(s) = g_{\rho\pi\pi}^2 \Pi''(s) + 2g_{\rho KK}^2 \Pi_{+/0}(s) + [g_{\rho^\pm\omega\pi^\pm}^2 \Pi_{\omega\pi}(s) + \dots] \quad (9)$$

with a partial display of the anomalous loop contributions between the square brackets. This expression actually differs little from the neutral ρ self-mass ; indeed, the effect of having

different masses for neutral and charged particles in these loop computations is tiny. Of course, in the Isospin Symmetry limit, we have $\Pi'_{\rho\rho}(s) = \Pi_{\rho\rho}(s)$ and then the ρ^0 and ρ^\pm propagators (and their poles) coincide.

As is clear from the Lagrangian pieces given by Eqs. (1) and (2), the fields ρ^I , ω^I , ϕ^I as well as ρ^\pm are certainly mass eigenstates at tree level. This statement remains true for ρ^\pm at one-loop order as there is no transition loop from this meson to any other one. This is, however, clearly not true for ρ^I , ω^I , ϕ^I fields which undergo mixing with each other, as can be seen from Eqs. (8). Moreover, as for the self-masses, these transition amplitudes are invariant mass dependent as already noted [9, 16] !

This implies that physical fields associated with the ρ^0 , ω , ϕ mesons do not coincide with their ideal combinations as soon as one-loop corrections are considered. Moreover, the precise content of the physical fields in terms of ideal fields varies with s , or more precisely with the invariant mass flowing through the physical field under consideration. This does not prevent in standard approaches to use ρ^I , ω^I , ϕ^I in physical amplitudes [55]. However, as one loop effects have certainly to be considered even only in order to shift the vector meson poles off the physical region, they should legitimately be considered also for field mixing.

We raise the question of taking these loop effects properly into account and proceeding to the appropriate field redefinition to physical fields. In order to deal with this problem, let us define as effective Lagrangian the Lagrangian in Eq. (3) supplemented with the self-masses and transition terms occurring at one-loop order ; this turns out to replace the simple vector meson mass term in the HLS Lagrangian by ($m^2 = ag^2 f_\pi^2$):

$$\mathcal{L}_{mass} = \left\{ \begin{array}{l} \frac{1}{2} \{ [m^2 + \Pi_{\rho\rho}(s)]\rho_I^2 + [m^2 + \Pi_{\omega\omega}(s)]\omega_I^2 + [z_V m^2 + \Pi_{\phi\phi}(s)]\phi_I^2 \\ + 2\Pi_{\rho\omega}(s)\rho_I\omega_I + 2\Pi_{\rho\phi}(s)\rho_I\phi_I + 2\Pi_{\omega\phi}(s)\omega_I\phi_I \} + [m^2 + \Pi'_{\rho\rho}(s)]\rho^+\rho^- \end{array} \right. \quad (10)$$

The K^* mass term, which should also be modified correspondingly, is not shown as it plays no role in the present problem.

Even if anomalous VP contributions seem to play some role visible [51] (and, nevertheless, marginal) in pion form factor data, qualitatively their explicit form is really active only above the $\omega\pi$ threshold, which is the lowest mass VP threshold ; all others are far above the GeV region². Below the threshold, the main effect is due to their subtraction polynomials which can be well absorbed in the subtraction polynomials of the accompanying pion and kaon loops in order to put the poles of the ρ propagator at the place requested by the data.

Beside the (non-anomalous) pion and kaon loops, all transition amplitudes involve only K^*K loops, the threshold of which being at $\simeq 1.4$ GeV. This means that, besides their subtraction polynomials (minimally of degree 2), in the region below the GeV, their behavior [9] is a real logarithmic function (below $\sqrt{s} \simeq 0.4$ GeV) or an arctangent function ($0.4 \leq \sqrt{s} \leq 1.4$ GeV). This also can be numerically absorbed in a fitted subtraction polynomial.

Therefore, there is some sense in neglecting the (explicit) contributions of the anomalous loops, being understood that their effect is mostly concentrated in their subtraction polynomials. Moreover, as these come always together with pion and kaon loops, they can be accounted

²Their threshold masses are spread out between $\simeq 1.3$ GeV and $\simeq 2$ GeV.

for by simply letting the (free) pion and kaon loop subtraction polynomials to be second degree. Therefore, we shall neglect their (explicit) contributions, pointing at the appropriate places to their possible influence. Thus, from now on, the self–energies and transition amplitudes should be understood as Eqs. (8–9) amputated from the square bracket terms.

The use of the modified HLS Lagrangian has already been discussed in [9] where it was shown, for instance, that this method allows one to recover vector meson propagators usually derived through the Schwinger–Dyson resummation procedure, which turns out to sum up an infinite series, which is not necessarily convergent. However, we show shortly that introducing this modified mass term allows us to also account for the other transition effects which would be more difficult to derive from the Schwinger–Dyson resummation procedure (of course, this should be possible, merely tedious).

5 Mass Matrix Diagonalization And Physical Fields

As clear from Eq. (10), at one loop order, the mass term is diagonal in the charged vector meson sector and will not be discussed any longer. In the neutral vector meson sector, however, the mass matrix is not diagonal and the effective Lagrangian mass term is :

$$\mathcal{L}_{mass} = \frac{1}{2} \tilde{U} M^2(s) U \quad \text{with} \quad \tilde{U} = (\rho^I, \omega^I, \phi^I) \quad (11)$$

(the ideal fields being supposed real) and³ :

$$M^2(s) = \begin{pmatrix} m^2 + \Pi_{\pi\pi}(s) + \epsilon_2 & \epsilon_1 & -\mu\epsilon_1 \\ \epsilon_1 & m^2 + \epsilon_2 & -\mu\epsilon_2 \\ -\mu\epsilon_1 & -\mu\epsilon_2 & z_V m^2 + \mu^2 \epsilon_2 \end{pmatrix} \quad (\text{with } \mu \equiv z_V \sqrt{2}) \quad (12)$$

where we have defined :

$$\begin{cases} \epsilon_1 = g_{\rho KK}^2 (\Pi_+(s) - \Pi_0(s)) \\ \epsilon_2 = g_{\rho KK}^2 (\Pi_+(s) + \Pi_0(s)) \\ \Pi_{\pi\pi}(s) = g_{\rho\pi\pi}^2 \Pi'(s) \end{cases} \quad (13)$$

In the region where we work – invariant masses bounded essentially by the two–pion threshold and the ϕ mass–, the functions ϵ_1 and ϵ_2 are small and can be treated as perturbation parameters⁴ ; moreover, they are real for real s up to the two–kaon threshold region. In contrast, $\Pi_{\pi\pi}(s)$ is complex starting from the two–pion production threshold and is not expected to be small enough to be consistently treated as a perturbation parameter.

The physical vector meson mass eigenstates are the (s –dependent) eigenvectors of $M^2(s)$ and their masses are the corresponding eigenvalues, which are also s dependent ! Expressed

³For ease of reading, ϵ_1 ϵ_2 are not written with their explicit s dependence which is (or may be) understood throughout this paper.

⁴Actually, from their very expressions in terms of kaon and K^*K loops, one may expect ϵ_1 to be sensitively smaller than ϵ_2 in absolute magnitude.

this way, the notion of vector meson mass looks a little bit paradoxical, however, it is not really new : writing, as usual, the inverse ρ dressed propagator $D_V(s) = s - m^2 - \Pi_{\rho\rho}(s)$ can be interpreted as stating that the ρ mass squared is $m^2 + \Pi_{\rho\rho}(s)$ and includes an imaginary part. From a physics point of view, what is important is that the pole position associated with the ρ is always a zero of $s - m^2 - \Pi_{\rho\rho}(s)$ located on the unphysical sheet, close to the physical region⁵. The ρ pole position has been fitted long ago by [56] in e^+e^- data, and more recent fit values can be found in [57] ; this piece of information is actually highly model independent, in contrast with any other definition [53]. We shall revisit this issue with our fit results.

One may wonder about the hermiticity properties of the Lagrangian modified as proposed. As below the two-pion threshold, the loops defined above are all real for real s , we still indeed have $\mathcal{L}(s) = \mathcal{L}^\dagger(s)$, however, above this point, the hermiticity should be redefined as $\mathcal{L}(s) = \mathcal{L}^\dagger(s^*)$. This property known as hermitian analyticity [58] is fulfilled by our modified Lagrangian as it is already fulfilled by the loop functions.

Now, in order to define the physical ρ , ω , ϕ in terms of their ideal partners, one has to find the eigenstates of the squared mass matrix given by Eq. (12). Let us take advantage of the smallness of ϵ_1 and ϵ_2 to solve the problem perturbatively in order to avoid dealing with untractable expressions. Let us split up the squared mass matrix into two pieces and write it $M^2 = M_0^2 + \epsilon B$ with :

$$M_0^2 = \begin{pmatrix} m^2 + \Pi_{\pi\pi}(s) & 0 & 0 \\ 0 & m^2 + \epsilon_2 & 0 \\ 0 & 0 & z_V m^2 + \mu^2 \epsilon_2 \end{pmatrix}, \quad \epsilon B = \begin{pmatrix} \epsilon_2 & \epsilon_1 & -\mu\epsilon_1 \\ \epsilon_1 & 0 & -\mu\epsilon_2 \\ -\mu\epsilon_1 & -\mu\epsilon_2 & 0 \end{pmatrix} \quad (14)$$

In this splitting up, we have found appropriate to leave a part of the actual perturbation inside M_0^2 . In this way, we avoid to some extent the problem of having the unperturbed eigenvalues degenerated twice or even three times (when $z_V = 1$) for some values of s and some z_V . However, while assuming that $\Pi_{\pi\pi}(s)$, $\epsilon_1(s)$ and $\epsilon_2(s)$ vanish at origin, one cannot avoid to have a twofold degeneracy at $s \equiv 0$; this degeneracy is resolved as soon as s departs from zero by an arbitrary small quantity. This issue, which affects strictly the point $s = 0$ (where the exact solution is trivial !), does not raise any problem with our data which are all at $s \neq 0$, even if close to zero, as the NA7 spacelike form factor data [59]. Another solution to this problem would be that the HK masses for ρ_I and ω_I would be slightly different ; such a mechanism remains to be found⁶.

The unperturbed solution is then trivial, as the eigenvectors are the canonical ideal combinations of the neutral vector meson fields, with eigenvalues as can be read off the diagonal of M_0^2 . Then, one has to solve the following system for the perturbations δv_i and $\delta \lambda_i$:

⁵The upper lip of the physical region– the $s \geq 4m_\pi^2$ semi-axis – located on the physical sheet is topologically close to the lower lip in the unphysical sheet of the Riemann surface ; in contrast, the lower lip in the physical sheet is topologically far from the upper lip in the same sheet.

⁶ A way to get it would have been to use as breaking matrix $X_V = \text{Diag}(1 + \epsilon_u, 1 + \epsilon_d, \sqrt{z_V})$ instead of $X_V = \text{Diag}(1, 1, \sqrt{z_V})$ while computing \mathcal{L}_V (see Appendix C). This, actually, generates a mass difference between ρ^0 and ρ^\pm , but the HK mass for the ω meson remains equal to that of the ρ^0 meson.

$$\begin{cases} M_0^2 v_i = \lambda_i v_i \quad , \quad \tilde{v}_i \cdot v_i = 1, \quad (i = 1, 2, 3) \\ M^2(v_i + \delta v_i) = (\lambda_i + \delta \lambda_i)(v_i + \delta v_i) \quad , \quad \text{with[60]} \quad : \quad \tilde{v}_i \cdot \delta v_i = 0 \end{cases} \quad (15)$$

for each $i = (\rho, \omega, \phi) = (1, 2, 3)$. The solution can be written :

$$\begin{pmatrix} \rho^0 \\ \omega \\ \phi \end{pmatrix} = R(s) \begin{pmatrix} \rho_I^0 \\ \omega_I \\ \phi_I \end{pmatrix} \quad , \quad \begin{pmatrix} \rho_I^0 \\ \omega_I \\ \phi_I \end{pmatrix} = \tilde{R}(s) \begin{pmatrix} \rho^0 \\ \omega \\ \phi \end{pmatrix} \quad (16)$$

where (recall $\epsilon_i \equiv \epsilon_i(s)$ are analytic functions of s) :

$$R = \begin{pmatrix} 1 & \frac{\epsilon_1}{\Pi_{\pi\pi}(s) - \epsilon_2} & -\frac{\mu\epsilon_1}{(1 - z_V)m^2 + \Pi_{\pi\pi}(s) - \mu^2\epsilon_2} \\ -\frac{\epsilon_1}{\Pi_{\pi\pi}(s) - \epsilon_2} & 1 & -\frac{\mu\epsilon_2}{(1 - z_V)m^2 + (1 - \mu^2)\epsilon_2} \\ \frac{\mu\epsilon_1}{(1 - z_V)m^2 + \Pi_{\pi\pi}(s) - \mu^2\epsilon_2} & \frac{\mu\epsilon_2}{(1 - z_V)m^2 + (1 - \mu^2)\epsilon_2} & 1 \end{pmatrix} \quad (17)$$

The matrix R is orthogonal up to (neglected) second order terms (see Section 6 in [16]) and its elements are, actually, meromorphic functions of s ; this, for instance, means that one has to check that they do not develop singularities in the region of physical interest for our model.

On the other hand, one may wonder getting $\tilde{R}(s)$ with no complex conjugation in the field transformation ($\Pi_{\pi\pi}(s)$ is complex and fulfills the real analyticity condition $\Pi_{\pi\pi}(s) = \Pi_{\pi\pi}^*(s^*)$). This is due to the unitarity condition which writes [9] :

$$R(s + i\varepsilon) \cdot R^\dagger((s + i\varepsilon)^*) = R(s + i\varepsilon) \cdot R^\dagger(s - i\varepsilon) = 1 \quad (18)$$

for real s above threshold and $\varepsilon > 0$. The real analyticity property fulfilled by the matrix function R then gives $R^*(s - i\varepsilon) = R(s + i\varepsilon)$ and then Eq. (18) becomes :

$$R(s + i\varepsilon)\tilde{R}(s + i\varepsilon) = 1 \quad (19)$$

as can be checked directly with the R matrix above.

At first order, the corrections for eigenvalues are not changed with respect to their unperturbed values for ω and ϕ , while for ρ^0 , the first order correction is such that the eigenvalue is restored to $m^2 + \Pi_{\pi\pi}(s) + \epsilon_2$ and is formally identical to the ρ^\pm mass squared⁷. Therefore, in order to deal with the physical eigenstates ρ^0 , ω and ϕ , one has to introduce in the Lagrangian (3) above, the physical fields as defined by Eqs. (16) using Eqs. (17). For coupling constants, one has to perform exactly as explained in Section 6.3 of [16], using the R matrix above and, where appropriate, the ideal coupling constants given in Appendix D.

In order that this solution should be considered valid, one has to check that the non-diagonal elements of the matrix R are small compared to 1 in the whole range of application of our model. As they depend on fit parameters, this check can only be performed with the fit solution.

⁷For ρ^\pm , the mass squared value contains what was named $\Pi'_{\pi\pi}(s)$ and $\epsilon_2 \rightarrow 2g_{\rho KK}^2 \Pi_{+/0}(s)$.

We will not go into more details with expressing the full Lagrangian (3) in the basis of physical neutral vector meson fields, as formulae become really complicated (even if they can be readily and tediously written down). Let us only give the most interesting piece in terms of physical fields for illustrative purposes :

$$\frac{iag}{2} \rho^I \cdot \pi^- \overleftrightarrow{\partial} \pi^+ = \frac{iag}{2} \left[\rho^0 - \frac{\epsilon_1}{\Pi_{\pi\pi}(s) - \epsilon_2} \omega + \frac{\mu\epsilon_1}{(1 - z_V)m^2 + \Pi_{\pi\pi}(s) - \mu^2\epsilon_2} \phi \right] \cdot \pi^- \overleftrightarrow{\partial} \pi^+ \quad (20)$$

This clearly shows how kaon loops generate couplings of the *physical* ω and ϕ fields to $\pi^- \pi^+$ which vanish (with ϵ_1) in the Isospin Symmetry limit. As $\Pi_{\pi\pi}(s)$ has a large imaginary part, it is clear that the phase of the ω coupling compared with ρ^0 will be very large at the ρ peak. It should also be mentioned that the couplings shown here (and the matrix elements of R) have all a finite limit at $s = 0$ even if the loops individually vanish at $s = 0$ as the pseudoscalar pairs couple to conserved currents [52, 17].

The effects of the neglected loops could be briefly mentioned here. The most important effect in the expression for $R(s)$ (Eq. (17)) is on the denominators of $R_{12}(s)$ and $R_{21}(s)$ where the difference of the anomalous contributions to self-energies for the ρ and ω mesons will add to the present $\Pi_{\pi\pi}(s) - \epsilon_2(s)$; this could change a little bit the behaviour near $s = 0$ where all loops tend to zero.

As stated above, at first order in perturbations, the squared mass eigenvalues are the entries in the diagonal of $M^2(s)$ given in Eq. (12). For further use, let us also give the second order corrections to the eigenvalues (and thus to the running squared meson masses) :

$$\left\{ \begin{array}{l} \delta_2 \lambda_\rho = \frac{\epsilon_1^2}{\Pi_{\pi\pi}(s) - \epsilon_2} + \frac{\mu^2 \epsilon_1^2}{(1 - z_V)m^2 + \Pi_{\pi\pi}(s) - \mu^2 \epsilon_2} \\ \delta_2 \lambda_\omega = -\frac{\epsilon_1^2}{\Pi_{\pi\pi}(s) - \epsilon_2} + \frac{\mu^2 \epsilon_2^2}{(1 - z_V)m^2 + (1 - \mu^2)\epsilon_2} \\ \delta_2 \lambda_\phi = -\frac{\mu^2 \epsilon_1^2}{(1 - z_V)m^2 + \Pi_{\pi\pi}(s) - \mu^2 \epsilon_2} - \frac{\mu^2 \epsilon_2^2}{(1 - z_V)m^2 + (1 - \mu^2)\epsilon_2} \end{array} \right. \quad (21)$$

In the mass range where we are working (from the two-pion threshold to the ϕ mass), the mass eigenvalues at first order are real⁸ for the ω and ϕ , excluding a width. At second order, one observes that the pion loop generates an imaginary part to these mass eigenvalues. Let us remind the reader that, as the pole positions are the solutions of $s - \lambda_i(s) = 0$, one expects to find the ρ pole position close to the value found by [56]. However there is little chance that the ω width happens to carry the correct width value as this should be generated by the anomalous $\rho\pi$ loop with additional insertion of a pion loop on the ρ leg (or simply considering the dressed ρ propagator) or directly through a possible $\omega \rightarrow 3\pi \rightarrow \omega$ (double) loop. Finally, in the model we use, it is only at second order that a difference between neutral and charged ρ propagators (and thus masses) occurs and this is a net (and small) effect of the neutral vector meson mixing. This comes in addition to other sources of $\rho^0 - \rho^\pm$ mass difference (see footnote 6).

⁸Actually, near the accepted ϕ mass our perturbation parameters start to carry a tiny imaginary part.

6 Renormalization Conditions On Loops

With the approximations we choosed (neglecting the anomalous loop contributions), the loop expressions needed in order to construct the pion form factor $F_\pi(s)$ are only the $\pi\pi$ and $K\bar{K}$ loops. They can be computed by means of Dispersion Relations [52, 9] and can be derived without explicit integration, relying only on properties of analytic functions, especially the uniqueness property of the analytic continuation (see⁹ Appendix A in [9]).

From general principles, any loop $\Pi(s)$ is a so-called real analytic function (see the Section just above), the imaginary part of which is calculable using the Cutkosky rules, or in the simple case of single loops, using the partial width of the process $V \rightarrow PP'$. Indeed :

$$\text{Im}\Pi(s) = -\sqrt{s}\Gamma(s) \quad , \quad s > s_0 \quad (22)$$

where s is the pair invariant mass squared, s_0 the threshold mass squared of the pair and Γ the partial width of the decay. With this at hand, the full loop is the solution of the integral equation :

$$\Pi(s) = P_{n-1}(s) + \frac{s^n}{\pi} \int_{s_0}^{\infty} \frac{\text{Im}\Pi(z)}{z^n(z-s+i\epsilon)} dz \quad (23)$$

where $P_{n-1}(s)$ is a polynomial of degree $n-1$ with real coefficients and the integral runs over the right-hand cut (the physical region). These coefficients should be fixed by means of condition to be fulfilled by $\Pi(s)$, the so-called subtraction polynomial, which are nothing but renormalization conditions chosen externally and depending on the problem at hand.

A priori, the number of subtractions, *i.e.* the number of conditions can be arbitrary, however, in order that the integral in Eq. (23) converges, there is a minimal number of subtractions to perform : For PP' loops $n \geq 2$, for VP loops $n \geq 3$.

In the most general form of the pion form factor following from the Lagrangian (3), and using the modified one-loop mass term (10) the relevant basic loops are only the pion loops and the kaon loops. These imply that at least $n = 2$; however, the very existence of VP neglected loops implies that we are still minimally subtracting with using $n = 3$ for all loop functions in this paper. As discussed in Section 4, in this way, the subtraction polynomials carry some (unknown) information on the anomalous loop contribution.

Additionally, we request all polynomials $P_{n-1}(s)$ to fulfill $P_{n-1}(0) = 0$ reflecting this way current conservation [52, 17, 61, 51] when needed and an appropriate constraint otherwise¹⁰.

In usual approaches[61, 57, 52], the renormalization conditions are defined from start and, then, one leaves free other parameters like meson mass and width in order to accomodate the experimental data. As already done in [51], we proceed in the opposite way : as masses and couplings are fixed consistently in our Lagrangian, we leave free the subtraction polynomials in the loops $\Pi_{\pi\pi}(s)$, $\epsilon_1(s)$ and $\epsilon_2(s)$. This way allows the full data set to contribute to fixing the subtraction constants. The basic (pion and kaon) loop expressions are given in Appendix E and are used only subtracted once (in order that they vanish at $s = 0$) ; they are supplemented with

⁹In this reference, the loop expressions for equal mass pseudoscalar meson pairs and vector-pseudoscalar pairs are already given and are correct ; the function given for unequal mass pseudoscalar meson pairs is not correct as the contribution of the gauge term has been omitted ; we apologize for this error and correct for in the present paper (see Appendix E).

¹⁰In this case, the constant term in the ρ propagators is the squared (HK) mass occuring in the Lagrangian with no modification.

second degree polynomials vanishing at $s = 0$ and having coefficients to be fixed by fitting the data.

7 The Model Pion Form Factor In τ Decay

Introducing pion and kaon loop effects gives the ρ^\pm a self-mass, but, nevertheless, the ρ^\pm fields remain mass eigenstates. To stay consistent with using ρ self-mass, one has also to account for loops in the $W - \rho$ transition amplitude. In τ decay the relevant loop effects, while neglecting anomalous VP loops, are the $\pi^\pm\pi^0$ and K^0K^\pm contributions. Accounting for this modifies Eq. (4) to :

$$F_\pi^\tau(s) = \left[\left(1 - \frac{a}{2}\right) - F_\rho^\tau g_{\rho\pi\pi} \frac{1}{D_\rho(s)} \right] \quad (24)$$

with :

$$\begin{cases} F_\rho^\tau = f_\rho^\tau - \Pi_W(s) \\ D_\rho(s) = s - m^2 - \Pi'_{\rho\rho}(s) \\ f_\rho^\tau = agf_\pi^2 \end{cases} \quad (25)$$

where $m^2 = ag^2f_\pi^2$ and the charged ρ self-mass $\Pi'_{\rho\rho}(s)$ has been defined in Section 4 by Eq. (9) and used in Eq. (10) (recall we neglect VP loops). One should note that Eqs. (24) and (25) are not affected by any breaking mechanism. The diagrams contributing to the pion form factor in τ decays are sketched in Figure 1.

Let us denote for a moment the pion and kaon amputated (*i.e.* computed with unit coupling constants) loops by $\ell_\pi(s)$ and $\ell_K(s)$, assuming they are already subtracted once in order that they identically vanish at $s = 0$ (see Appendix E). The $W - \rho$ transition amplitude and the ρ^\pm self-energy occuring in the pion form factor have the following expressions in terms of pion and kaon amputated loops :

$$\begin{cases} \Pi_W(s) = g_{\rho\pi\pi} \left[\left(1 - \frac{a}{2}\right)\ell_\pi(s) + \frac{1}{2z_A^2} \left(z_A - \frac{a}{2}\right)\ell_K(s) \right] + P_W(s) \\ \Pi'_{\rho\rho}(s) = g_{\rho\pi\pi}^2 \left[\ell_\pi(s) + \frac{1}{2z_A^2}\ell_K(s) \right] + P_\rho(s) \end{cases} \quad (26)$$

where $g_{\rho\pi\pi} = ag/2$ and $P_W(s)$ and $P_\rho(s)$ being subtraction polynomials with real coefficients to be fixed by external renormalization. As emphasized in [51] and [61], the polynomials $P_W(s)$ and $P_\rho(s)$ can be chosen independent. Indeed, $\text{Im } \Pi_W(s)$ and $\text{Im } \Pi'_{\rho\rho}(s)$ are even not proportional as soon as SU(3) is broken ($z_A \neq 1$) ; moreover, the transition amplitude $\Pi_W(s)$ is non zero even if $a = 2$ as soon as SU(3) symmetry is broken. We choose to constrain $P_W(s)$ and $P_\rho(s)$ to be second degree and vanishing at $s = 0$, as discussed in Section 6.

For the sake of simplicity, we have also chosen to use the pseudoscalar meson loops assuming $m_{\pi^\pm} = m_{\pi^0}$ and $m_{K^\pm} = m_{K^0}$ after having checked that this is numerically harmless while dealing with all form factor data. Under this approximation¹¹, we have $\Pi'_{\rho\rho}(s) = \Pi_{\rho\rho}(s)$ (see

¹¹ This implies that the ρ^0 and ρ^\pm widths do not significantly differ. This statement is supported by the various experimental data collected in [54].

Eq. (8)) and all symmetry breaking effects due to the pion mass difference are concentrated in the phase space factors (see Eqs. (5–7)) for cross sections and partial widths where these have a sizable effect.

Therefore, one can rewrite Eqs. (26) under a form more appropriate for our fitting procedure :

$$\begin{cases} \Pi_W(s) = \frac{1}{g_{\rho\pi\pi}} \left[\left(1 - \frac{a}{2}\right) \Pi_{\pi\pi}^W(s) + \left(z_A - \frac{a}{2}\right) \epsilon_2(s) \right] \\ \Pi'_{\rho\rho}(s) = \Pi_{\rho\rho}(s) = \Pi_{\pi\pi}^\rho(s) + \epsilon_2(s) \end{cases} \quad (27)$$

where $\epsilon_2(s)$, already defined in Eqs. (13), carries its own subtraction polynomial, and having defined :

$$\begin{cases} \Pi_{\pi\pi}^W(s) \equiv g_{\rho\pi\pi}^2 \ell_\pi(s) + Q_W(s) \\ \Pi_{\pi\pi}^\rho(s) \equiv g_{\rho\pi\pi}^2 \ell_\pi(s) + Q_\rho(s) \end{cases} \quad (28)$$

where $Q_W(s)$ and $Q_\rho(s)$ are second degree polynomials¹² (with real coefficients to be fitted) and vanishing at origin. Possible correlations among them, if any, would be an outcome of the fit procedure and can be detected from inspecting the fit error covariance matrix. Finally, one can check that the condition $F_\pi^\tau(0) = 1$ is automatically fulfilled

Before turning to the pion form factor in e^+e^- annihilations, let us also remind the reader that the τ partial width expression in Eqs. (5) has to be further corrected for isospin breaking effects by multiplying it by¹³ $S_{EW} = 1.0232$ which accounts for short range radiative corrections [37]. Long range radiative corrections have been derived in [38, 39, 40] and come as a further factor $G_{EM}(s)$; another estimate taking into account additional Feynman diagrams can be found [41, 42] and a corresponding numerical parametrization of $G_{EM}(s)$ has been provided in [43]. This means that in all our fits we perform the substitution :

$$F_\pi^\tau(s) \implies S_{EW} G_{EM}(s) F_\pi^\tau(s) \quad (29)$$

which, therefore, accounts for all reported Isospin Symmetry breaking effects specific to the τ sector. Another isospin breaking effect might have to be considered, namely a $\rho^0 - \rho^\pm$ mass difference. This can be generated, for instance, by means of the mechanism sketched in footnote 6. It can be shown that this turns out to modify Eqs. (25) to :

$$\begin{cases} F_\rho^\tau = f_\rho^\tau - \Pi_W(s) \\ D_\rho(s) = s - m^2 - \delta m^2 - \Pi'_{\rho\rho}(s) \\ f_\rho^\tau = ag f_\pi^2 + \frac{\delta m^2}{g} \end{cases} \quad (30)$$

where δm^2 is left free. The modified Eqs.(30) allows Eq. (24) to still fulfill $F_\pi^\tau(0) = 1$ automatically¹⁴.

¹²We recall here, that these polynomials may account for the neglected anomalous loop effects not introduced explicitly.

¹³Actually, this numerical value has been derived for the pion final state ; in practical applications, it is usually assumed that this value holds also for the ρ final state – see for instance [34, 35, 36].

¹⁴Anticipating somewhat the fit results, a possible δm^2 can be detected on ALEPH data [31] (not on CLEO data [33]) and amounts to $\simeq -0.25 \text{ GeV}^2$. $f_\rho^\tau (\simeq 0.7 \text{ GeV}^2)$ is then increased by $0.5 \cdot 10^{-3} \text{ GeV}^2$, a quite negligible quantity.

8 The Model Pion Form Factor In e^+e^- Annihilations

In τ decay, the pion form factor, as just seen, is free from any vector meson mixing effect. Instead, the pion form factor $F_\pi^e(s)$ expression is sharply influenced by the vector meson mixing mechanism constructed explicitly in Section 5 which leads us to make the transformation from ideal to physical vector meson fields. After this transformation, we get from our effective Lagrangian the diagrams shown in Figure 1 and the corresponding expression :

$$F_\pi^e(s) = \left[\left(1 - \frac{a}{2}\right) - F_\rho^e(s)g_{\rho\pi\pi}\frac{1}{D_\rho(s)} - F_\omega^e(s)g_{\omega\pi\pi}\frac{1}{D_\omega(s)} - F_\phi^e(s)g_{\phi\pi\pi}\frac{1}{D_\phi(s)} \right] \quad (31)$$

where $D_\rho(s)$ (see Eq. (25) for its charged partner), $D_\omega(s)$ and $D_\phi(s)$ are the inverse propagators of the corresponding (physical) vector mesons. We have now :

$$D_\rho(s) = s - m^2 - \Pi_{\rho\rho}(s) \quad (32)$$

(recall that our assumptions on pseudoscalar meson masses implies $\Pi_{\rho\rho}(s) = \Pi'_{\rho\rho}(s)$, which reduces the number of free parameters in our model). The vector meson couplings to a pion pair after symmetry breaking are :

$$\left\{ \begin{array}{l} g_{\rho\pi\pi} = \frac{ag}{2} \\ g_{\omega\pi\pi} = -\frac{ag}{2} \frac{\epsilon_1}{\Pi_{\pi\pi}^\rho(s) - \epsilon_2} \\ g_{\phi\pi\pi} = \frac{ag}{2} \frac{\mu\epsilon_1}{(1 - z_V)m^2 + \Pi_{\pi\pi}^\rho(s) - \mu^2\epsilon_2} \end{array} \right. \quad (33)$$

where $\Pi_{\pi\pi}^\rho(s)$ has been defined in the previous Section. One should note that the quantity named $\Pi_{\pi\pi}(s)$ in the definition of the matrix $R(s)$ (see Section 5) coincides with the presently defined $\Pi_{\pi\pi}^\rho(s)$.

The quantities F_V^e can be written :

$$F_V^e(s) = f_V^e - \Pi_{V\gamma}(s) \quad (34)$$

Collecting the various couplings of the ideal fields suitably weighted by elements of the matrix transformation $R(s)$ (see Eq. (17)), we get :

$$\left\{ \begin{array}{l} f_\rho^e = agf_\pi^2 \left[1 + \frac{1}{3} \frac{\epsilon_1}{\Pi_{\pi\pi}^\rho(s) - \epsilon_2} + \frac{1}{3} \frac{\mu^2\epsilon_1}{(1 - z_V)m^2 + \Pi_{\pi\pi}^\rho(s) - \mu^2\epsilon_2} \right] \\ f_\omega^e = agf_\pi^2 \left[\frac{1}{3} - \frac{\epsilon_1}{\Pi_{\pi\pi}^\rho(s) - \epsilon_2} + \frac{1}{3} \frac{\mu^2\epsilon_2}{(1 - z_V)m^2 + (1 - \mu^2)\epsilon_2} \right] \\ f_\phi^e = agf_\pi^2 \left[-\frac{\mu}{3} + \frac{\mu\epsilon_1}{(1 - z_V)m^2 + \Pi_{\pi\pi}^\rho(s) - \mu^2\epsilon_2} + \frac{\mu}{3} \frac{\epsilon_2}{(1 - z_V)m^2 + (1 - \mu^2)\epsilon_2} \right] \end{array} \right. \quad (35)$$

and, keeping only the leading (first order) terms, the loop corrections $\Pi_{V\gamma}(s)$ (see the definitions in Eqs. (13), and the expression for μ in Eqs. (12)) are :

$$\left\{ \begin{array}{l} \Pi_{\rho^0\gamma}(s) = (1 - \frac{a}{2}) \frac{\Pi_{\pi\pi}^\gamma(s)}{g_{\rho\pi\pi}} + (z_A - \frac{a}{2} - b) \frac{\epsilon_1 + \epsilon_2}{g_{\rho\pi\pi}} + b \frac{\epsilon_2 - \epsilon_1}{g_{\rho\pi\pi}} \\ \Pi_{\omega\gamma}(s) = -(1 - \frac{a}{2}) \frac{\epsilon_1}{\Pi_{\pi\pi}^\rho(s) - \epsilon_2} \frac{\Pi_{\pi\pi}^\gamma(s)}{g_{\rho\pi\pi}} + (z_A - \frac{a}{2} - b) \frac{\epsilon_1 + \epsilon_2}{g_{\rho\pi\pi}} - b \frac{\epsilon_2 - \epsilon_1}{g_{\rho\pi\pi}} \\ \Pi_{\phi\gamma}(s) = (1 - \frac{a}{2}) \frac{\mu\epsilon_1}{(1 - z_V)m^2 + \Pi_{\pi\pi}^\rho(s) - \mu^2\epsilon_2} \frac{\Pi_{\pi\pi}^\gamma(s)}{g_{\rho\pi\pi}} - (z_A - \frac{a}{2} - b) \mu \frac{\epsilon_1 + \epsilon_2}{g_{\rho\pi\pi}} + b \mu \frac{\epsilon_2 - \epsilon_1}{g_{\rho\pi\pi}} \end{array} \right. \quad (36)$$

The first term for each transition loop is the pion loop contribution while the others are resp. the charged and neutral kaon loops. Of course, the functions occurring there are the same as for F_π^τ . We have denoted by $\Pi_{\pi\pi}^\gamma(s)$ the transition amplitude for $\gamma - \rho^I$, which is in correspondence with the $W - \rho^\pm$ transition amplitude introduced in the previous Section (see Eq. (28)). A priori, the subtraction polynomials of $\Pi_{\pi\pi}^\gamma(s)$ and $\Pi_{\pi\pi}^W(s)$ might be slightly different. However, in an attempt to reduce further the number of free parameters of the model, we assume that they coincide, which turns out to identify the amputated $W - \rho^\pm$ and $\gamma - \rho^I$ transition amplitudes¹⁵. We shall see that this assumption is well accepted by the data and, moreover, make clearer the switching to the τ form factor expression.

In addition to the explicit dependence of our model on the HLS basic parameters a, g , and on the breaking parameters x, z_A, z_V, z_T and δm^2 , there is a further dependence on subtraction parameters hidden inside $\Pi_{\pi\pi}^\rho(s), \Pi_{\pi\pi}^{W/\gamma}(s), \epsilon_1(s)$ and $\epsilon_2(s)$. Isospin symmetry breaking is reflected in having a non-zero $\epsilon_1(s)$ function. We have :

$$\left\{ \begin{array}{l} \Pi_{\pi\pi}^{W/\gamma}(s) = Q_0(s) + \ell_\pi(s) \\ \Pi_{\pi\pi}^\rho(s) = P_0(s) + \ell_\pi(s) \\ \epsilon_1(s) = P_-(s) + \ell_{K^\pm}(s) - \ell_{K^0}(s) \\ \epsilon_2(s) = P_+(s) + \ell_{K^\pm}(s) + \ell_{K^0}(s) \end{array} \right. \quad (37)$$

where $\ell_\pi(s), \ell_{K^\pm}(s)$ and $\ell_{K^0}(s)$ are now the *non-amputated* $\pi^+\pi^-$, K^+K^- and $K^0\bar{K}^0$ loops, subtracted in order that these loops vanish at the origin. The parameter polynomials $Q_0(s), P_0(s), P_-(s)$ and $P_+(s)$ are chosen to be second degree with zero constant terms in order to stay consistent with the Node theorem [17, 52].

One can check that $F_\pi^e(0) = 1 + \mathcal{O}(\epsilon_1^2)$, which could have been expected from having neglected terms of order greater than 1 in our diagonalization procedure¹⁶.

As the form factor data collected at the ϕ are not currently available, the last term in Eq. (31) could have been removed. However, in order to account for tails effects, we preferred keeping it and use a fixed width Breit-Wigner expression incorporating the Particle Data Group mass and width recommended values [54]. Due to the narrowness of the ω mass distribution, we also

¹⁵This is a nothing but a strong CVC assumption.

¹⁶ Actually, it depends on the (1,2) rotation matrix element : $1 + \mathcal{O}([R_{12}(s=0)]^2)$ and, numerically, the neglected term is $\simeq 1.5 \cdot 10^{-3}$.

have replaced in our fits the ω propagator by a fixed width Breit–Wigner constructed using the recommended mass and width from [54].

Let us also recall that Eq. (31) is our formula for the pion form factor $F_\pi^e(s)$ in both the spacelike and timelike regions. Indeed, for consistency, we will not remove the ω and ϕ meson contributions while going to negative s .

Using the first order correction to the ρ mass eigenvalue, the inverse propagator could have been written $D_{\rho^0}(s) = s - \lambda_\rho(s)$, as the leading order squared mass eigenvalue is :

$$\lambda_\rho(s) = m^2 + \Pi_{\pi\pi}^\rho(s) + \epsilon_2(s) \quad (38)$$

As far as e^+e^- data are concerned, we shall modify the eigenvalue expression to :

$$\lambda_\rho(s) = m^2 + \Pi_{\pi\pi}^\rho(s) + \epsilon_2(s) + \delta_2\lambda_\rho(s) \quad (39)$$

by adding the second order correction given in Eqs. (21). This does not add any more freedom in the model, but rather allows some check of the diagonalization method.

Therefore, the difference between $F_\pi^e(s)$ and $F_\pi^\tau(s)$ is solely concentrated in the coupling changes from ideal to physical fields given by the varying matrix $R(s)$ (see Eq.(17)) which only affects $F_\pi^e(s)$. Stated otherwise, modifying the function $F_\pi^\tau(s)$ in order to incorporate isospin breaking effects is strictly equivalent to using our expression for $F_\pi^e(s)$ directly, the factor $S_{EW}G_{EM}(s)$ being removed and δm^2 being made identically zero.

9 The Photon Vacuum Polarization

The raw data on the pion form factor $F_\pi^e(s)$ should be “undressed” by unfolding the contributions due to radiative corrections and to the photon vacuum polarization (VP) before any comparison with τ data (we refer the reader to [62, 63] for a comprehensive analysis of these factors and for previous references). Quite generally, available experimental data on $F_\pi^e(s)$ have already been unfolded from radiative corrections [27, 28, 29, 30, 59, 64]. All the data sets just referred to are not unfolded from photon vacuum polarization (VP) effects, except for KLOE data [64]. Therefore, one has to account for VP effects by including the corresponding factor when comparing a pion form factor model with experimental data. Traditionally (see for instance [65] and references quoted therein), this results in the change¹⁷ :

$$F_\pi^e(s) \longrightarrow (1 - \Pi_{VP}(s))F_\pi^e(s) \quad (40)$$

when comparing with most data sets.

The VP function $\Pi_{VP}(s)$ contains two parts. The first one is the sum of the leptonic loops which can be computed in closed form at leading order (In Appendix E, we recall the explicit form at order α and give its expression along the real s axis). The second part is the one particle irreducible hadronic contribution to the photon self–energy which is derived by means of a dispersion relation ; at low energy, where non–perturbative effects are dominant, this is

¹⁷ If one denotes by $\Sigma(s)$ the photon self–mass, the inverse photon propagator is given by $D_\gamma^{-1}(s) = s - \Sigma(s) = s(1 - \Sigma(s)/s)$. Therefore, compared with traditional notation, we have $\Pi_{VP}(s) = -\Sigma(s)/s$. This is not a problem but should be kept in mind.

estimated using the experimentally measured e^+e^- cross section (see, for instance, [65, 66, 67, 62]), while the high energy tail is calculated using perturbative QCD.

For our purpose, we use the sum of the leptonic VP as given in Appendix E for e^+e^- , $\mu^+\mu^-$ and $\tau^+\tau^-$ together with a numerical parametrization of the hadronic contribution. From the 2-pion threshold to the ϕ mass, we benefited from a parametrization¹⁸ provided by M. Davier [68]. Below the 2-pion threshold and down to $s = -0.25 \text{ GeV}^2$, we use instead a (real valued) parametrization provided by H. Burkhardt [69].

10 Decay Widths Of Light Mesons

In order to compute decay widths and fit data, one has to define the couplings allowing the decays of the light mesons involved. For the two-photon decays of the η and η' mesons, as well as for the radiative decays of the ρ^\pm and K^* 's mesons, the couplings defined after SU(3) and Nonet Symmetry breaking (see Eqs. (90) and (93)) are the couplings coming directly in the decay widths formulae (see Subsection D.3) and do not depend on further Isospin Symmetry breaking effects than mass values in phase space factors.

The isospin breaking procedure we presented plays only for the ρ^0 , ω and ϕ mesons. In order to compute the leptonic decays of these, one has to use the full couplings $F_V^e(s)$ as given by Eqs. (35) and (36) in Eq. (97) and computed at the appropriate vector meson masses $F_V^e(m_V^2)$. As the loop functions are slowly varying, one can choose m_V as the Higgs-Kibble masses occurring in the Lagrangian (see Eq.(1)), which moreover simplifies the fitting procedure.

For the other (radiative decay) coupling constants one has to combine the ideal coupling constants (given in Appendix D) using the transformation $R(s)$ to derive the physical coupling constants, as was described in [16] ; the context, compared to [16], slightly differs due to the fact that, now, the mixing parameters are functions to be computed at the appropriate m_V^2 values in order to go to the vector meson mass shell.

Traditionally, the ρ^0 is decoupled from mixing and treated as the ρ^\pm and mixing effects are only considered in the $(\omega - \phi)$ sector. Additionally, it is usual to treat the $(\omega - \phi)$ mixing angle as a constant to be fit (see [4, 5, 8, 9, 16] and the references quoted therein). The approach in the present study is different : One considers a full $(\rho^0 - \omega - \phi)$ mixing scheme (as in [16]), however – for the first time – the mixing parameters are functionally related and the same functions have to be computed at each vector meson mass. For instance, the $(\omega - \phi)$ mixing “angle” has not the same numerical value at the ω mass and at the ϕ mass. This only reflects that the mixing is actually invariant mass dependent. When, as for $\omega \rightarrow \pi\pi$ [50] and $\phi \rightarrow \pi\pi$ [70, 71] data exist on the phase of the coupling constant, these phases can be introduced in the fit with the same functions, the moduli of which determine the branching fractions.

11 The Full Set of Data Submitted To Fit

In order to work out the model presented in the Sections above, we use several kinds of data sets. In this Section, we list them and give some details on the way they are dealt with in our fit procedure.

¹⁸This parametrization neglects the (very small) imaginary part of the hadronic VP contribution.

11.1 Partial Width Decays

As a general statement, the decay data submitted to fits have been chosen as the so-called “fit” values recommended by the Particle Data Group (PDG) in the latest (2006) issue of the Review of Particle Properties [54].

This covers, with no exception, the leptonic decay widths of the ρ^0 , ω and ϕ mesons, the two-photon decay widths of the η and η' mesons and the $\pi^+\pi^-$ partial width of the ϕ meson. There are two measurements of the phase of the coupling constant $g_{\phi\pi^+\pi^-}$ reported in the literature ; the older one [70] is $\psi_\phi = -42^\circ \pm 13^\circ$ and more recently [71] $\psi_\phi = -34^\circ \pm 4^\circ \pm 3^\circ$. Summing up in quadrature the errors, we choose as reference value in our fits $\psi_\phi = -34^\circ \pm 5^\circ$. We could have chosen to include in our fits the 2-photon decay width of the π^0 ; however, we preferred replacing this piece of information by the pion decay constant value $f_\pi = 92.42$ MeV and did not let it vary, as this is supposed to carry a very small error¹⁹ [54].

The RPP pieces of information [54] concerning the ρ and ω decay width to $\pi^+\pi^-$ and the partial width $\rho^0 \rightarrow e^+e^-$ are not considered as data to be submitted to fits, as they have all been extracted from fitting the same pion form factor timelike data which we are included in our fit procedure (see the Subsection below) ; this does not prevent us from comparing our results to the RPP available information. This is also true for the relative phase of the couplings $g_{\omega\pi^+\pi^-}$ to $g_{\rho\pi^+\pi^-}$ (the so-called Orsay phase) which has been measured [50] with the result $\psi_\omega = 104.7^\circ \pm 4.1^\circ$.

Instead, it is quite legitimate to include the $\omega \rightarrow e^+e^-$ partial width in our fit procedure as, even if this mode could have been marginally influenced by the pion form factor data, it is mostly extracted from $e^+e^- \rightarrow \pi^+\pi^-\pi^0$ data [54]. As the pion form factor spectrum around the ϕ mass is not currently available, the $\phi \rightarrow e^+e^-$ partial width is quite legitimately included in our fit data set.

In the present work, as in our previous works on the same subject [8, 9, 16], we do not intend to use the decay widths $K^* \rightarrow K\pi$. Actually, as for the width $\rho \rightarrow \pi\pi$ which is inherently fitted with the pion form factor, the choice of the mass value for a very broad object makes the extraction of coupling constants a delicate matter. It should be more appropriately discussed with the $K\pi$ form factor in τ decays when the corresponding data will become available.

The data on the two kaon partial widths of the ϕ meson are also left outside our fit procedure. In a previous work of some of us [16], as in other works [72] the issue of accomodating the $\phi \rightarrow K^+K^-$ partial width was raised. A recent work [73] claims that the ratio of these partial widths can be accomodated by introducing corrections to the decay widths which increase both partial widths as derived from the matrix elements of the transitions. As then, the problem may affect both the $\phi \rightarrow K^+K^-$ and the $\phi \rightarrow K^0\bar{K}^0$ partial widths, we have preferred leaving both modes outside the fit procedure. We will discuss this issue below in a devoted Subsection.

We also use all radiative decay partial widths of light flavor mesons of the form $V \rightarrow P\gamma$ or $P \rightarrow V\gamma$. As a general rule, we chose as reference data the “fit” values recommended by the PDG as given in the latest RPP issue [54]. There are two exceptions to this statement : the partial widths for $\omega \rightarrow \eta\gamma$ and $\omega \rightarrow \pi^0\gamma$.

Indeed, as already noted in [16], there is some difficulty in accomodating the present $\omega \rightarrow \eta\gamma$ “fit” branching fraction $((4.9 \pm 0.5) 10^{-4})$ while the so-called “average” value [54] $((6.3 \pm 1.3) 10^{-4})$ is much better accepted by our model fitting.

¹⁹Possible Isospin Symmetry breaking effects might have to be considered.

On the other hand, the new “fit” value for the branching fraction $\omega \rightarrow \pi^0\gamma$ ($(8.90 \pm 0.25) 10^{-2}$) is also hard to accommodate in our model. We show that the previous PDG “fit” value ($[8.50 \pm 0.50] 10^{-2}$) seems in better consistency with the rest of the data we handle ; we also checked that the value ($[8.39 \pm 0.25] 10^{-2}$) produced by a fit [74] performed in a completely different context was also well accepted, pointing to a possible overestimate of the central value for this mode²⁰. The questions raised by the values of these two decay widths will be discussed at the appropriate place below.

Finally, we also introduce in the fit procedure the ratio of the kaon to pion decay constants as they are reported in the latest RPP [54]. This actually coincides with our SU(3) breaking parameter ($z_A = [f_K/f_\pi]^2$).

11.2 Timelike Form Factor Data in e^+e^- Annihilations

Beside the decay data listed just above, we have included in our fit all data on the pion form factor collected in e^+e^- annihilations by the OLYA and CMD Collaborations as tabulated in [75] and the DM1 data [76] collected at ACO (Orsay). These data will be referred to globally as “old timelike data”. When included into a χ^2 expression, systematic errors have to be combined with the published statistical errors ; they have been first added in quadrature to the statistical errors for OLYA data (4%) and CMD (2%) following expert advice [77]. However, for sake of consistency with the new data discussed just below, we preferred extracting the correlated part of the systematic errors, estimated [77] to 1% and have performed the same treatment as for the new data (see just below). The accuracy of the DM1 data making the influence of this data set marginal, we did not add any further contribution to the published errors. We only use the data points located below the ϕ meson mass in order to avoid being sensitive to higher mass vector mesons, not included in the present model.

Four additional data sets have been collected later at Novosibirsk on the VEPP2M ring. The first one, covering the region from about 600 to 960 MeV, collected by the CMD2 collaboration [78] and recently corrected [27], is claimed to have the lowest systematic error (0.6%) ever reached in this field.

CMD2 has collected in 1998 and recently published two additional data sets, one [28] covering the energy region from 600 to 970 MeV is claimed to reach a systematic error of 0.8%, and a second set [29] covering the threshold region (from 370 to 520 MeV) with an estimated systematic error of 0.7%. On the other hand, the SND collaboration has recently published [30] a new data set covering the invariant mass region from 370 to 970 MeV with a systematic error of 1.3% over the whole data set except for the very low mass region where the (first) 2 points carry a systematic error of 3.2%.

Concerning these four data sets (which will be referred to globally as “new timelike data”), we could, as per usual, add in quadrature the systematic and statistical errors and then get a diagonal error matrix which can be used in χ^2 fits in a trivial way.

However, an important part of the systematic uncertainties for these data sets is expected to be a common global scale uncertainty [77] which has been estimated to 0.4% and generates bin to bin correlated errors. In principle, one should take the latter information into account in

²⁰Looking at [54], the role of the data and analyses for the $e^+e^- \rightarrow \pi^0\gamma$ process itself to get the “fit” value for the partial width $\omega \rightarrow \pi^0\gamma$ is unclear.

fits ; this implies dealing with systematic and statistical errors in a way slightly more elaborate than simply adding in quadrature statistical and systematic errors.

Firstly, the (bin per bin) uncorrelated part of the systematic error is derived by subtracting in quadrature 0.4% from the already quoted systematic errors. This uncorrelated part of the systematic error (*i.e.* $\sqrt{\sigma_{\text{sys.}}^2 - (0.4\%)^2}$, depending on the data set considered) can certainly be added in quadrature to the statistical error bin per bin, giving a combined standard deviation named σ_i for the measurement m_i in the energy bin i ; the σ_i are uncorrelated errors and define a diagonal error matrix. The question then becomes how to redefine the full covariance matrix for each experiment, being understood that the quantity to be compared with the theoretical pion form factor f_i^{th} for each energy bin i is related with the measurement m_i by :

$$f_i^{\text{th}} \longmapsto m'_i = (1 + \delta\lambda)m_i \quad (41)$$

where $\delta\lambda$ is considered a gaussian random variable with zero mean and standard deviation $\lambda = 0.4 \times 10^{-2}$. With this assumption it is possible to model reasonably well the covariance matrix, which is no longer diagonal.

Secondly, one has to treat these correlations. The quoted correlated systematic error is a conservative estimate of the accuracy of radiative corrections performed on the four data sets using the same Monte Carlo generator [77]. Therefore, the fit parameter introduced in order to optimize the absolute scale should be the same for all data sets ; in statistical terms, this fit parameter value can be considered as only *one* sampling of the gaussian random variable $\delta\lambda$ defined just above and should be valid for all data collected by the CMD2 and SND Collaborations.

If the correlated part of the systematic error was strictly zero, the error covariance matrix for each data set would simply be given by :

$$V_{ij} = \sigma_i^2 \delta_{ij} \quad (42)$$

where i and j label energy bins in the data set. In the case of existing correlations, having defined σ_i as the sum in quadrature of the statistical error and the uncorrelated systematic error in the i^{th} bin , the error covariance matrix elements can be written :

$$V_{ij} = \sum_{k,l} M_{ik} W_{kl} M_{lj} \quad , \quad (43)$$

where :

$$M_{ij} = \sigma_i \delta_{ij} \quad \text{and} \quad , W_{ij} = \delta_{ij} + \lambda^2 e_i e_j \quad (44)$$

$\lambda = 0.4 \cdot 10^{-2}$ being the standard deviation of the correlated error function and the vector e being defined by its components on the various energy bins i as the ratio of the corresponding measurement to its uncorrelated error :

$$e_i = m_i / \sigma_i \quad , \quad \forall i \in [1, \dots, n_{\text{measur.}}] \quad (45)$$

However, what is relevant for χ^2 fitting, is not so much the covariance matrix Eq. (44) as its inverse. It happens that the matrix W can be inverted in closed form :

$$W_{ij}^{-1} = \delta_{ij} - \mu^2 e_i e_j \quad \text{and} \quad \mu^2 = \frac{\lambda^2}{1 + \lambda^2 \sum_{i=1}^{n_{\text{measur.}}} e_i^2} \quad (46)$$

and the full error covariance matrix is also inverted in closed form :

$$V_{ij}^{-1} = \sum_{k,l} M_{ik}^{-1} W_{kl}^{-1} M_{lj}^{-1} \quad . \quad (47)$$

This is, together with the measured values, the main ingredient of the χ^2 calculation which will be performed with the four new timelike data sets. Finally, while fitting the new data, a term has to be added to the χ^2 ; naming λ_{fit} the fit parameter for the global scale common to all the new Novosibirsk data sets and $\lambda_{exp} = 0.4 \times 10^{-2}$ the scale uncertainty on the measured form factor estimated by the experiments, this additional contribution to the χ^2 is simply $[\lambda_{fit}/\lambda_{exp}]^2$.

Mutatis mutandis, the same method has been applied to the old Novosibirsk data sets using another global scaling factor λ'_{fit} with $\lambda'_{exp} = 1.0 \cdot 10^{-2}$, as recommended by informed people [77], and the same procedure to construct the final inverse covariance matrix to be used in fits.

A new data set has been recently collected by the KLOE collaboration [64] using the Radiative Return Method. Existing analyses (see, for instance, the short account in [26]) however report a disagreement between KLOE data and the recently collected data sets at Novosibirsk due to some systematic effect presently not understood. A recent study of a parametrization of the pion factor [79] argues about a possible systematic energy shift in the data which would be detected by fitting the ω mass. In view of this unclear situation, we have found it appropriate to postpone including the existing KLOE set among our fitting data samples.

11.3 Spacelike Pion Form Factor Data

In order to further constrain the pion form factor in the timelike region, information on the close spacelike region is valuable. Reliable data on the pion form factor in the negative s region are somewhat old [59, 80]. The Fermilab data set [80] consists of 14 measurements of $|F_\pi(s)|^2$ between $s = -0.039 \text{ GeV}^2/c$ and $s = -0.092 \text{ GeV}^2/c$ with $2 \div 7 \%$ statistical error ; an estimated systematic error (overall normalization) of 1 % is provided in [80]. The NA7 data cover the region between $s = -0.015 \text{ GeV}^2/c$ and $s = -0.253 \text{ GeV}^2$ with 45 measurement points and an overall statistical precision better than those of the Fermilab data. However, NA7 data are also claimed to undergo an overall scale error of 0.9 % rms.

One will use these two data sets and treat these correlated systematic errors exactly as explained above for the timelike pion form factor data.

Data have more recently been collected at the Jefferson Accelerator Facility [81] and reanalyzed in order to optimize the extraction of the pion form factor data in the region for s between -0.60 and -1.60 GeV^2 with a quoted uncertainty of about 10%. No precise information about the correlated–uncorrelated sharing of the systematic error is reported. Including these data involves some more studies and modelling which goes beyond the main task of the present work, namely, to check the consistency of e^+e^- and τ data.

11.4 Phase Pion Form Factor Data

There are several data sets available which provide measurements of the isospin 1 part of the $\pi\pi$ amplitude phase shift. The most precise set is the CERN/Munich one [82], but the older Fermilab data set [83] is still useful. However, systematic errors here are not completely controlled. Moreover, as we neglect vertex corrections at the $\pi\pi$ vertex and t -channel resonance

exchanges which may carry some unknown imaginary part, one cannot draw firm conclusions when comparing the phase information of our pion form factor with phase shift data. Therefore, we have left these data outside our fitting procedure and limited ourselves to simply compare graphically with the phase of our pion form factor.

11.5 Pion Form Factor Data From τ decays

There are presently three available data sets concerning the pion form factor. These have been collected at LEP by ALEPH [31] and OPAL [32] Collaborations and at much lower energy by the CLEO Collaboration [33].

The data provided by the ALEPH Collaboration [31] include the covariance matrices for statistical and systematic errors which should be added before inversion in order to be used in a χ^2 minimization. There is some disagreement between ALEPH [31] and OPAL data [32] which has led most works to discard this data set ; we shall do likewise.

The CLEO data [33] on the pion form factor are also provided with their full error matrix, but one that accounts for statistical errors only. Statistical errors dominate most of the systematic uncertainties except for those contributing to the absolute energy scale for determining \sqrt{s} [84]. These were quantified by CLEO as a systematic uncertainty on the value of the ρ^\pm mass obtained in their fits to form factor models, estimated to be 0.9 MeV. This error, not accounted for by the CLEO error covariance matrix, is a systematic error which correlates the various bin energy values. In contrast with the Novosibirsk data, it is not easy to rigorously account for this correlated systematic error²¹. As an approximation we allow the central bin \sqrt{s} value to vary by some ε MeV and add $[\varepsilon/0.9]^2$ to the CLEO data χ^2 . This approach provides a simple and reasonable way to deal with the data and errors [84].

In order to stay consistent with our dealing with e^+e^- data, we have limited our fitting range to the ϕ mass and then removed all points above $s = 0.9 \text{ GeV}^2$. This leaves us with 33 measurements from ALEPH and 25 from CLEO, largely unaffected by higher mass vector meson effects, as will be checked.

12 The Main Global Fit To The Data Sets

12.1 General Comments About The Fits

Our global model has seven parameters carrying an obvious physical meaning :

- The universal vector coupling g ,
- the SU(3) breaking parameter z_A (expected to coincide with $[f_K/f_\pi]^2$ within errors),
- the Nonet Symmetry breaking parameter x ,
- the basic HLS parameter a (expected close to 2),
- the parameter z_V which mostly governs the mass difference between the $\rho^0 - \omega$ system and the ϕ meson but also plays a role in some coupling constants,

²¹It affects the position of the measurement, not the measured value itself.

- z_T which affects only the K^* radiative decay sector in the data used,
- and, finally, the $\rho^0 - \rho^\pm$ squared mass shift δm^2 .

These have been already fitted in isolation in related previous works [8, 9, 16, 51, 85] and we expect to find fit values close to the already published ones. Within our approach, the pseudoscalar mixing angle is not free, but is derived from the previous parameters using Eq. (95) and is expected close to -10.5° degrees from previous fits [8, 16]. This has been found in perfect agreement with the two-angle formulation [11] expressed in the framework of Extended Chiral Perturbation Theory [13, 14].

Beside these parameters carrying a clear physical meaning, one has the subtraction polynomial of the pion loop (mostly associated with the ρ meson self-energy), assumed to be written $c_1 s + c_2 s^2$ with c_1 and c_2 to be fitted. Two additional subtraction polynomials carrying the same form and associated with the difference ($\epsilon_1(s)$) and the sum ($\epsilon_2(s)$) of the $K^+ K^-$ and $K^0 \bar{K}^0$ loops introduce 4 more parameters²² to be fit. Finally, two more subtraction parameters come from the specific subtraction of the $\gamma\rho$ (or $W\rho^\pm$) transition amplitude. We thus end up with 15 parameters²³ for a number of data of 344 (18 decay modes, 127 data points from the new timelike pion form factor data, 82 from the old timelike pion form factor data, 59 data points in the spacelike region, 33 data points coming from ALEPH data and 25 from CLEO).

In addition to these parameters which define our model, we have to account for correlated systematic errors in several experiments by fitting the corresponding scale factors and using the experimental pieces of information as constraints. These additional degrees of freedom are therefore exactly compensated in number by the constraints. This covers the global scale factor of the former Novosibirsk experiments as reported in [75] (estimated to 1.0% r.m.s.), the global scale factor of the new Novosibirsk experiments as reported in [27, 28, 29, 30] (estimated to 0.4% r.m.s), the scale factor for the NA7 [59] and Fermilab [80] data (estimated respectively to 0.9 % and 1.0 % r.m.s.). Finally, the CLEO data set is expected to carry a systematic energy shift which will be fitted and is expected [84] of the order 0.9 MeV.

We have performed various kinds of fits. In all of them, as detailed above, we have introduced all usual symmetry breaking effects in the value of meson masses, the prominent effects of $\rho^0 - \omega - \phi$ mixing (for e^+e^- data) and the long- and short-range [37] IB correction factors (for the τ spectra). We have observed that the two proposed ways to account for long range corrections by either of [38, 39, 40] and [41, 42, 43] approaches provide quite similar effects and that, on the basis of probabilities, the difference was never observed significant in any of our fits. For definiteness, we choose to use the function of [38, 39, 40] for all results presented here.

On the other hand, it was useful to check the effect of excluding the photon vacuum polarisation (VP), by fixing the corresponding factor to 1. We also found it of interest to perform fits by excluding either the τ data or the spacelike data ; this gives information on the effect of these on the global fit quality and on the stability of the fit parameter values. Finally, it has also

²²We note that we approximate the $K^\pm K^0$ loop by the average value of the $K^+ K^-$ and $K^0 \bar{K}^0$ loops in order to limit the number of free parameters.

²³It will be emphasized later on that one among these subtraction parameters does not influence the fit and can be safely fixed to zero.

Data Set	Without VP	With Vacuum polarisation (VP)			
# (data + conditions)	Full Fit	Full Fit	No τ	No Spacelike	No ρ mass shift
Decays (18+1)	11.46	11.13	11.52	11.48	11.25
New Timelike (127+1)	132.81	128.10	122.02	125.76	132.23
Old Timelike (82+1)	62.22	59.05	54.68	55.20	60.15
Spacelike(59+2)	68.53	65.70	55.20	89.82/(59)	65.13
τ ALEPH (33)	27.06	23.86	42.27/(33)	20.80	24.48
τ CLEO (25+1)	25.53	26.06	26.16/(25)	29.72	28.55
χ^2/dof	327.40/331	313.83/331	257.73/274	238.81/272	321.75/332
Probability	54.6 %	74.3 %	75.2%	92.7%	64.7%

Table 1: The first column lists the subset named as defined in the text together with its number of measurements and condition(s) if any. Each row displays the corresponding χ^2 contribution under the condition quoted in the title of the data column. The last row gives the total χ^2 /(number of degree of freedom), followed by the fit probability. Information written bold-face indicates the χ^2 distance of the fit function to a data set left outside from the fit procedure together with its number of data points. In this case, the condition parameter associated with the corresponding data set (scale or mass shift) is fixed to the value returned by the full global fit reported in the second data column and given in Table 2.

been of interest to check the mass shift effect between the ρ^0 and the ρ^\pm mesons, by fixing the corresponding parameter to zero. The results summarizing the statistical qualities are gathered in Table 1 and the fit parameter values can be found in the appropriate data column of Tables 2 and 3.

12.2 Discussion Of The Fit Information

Table 1 reports the statistical information about our fits under various conditions. As a general statement, the fit quality is always either reasonable or very good as clear from the last row in Table 1.

Parameter	Full Fit	No τ	No Spacelike	No ρ mass shift
Scale New Timelike	1.006 ± 0.004	1.000 ± 0.003	1.004 ± 0.003	1.007 ± 0.003
Scale Old Timelike	1.012 ± 0.009	1.010 ± 0.009	1.011 ± 0.009	1.013 ± 0.009
Scale NA7	1.008 ± 0.007	1.008 ± 0.007	1.008	1.011 ± 0.006
Scale Fermilab	1.006 ± 0.007	1.006 ± 0.008	1.006	1.008 ± 0.007
CLEO Shift (MeV)	0.40 ± 0.52	0.40	0.36 ± 0.52	1.37 ± 0.39
δm^2 (10^2 GeV 2)	-0.268 ± 0.095	-0.268	-0.285 ± 0.096	0
a	2.303 ± 0.012	2.297 ± 0.012	2.292 ± 0.012	2.306 ± 0.012
g	5.576 ± 0.015	5.578 ± 0.017	5.597 ± 0.016	5.573 ± 0.015
x	0.903 ± 0.013	0.902 ± 0.013	0.902 ± 0.013	0.903 ± 0.013
z_A	1.503 ± 0.010	1.505 ± 0.010	1.507 ± 0.010	1.503 ± 0.010
z_V	1.459 ± 0.014	1.466 ± 0.014	1.453 ± 0.014	1.460 ± 0.014
z_T	1.246 ± 0.049	1.245 ± 0.049	1.243 ± 0.049	1.246 ± 0.049

Table 2: Parameter values in fits performed including photon VP. Three data columns are associated with all data (first data column), removing only the τ data (second data column) and removing only the spacelike data (third data column). The last data column reports parameter values returned while fitting all data sets by fixing $\delta m^2 \equiv 0$. Information written boldface displays values not allowed to vary in the fit procedure.

As a first remark, neglecting to account for photon vacuum polarization effects does not end up with a dramatic failure ; however, there is a general improvement while introducing the corresponding function. The negligible degradation observed for CLEO data is entirely produced by the value found for the CLEO mass shift parameter (which contributes to the χ^2 , as explained above) will be discussed below. This statement clearly follows from comparing the two data columns named “full fit” in Table 1.

The gain in χ^2 , while including the photon VP, is 13.5 units without any additional parameter freedom and, in terms of fit probability, one wins 20 %. Therefore, one may conclude that the data description prefers including explicitly the photon VP while fitting the e^+e^- data. Under realistic conditions, the fit probability is then always of the order 75 % or better.

The fits have always been performed using the package MINUIT [86] and the errors quoted are always the improved errors returned by the routine MINOS. This has allowed us to check that the minimum χ^2 was always locally parabolic, which provides symmetric errors.

Tables 2 and 3 display the fit parameter values as returned by the fit under the conditions defined by the various titles of the data columns. We only provide results including the photon VP inside the definition of the pion form factor for e^+e^- annihilations. Comparing the various data columns in Tables 2 and 3, clearly illustrates that the fit parameter values stay close together, and generally widely within their (MINOS) errors. The single exception is obtained while removing the spacelike data ; in this case, the coefficients for the subtraction polynomial $\Pi_{\pi\pi}^{W/\gamma}$ differ significantly from all other cases. This could have influenced the predicted values for the $V \rightarrow e^+e^-$ partial widths ; however, we have checked that this is not the case numerically.

Among the fit parameter values given in Table 2, the most interesting are clearly the fit values for the scale factors which are nicely consistent with the corresponding experimental information recalled at the beginning of this Section.

The single exception is the CLEO global invariant mass shift which is found consistent with zero. Taking into account the way the 0.9 MeV expected shift has been determined²⁴ [84], this information is interesting. It will be rediscussed when examining the fit residuals which provides an additional important information.

The third and fourth data columns in Table 1 provide further information :

i/ Removing the τ data from the fit sample, one can construct the *predicted* distributions for the ALEPH and CLEO data which are fully derived using our model together with only light meson partial width decays and the e^+e^- data. The predicted χ^2 distance to CLEO data is practically unchanged with respect to fitting with them, while the prediction for the ALEPH data is not as good even if it remains reasonable²⁵. This indicates that CLEO data are in so nice agreement with predictions (especially e^+e^- data) that they do not really constrain the fit ! In this respect, ALEPH data, while introduced in the fit data set, clearly influence the procedure.

ii/ Removing only the spacelike data looks a little bit more appealing. The χ^2 distance of the NA7 and Fermilab data altogether is degraded by $\simeq 24$ units, and the fit probability grows from 74 % to 93%, pointing to some slight difficulty in accomodating these data sets. However, this result is by no way problematic enough to either force us to remove the spacelike data or to deeply question their quality.

In order to compute the χ^2 distance of the data samples left out from fit, one had to choose either the CLEO energy shift or the NA7 and Fermilab scale factors, as they can no longer be fit. We choose to fix them to their fit value as given in the first data column in Table 2.

Tables 2 and 3, mostly illustrate that, whatever the fit conditions examined, the location of the minimum in the fit parameter space remains practically unchanged²⁶. The results obtained by removing τ data from the fit sample, those by removing the spacelike data, those corresponding to removing the photon VP function (not shown) ... are consistent with each other. Let us note that the fit parameter c_1 in the $\Pi_{\pi\pi}^\rho(s)$ function (*i.e.* essentially the ρ self-mass function) has been fixed to zero, as it was not found to sensitively affect the fits in the energy range we are fitting. The various fit conditions only affect the fit quality which varies from

²⁴ In order to make consistent the ρ parameters derived from fits to the ALEPH and CLEO data.

²⁵The χ^2 distance for the fitted part of the ALEPH spectrum corresponds to an average χ^2 per point of 1.28 and thus to an average distance of 1.13σ per data point. As a prediction, it is already a good starting point, which is improved by the global fit to an average distance of 0.90σ per data point.

²⁶With the exception mentioned above.

good to very good, while including the photon VP.

Parameter	Full Fit	No τ	No Spacelike	No ρ mass shift
Subtraction Polynomial : $\Pi_{\pi\pi}^{\rho}(s)$				
c_1	0	0	0	0
c_2	-0.467 ± 0.013	-0.463 ± 0.014	-0.472 ± 0.013	-0.470 ± 0.013
Subtraction Polynomial : $\epsilon_2(s)$				
c_1	-0.071 ± 0.003	-0.071 ± 0.003	-0.072 ± 0.003	-0.072 ± 0.003
c_2	0.045 ± 0.004	0.045 ± 0.004	0.046 ± 0.004	0.045 ± 0.004
Subtraction Polynomial : $\epsilon_1(s)$				
c_1	-0.017 ± 0.001	-0.017 ± 0.001	-0.017 ± 0.001	-0.017 ± 0.001
c_2	0.020 ± 0.001	0.020 ± 0.001	0.020 ± 0.001	0.020 ± 0.001
Subtraction Polynomial $\Pi_{\pi\pi}^{W/\gamma}(s)$				
c_1	0.918 ± 0.061	0.944 ± 0.068	0.727 ± 0.074	0.915 ± 0.060
c_2	0.433 ± 0.106	0.361 ± 0.115	0.831 ± 0.145	0.440 ± 0.105

Table 3: Parameter values under various strategies (cont'd). Boldface parameters are not allowed to vary. Each subtraction polynomial is supposed to be written $c_1 s + c_2 s^2$.

Figure 2 shows the fit with the e^+e^- data in the timelike region superimposed. The global scale factor effects are accounted for. In the ϕ mass region, the lineshape is a prediction essentially derived from the phase and branching fraction of the $\phi \rightarrow \pi^+\pi^-$ decay mode as measured by the SND Collaboration [71]. Information on the full (local) invariant mass spectrum (when available) would certainly improve this prediction.

Likewise, Figure 3 shows the fit function and the ALEPH [31] and CLEO [33] data superimposed. One may note that the highest data point from CLEO data lies at $\simeq 2\sigma$ of the fitting curve. Actually, the lineshape of the CLEO data in the neighborhood of the maximum raises some difficulty while fitting, as will be seen shortly with the fit residuals.

Leftside Figure 4 shows the spacelike data [59, 80] together with the fit function. One may note a small, but systematic upwards shift of the fit compared with the NA7 data which certainly explains the jump in the fit probability while removing this data sample from the fit.

One may conclude from Figures 2, 3, 4 and the fit probabilities that the agreement of the

data with the model functions is good and that no obvious drawback shows up.

In order to refine this statement, we have had a closer look at the fit residual plots. The fit residuals for the new Novosibirsk data are shown in Figure 5. The plotted experimental errors do not take into account the bin-to-bin correlations generated by the global scale error common to all data sets. As the difference between our pion form factor functions $F_{\pi}^e(s)$ and $F_{\pi}^{\tau}(s)$ essentially lies in the $\rho^0 - \omega - \phi$ mixing scheme produced by breaking Isospin Symmetry, this Figure can be compared with Figure 1 in [26] or Figure 9 in [36] where a systematic s -dependent effect was pointing towards a consistency problem between e^+e^- and τ data ; such an effect is no longer observed, pointing to a presently more adequate manner of performing the breaking of Isospin Symmetry. One may possibly note that the dispersion of the residuals is very small everywhere for the 1998 CMD2 data [28, 29], while it is larger for the 1995 CMD2 [27] and SND [30] data which additionally, are moved in opposite directions in the ω mass region. This indicates that our fit parameter values are dominated by the 1998 CMD2 data.

The residual distributions for τ data –the upper plots in Figure 6– look more interesting. The arrows indicate the limit of the fit regions. The errors plotted are certainly underestimated, as the bin-to-bin correlations are not accounted for in the drawings ; moreover, the errors produced by identifying invariant mass coordinate and central bin value also are not considered.

One observes now a small but clear s -dependent structure above the ρ peak location (more precisely above $s \simeq 850$ MeV) which certainly reflects the influence of the unaccounted for higher mass vector mesons. One has also examined the effect of removing the parameter δm^2 by fixing it identically to zero while fitting. The τ data residuals are given by the lower plots in Figure 6. One clearly observes the rise of a structure at the ρ peak location in ALEPH data which is therefore a clear signal of a $\rho^0 - \rho^{\pm}$ mass difference. The mass shift observed is :

$$m_{\rho^0}^2 - m_{\rho^{\pm}}^2 = -\delta m^2 = (0.27 \pm 0.10) 10^{-2} \text{ GeV}^2 \iff m_{\rho^0} - m_{\rho^{\pm}} = 1.73 \pm 0.60 \text{ MeV} \quad (48)$$

in good agreement with several other reported values [54].

In contrast, the shape of the CLEO residual distribution rather indicates a systematic effect in CLEO data located only in the ρ peak region. The global Cleo energy shift of 0.9 MeV serves to recover from the disagreement with ALEPH data. However, these plots clearly show that the problem of systematics is not global but local and that there is no evidence for a significant global invariant mass shift within the CLEO data in our fitted range. As this residual behaviour is also observed in the standalone fit performed in [33] (see Figure 10 therein), it should not follow from the constraints specific to our model.

Actually, there is a correlation between δm^2 and the Cleo energy shift ε which vanishes when performing a simultaneous fit of ALEPH and CLEO data. In order to check this statement, we have removed ALEPH data from the fit data set. Then, fixing $\delta m^2 \equiv 0$, we get $\varepsilon = 1.57 \pm 0.40$ MeV and, conversely, fixing $\varepsilon \equiv 0$ results in $-\delta m^2 = (0.36 \pm 0.08) 10^{-2} \text{ GeV}^2$, with no change in the χ^2 value and always the same residual shape as shown in Figure 6. Therefore, the value for δm^2 is set by the ALEPH data and should be confirmed by forthcoming data sets. Correspondingly, it is the use of ALEPH data which indicates that the Cleo energy shift could well be consistent with zero.

All the reported exercises also show, as clear from Tables 2 and 3, that the fit parameter values are stable (with the exception already mentioned). This means that our model is overconstrained and that, practically, only the fit quality (*i.e.* the height of the minimum) is

affected by the various conditions we have imagined. We were also aware of possible correlations between the subtraction polynomials. Looking at the fit error covariance matrix, we did not observe strong correlations between parameters belonging to different polynomials, which seems to indicate that they are indeed independent.

Therefore, one may consider that the description of all form factor data supports our mixing model, as reflected by the statistical fit qualities reported in Table 1 under various conditions.

Finally, rightside Figure 4 shows the *predicted* phase of the $I = 1$ part of the pion form factor together with the measured $P_{11}(\pi\pi)$ phase data from [82, 83]. Clearly, the description is good, keeping in mind that some contributions have not been included, especially the exchange of a spacelike ρ . Therefore, this Figure indicates that the neglected diagrams should contribute not more than a few degrees to the phase.

Therefore, the description of all form factor data can be considered as satisfactory and provides a solid ground to our main assumptions :

j/ The bulk of Isospin Symmetry effects which create the difference between e^+e^- and τ form factor lineshapes is a $\rho^0 - \omega - \phi$ mixing scheme of dynamical (*i.e.* s -dependent) structure.

jj/ An appropriate subset of meson partial width decays and the e^+e^- form factor data mostly suffice to set up a $\rho^0 - \omega - \phi$ mixing scheme able to derive the τ spectrum with good precision.

jjj/ The effects of higher mass vector mesons in the mass region below 1.0 GeV, even if somewhat visible on the upper wing of the ρ peak, are negligible.

jv/ The (observed) $\rho^0 - \rho^\pm$ mass shift is very small and of negligible effect. New τ data may confirm its relevance, as this follows only from ALEPH data.

12.3 Light Meson Decays

As a preliminary remark, when fitting partial widths (actually coupling constants), the recommended data used are the partial widths taken from the RPP [54], when available. If not, they are derived from the branching ratios and the full widths. Sometimes, this procedure reveals a surprising information. For instance, for $\eta \rightarrow \gamma\gamma$, the ratio of the “fit” partial width error to the corresponding central value is $0.026/0.51 = 0.05$, while the corresponding information derived from the quoted “fit” branching fraction is $0.26/39.98 = 0.007$, which might look somewhat optimistic.

The numerical estimates of branching fractions have been calculated using the information returned by the MINOS program and take into account the parameter error covariance matrix in the standard way (as recalled in Section 7.3 of [16], for instance). This is mandatory as some error correlation coefficients are very large, namely those among the two fit parameters hidden inside $\epsilon_1(s)$, or inside $\epsilon_2(s)$ are about 95%. Most other error correlation matrix elements are below the 10 % level. We therefore consider that our error estimates are accurate.

On the other hand, all partial width results we compute have been derived using the accepted values for all vector and pseudoscalar meson masses [54]. In order to produce the branching ratios as given in Table 4, we have also divided these partial widths by the accepted total widths reported in the latest issue of the Review of Particle Properties [54]. The errors on masses and widths have been taken into account in the computer code used in order to derive the reconstructed branching ratios.

12.3.1 Radiative Decays Of Light Mesons

The fit values for the branching fractions of light mesons radiative decays are displayed on top of Table 4. Most decay modes involving vector mesons are in nice correspondence with their recommended values [54].

The value returned for the $\omega \rightarrow \pi^0\gamma$ branching ratio is about 3σ from the presently recommended value [54], but is in good agreement with the former recommended value $((8.5 \pm 0.5) 10^{-2})$ as well as the value found in the fit of the $e^+e^- \rightarrow \pi^0\gamma$ reported in [74] $((8.39 \pm 0.25) 10^{-2})$. This indicates that the recommended central value for this decay mode can be questioned.

On the other hand, as already commented upon, the branching fraction found for $\omega \rightarrow \eta\gamma$, is in much better agreement with the average value proposed by the PDG in [54] $((6.3 \pm 1.3) 10^{-4})$ than their so-called fit value reported in Table 4 ; this result is also in perfect agreement with the Crystal Barrel [87] measurement $((6.6 \pm 1.7) 10^{-4})$ as well as the measurement obtained in a fit to the $e^+e^- \rightarrow \eta\gamma$ cross section [74] $((6.56 \pm 2.5) 10^{-4})$. We also consider confidently our result for this decay mode.

The ratio :

$$\frac{\Gamma(\omega \rightarrow \eta\gamma)}{\Gamma(\omega \rightarrow \pi^0\gamma)} = (0.802 \pm 0.011) 10^{-2} \quad (49)$$

depends only weakly on the mass and width definitions of the ω meson and is in agreement with all reported direct measurements in the RPP [54]. This also gives support to both fit results.

The contribution of the $\eta' \rightarrow \rho^0\gamma$ mode to the χ^2 is 2.23, while all others are smaller or of order 1. This may indicate that the box anomaly [85, 88] shows up and might have been accounted for.

The only difficult point of the model is the $\simeq 1.9\sigma$ departure of the partial width for $\eta \rightarrow \gamma\gamma$ from the expected value commented upon at the beginning of this Section. Whether this could be due to our assuming that the pion decay constant is not affected by Isospin Symmetry breaking effects is an open possibility. Instead, the partial width for $\eta' \rightarrow \gamma\gamma$ fits nicely its expected value, possibly because of its larger experimental uncertainty²⁷.

Finally, we should note that our model gives a precise indirect measurement of f_K/f_π :

$$\left[\frac{f_K}{f_\pi} \right]^2 = 1.503 \pm 0.010_{stat} \pm 0.002_{model} \implies \frac{f_K}{f_\pi} = 1.226 \pm 0.004_{stat} \pm 0.001_{model} \quad (50)$$

where the second quoted uncertainty reflects details of the model together with the effects of including the spacelike data in the fit. This is in balance with the corresponding quantity which can be derived from the reported world average data [54] as $f_K/f_\pi = 1.223 \pm 0.010$, assuming that the errors on f_π and f_K are uncorrelated.

12.3.2 Leptonic Decays Of Light Vector Mesons

Table 4 indicates that our model nicely accomodates the $\omega \rightarrow e^+e^-$ and $\phi \rightarrow e^+e^-$ partial widths giving values which coincide with their recommended values [54].

²⁷ In order to test this assumption, we have left f_{π^0} free in our fits and, for $f_{\pi^0} = 87.9 \pm 2.4$ MeV, we have reached a probability slightly above 80 %, with $\Gamma(\eta \rightarrow \gamma\gamma)$ at 0.55σ from its recommended value [54] and $\Gamma(\eta' \rightarrow \gamma\gamma)$ at only 0.37σ . This has to be compared with the reported value extracted from $\Gamma(\pi^0 \rightarrow \gamma\gamma)$ which provides $f_{\pi^0} = 91.92 \pm 3.54$ MeV.

Our result for $\rho^0 \rightarrow e^+e^-$ is derived from the same data which underly the other proposed values [54] and has been obtained with a careful account of all statistical and systematic reported errors. Therefore, this value can be confidently considered ; one should note that it exhibits a $\simeq 10\sigma$ distance to the presently accepted branching fraction [54]. A more straightforward information coming out from our fits is the corresponding partial width :

$$\Gamma(\rho \rightarrow e^+e^-) = (8.34 \pm 0.10 \pm 0.31)10^{-3} \text{ MeV} \quad (51)$$

where the first error merges all statistical and systematic uncertainties commented upon in the body of the text ; the second error takes into account the real uncertainty affecting the ρ mass used in order to derive the partial width from the coupling. It has conservatively been fixed to 10 MeV for reasons which will become clearer shortly. The corresponding partial width as given in [54] is $(7.02 \pm 0.11)10^{-3} \text{ MeV}$.

12.3.3 The $\omega/\phi \rightarrow \pi^+\pi^-$ Decays

The value found for the $\phi \rightarrow \pi^+\pi^-$ partial width compares well with its measured value [71]. Actually, one may suspect that this datum prominently influences some of our free parameters, certainly those in the expression for $\epsilon_1(s)$. The phase of the corresponding coupling constant being close enough (1.4σ) to expectation [71] might indicate that the data (modulus and phase) for this mode carry small systematic uncertainties.

The branching fraction we get for the $\omega \rightarrow \pi^+\pi^-$ mode is more appealing. It is derived from all data involved in this measurement with a precise account of all systematic uncertainties. Additionally, the quality of the measurement we propose probably does not suffer from significant model uncertainties, as the $\rho - \omega$ interference region is quite well described (see the insets in Figure 2). Therefore, our conclusion for this decay mode is either of :

$$\text{Br}(\omega \rightarrow \pi^+\pi^-) = (1.13 \pm 0.08)\% \quad , \quad \Gamma(\omega \rightarrow \pi^+\pi^-) = (9.59 \pm 0.80) 10^{-2} \text{ MeV} \quad (52)$$

using the recommended value for width and the mass of the ω meson [54].

This new datum may influence the global fit of all the ω decay modes in isolation. This is of concern for our purpose, as one has noticed that the disagreements observed between the PDG recommended values [54] and our results for the $VP\gamma$ modes refer mostly to the $\omega \rightarrow (\eta/\pi^0)\gamma$ branching ratios.

Finally, the unfitted Orsay phase for the coupling $\omega \rightarrow \pi^+\pi^-$ is found close to its expected value from a standalone fit to the so-called old timelike data [50], while our fit for the phase of the $\phi \rightarrow \pi^+\pi^-$ coupling is in good agreement with its measured value [71].

12.3.4 The $\phi \rightarrow K\bar{K}$ Decays

As explained in the body of the text, we have been led to leave both $\phi \rightarrow K\bar{K}$ decay widths outside our fit procedure, as there is some uncertainty with possible factors, like Coulomb corrections, which may affect the usual coupling constant contributions to both partial widths[72, 73, 89].

Therefore, the values reported in Table 4 are predictions only influenced by the other decay modes and without any additional correction factor to each of the $\phi \rightarrow K\bar{K}$ branching ratios. The numerical values found for these branching ratios clearly illustrate that our model

is overconstrained and provides precise values for the coupling constants of both $\phi \rightarrow K\bar{K}$ modes.

The ratio of the prediction to the recommended central values is 1.022 for the charged mode (*i.e.* a 1.8 σ distance) and 0.97 for the neutral decay mode (*i.e.* a 2.0 σ distance). Taking into account the model uncertainty reported in Table 4, the agreement could be considered as satisfactory.

Now, if correction factors have to be applied, they are expected to improve the prediction for the rates. Therefore, they should be of the order 0.976 and 1.031 for, respectively, the charged and neutral decay widths. This clearly invalidates the traditional 1.042 correction factor proposed in order to account for Coulomb interaction among the charged kaons²⁸.

Correcting both modes as argued in [73], even if able to provide a good account for the ratio $\Gamma(\phi \rightarrow K^+K^-)/\Gamma(\phi \rightarrow K^0\bar{K}^0)$, does not allow a good account of both modes separately, as the corrections proposed turn out to increase the expected rate for both modes.

Within the framework of our model, if correction factors have to be applied, they should not increase the charged decay mode by more than $\simeq 1\%$. There is more freedom with the neutral decay mode. Therefore, in order to fix one's ideas, one has let a correction factor for only the neutral decay mode to vary. In this case, of course, the correction factor ($= 1.047 \pm 0.024$) is found such that the neutral mode exactly coincides with its measured value, which could be expected beforehand. However, more interesting is that the χ^2 contribution of the charged decay mode (which does not explicitly depend on this factor) is only 0.3 (a 0.5 σ effect). This indeed confirms that only the predicted neutral decay width might have to be corrected significantly. Taking into account that systematic effects are harder to estimate for the charged mode than for the neutral one (see footnote 16 in [16]), this may look a physical effect. Whether the "mixed isoscalar and isovector source" scheme of [89] can account for such an effect would be interesting to explore.

As a summary, our analysis tends to disfavor a significant correction factor to the $\phi \rightarrow K^+K^-$ decay width (above the 1.01 level). It would rather favor a significant correction factor for only the neutral mode $\phi \rightarrow K^0\bar{K}^0$ (which could be as large as 1.047 for the rate). If the traditional scheme of Coulomb corrections should really apply, both measured widths for $\phi \rightarrow K\bar{K}$ are hard to understand, as already stated in [72, 16].

12.3.5 What Are The ρ Parameters?

For objects as broad as the ρ (or K^*) meson, the definition of mass and width is not a trivial matter [53]. Having defined in our model the ρ^0 and ρ^\pm propagators as analytic functions (or rather meromorphic functions on a 2-sheeted Riemann surface with branch point at threshold), one has at disposal the poles of the propagators. This has been shown to provide the most stable definition of the mass and width [53]. If one assumes that Analyticity of S-matrix elements is a basic principle, this is also the most model independent definition. Indeed, whatever are the working assumptions, the pole basically tells where the peak is really located and how wide is the invariant mass distribution around the peak (typically, close to the full width at half maximum). We remind that the data used for our fits is the most complete set of data allowing to define the mass and width of the ρ mesons.

²⁸ See [72] for a detailed account of the usual way to deal with Coulomb corrections and Isospin Symmetry breaking effects in ϕ decays.

For this purpose, our final results for the complex s locations of the ρ meson poles are :

$$\left\{ \begin{array}{l} s_{\rho^0} = (0.5782 - i 0.1099) \pm (0.9 + i 0.5) 10^{-3} \quad (\text{GeV}^2) \\ s_{\rho^\pm} = (0.5760 - i 0.1095) \pm (1.0 + i 0.5) 10^{-3} \quad (\text{GeV}^2) \\ s_{\rho^0} - s_{\rho^\pm} = (2.26 - i 0.38) 10^{-3} \pm (0.83 + i 0.14) 10^{-3} \quad (\text{GeV}^2) \end{array} \right. \quad (53)$$

All other results depend on how to define M_ρ and Γ_ρ starting from the relevant s_ρ pole location. Defining as in [56] $s_R = M_R^2 - iM_R\Gamma_R$ one gets :

$$\left\{ \begin{array}{l} M_{\rho^0} = 760.4 \pm 0.6 \text{ MeV} \quad , \quad \Gamma_{\rho^0} = 144.6 \pm 0.6 \text{ MeV} \\ M_{\rho^\pm} = 758.9 \pm 0.6 \text{ MeV} \quad , \quad \Gamma_{\rho^\pm} = 144.3 \pm 0.5 \text{ MeV} \\ M_{\rho^0} - M_{\rho^\pm} = 1.51 \pm 0.53 \text{ MeV} \end{array} \right. \quad (54)$$

which for ρ^0 are slightly larger than those found by [56] using only the so-called old timelike data. One may note that the mass difference is affected by a smaller uncertainty than the masses separately.

One may also choose [57] $s_R = (M_R - i\Gamma_R/2)^2$ and obtain slightly different values (not shown). Defining the mass by the location of the maximum of the distribution and the width by the full width at half maximum cannot be derived easily from Eqs. (53) ; they are :

$$\left\{ \begin{array}{l} M_{\rho^0} = 762.1 \pm 0.6 \text{ MeV} \quad , \quad \Gamma_{\rho^0} = 144.5 \pm 0.6 \text{ MeV} \\ M_{\rho^\pm} = 760.8 \pm 0.6 \text{ MeV} \quad , \quad \Gamma_{\rho^\pm} = 144.5 \pm 0.5 \text{ MeV} \\ M_{\rho^0} - M_{\rho^\pm} = 1.22 \pm 0.53 \text{ MeV} \end{array} \right. \quad (55)$$

One may consider these values as they are in consistency with the way the mass and width for objects like the ω and ϕ mesons are usually defined [54]. The difference between these results and those in Eqs. (54) could be attributed to the influence of the regular part of the invariant mass distribution.

If instead, one decides to parametrize the distributions with varying width Breit-Wigner shapes, one will recover the traditional values tabulated in [54] ($M_{\rho^0} = 775.5 \pm 0.4 \text{ MeV}$, for instance). However, their model dependence (not only their definition dependence) should be stressed.

Instead, one may also choose the HK mass values as they come from the Lagrangian²⁹ and our fitted parameters, one would rather get :

$$M_{\rho^0} = 782.1 \pm 2.1 \text{ MeV} \quad , \quad M_{\rho^\pm} = 780.4 \pm 2.2 \text{ MeV} \quad , \quad M_{\rho^0} - M_{\rho^\pm} = 1.74 \pm 0.60 \text{ MeV} \quad (56)$$

This result is also interesting. Indeed, as stated above (see Footnote 6), a reasonable breaking of Isospin Symmetry at Lagrangian level, while producing a (HK) mass difference between

²⁹ This has a clear physical meaning : It is the mass of the ρ meson while working at tree level (when possible, as maybe farther inside the spacelike region than we have gone).

the ρ^0 and ρ^\pm mesons returns $m_{\rho^0} = m_\omega$. As one may think that the HK ω mass is close to the tabulated value [54] ($m_\omega = 782.65 \pm 0.12$ MeV), it may have a meaning to find that the HK mass for the ρ^0 is consistent with the accepted ω mass. For this purpose, it should be noted that the ω mass value used in our fits was fixed at this accepted value, and then cannot directly influence the HK value for m_{ρ^0} .

Defining the width using Eq. (101) provides the same width for the charged and neutral ρ meson :

$$\Gamma_\rho = 172.13 \pm 1.42 \text{ MeV.} \quad (57)$$

which is substantially larger than all other definitions ; however this follows from the most usually employed formula for the two–body decay widths of the ω and ϕ mesons. One may argue that the larger mass and width exhibited by Eqs. (56) and (57) compared with usual is related with our having the HLS parameter a significantly different from 2 (a 3σ effect).

Therefore, one considers the best solution as being the s location of the poles as given by Eqs. (53) ; the definition of the mass and width using $s_R = M_R^2 - iM_R\Gamma_R$, or $s_R = (M_R - i\Gamma_R/2)^2$, or something else, can be derived algebraically.

12.4 A Few More Comments On The Model

In order to justify the change from ideal to physical fields, one should check that the functions in the non–diagonal elements of the R matrix in Eq. (17) are small compared to 1 in the relevant invariant mass range. For this purpose, we have had a closer look at the functions :

$$\left\{ \begin{array}{l} F_{\rho\omega}(s) = \frac{\epsilon_1(s)}{\Pi_{\pi\pi}(s) - \epsilon_2(s)} \\ F_{\rho\phi}(s) = \frac{\mu\epsilon_1(s)}{(1 - z_V)m^2 + \Pi_{\pi\pi}(s) - \mu^2\epsilon_2(s)} \\ F_{\omega\phi}(s) = \frac{\mu\epsilon_2(s)}{(1 - z_V)m^2 + (1 - \mu^2)\epsilon_2(s)} \end{array} \right. \quad (58)$$

computed with the fit parameter values. These are the entries of the $R(s)$ matrix which defines our transformation from ideal (bare) fields to physical fields.

Figure 7 shows the real and imaginary parts of these functions. They are all small compared to 1 all along the physical region : $|F_{\rho\omega}(s)| \simeq \mathcal{O}(10^{-2})$, $|F_{\rho\phi}(s)| \simeq \mathcal{O}(10^{-2})$ and $|F_{\omega\phi}(s)| \simeq \mathcal{O}(10^{-1})$. As expected $F_{\omega\phi}(s)$, which does not vanish in the Isospin Symmetry limit, is larger and one observes an order of magnitude difference.

$F_{\rho\omega}(s)$ represents the traditional $\rho - \omega$ mixing and its behaviour translates in our modelling the known large (Orsay) phase by the quasi–vanishing of its real part around m_ω^2 . It exhibits around $\sqrt{s} \simeq 0.3$ GeV, the two–pion threshold, an unexpected behaviour which is actually too small to influence numerically the pion form factor. Whether this local effect should be considered seriously is unclear, taking into account the approximations done in order to work out the model³⁰. The important point here is that, even if narrow, the amplitude does not exceed

³⁰This goes along with the remark that one would have preferred a solution for $F_{\rho\omega}(s)$ which vanishes at $s = 0$ as $F_{\rho\phi}(s)$ and $F_{\omega\phi}(s)$ do, even for non identically vanishing of ϵ_1 . This could also be a consequence of working at first order in ϵ_1 and ϵ_2 . However, the neglecting of the anomalous loop effects may play some role near $s = 0$.

a 10 % level there and a few percents all along the physical region (namely the ρ/ω and ϕ peak locations).

$F_{\rho\phi}(s)$ is of the same order of magnitude than $F_{\rho\omega}(s)$ but much smoother all along the physical region ; its real and imaginary parts around the ϕ meson mass are comparable ($\simeq 1$ %).

$F_{\omega\phi}(s)$ is more interesting as it represents what is traditionally attributed to a constant mixing angle of a few degrees [8]. It is indeed what is exhibited, as this function is real at m_ω^2 and close to real at m_ϕ^2 . However, the numerical values of the mixing angle vary significantly : at the ω mass the angle is 0.45° , while it is 4.64° at the ϕ mass. This tends to indicate that the notion of mixing angle has somewhat to be readdressed.

As explained above, the (not too small) magnitude of $F_{\omega\phi}(s)$ could have been inferred from the HLS model, as the transitions between ω_I and ϕ_I follow from kaon (and K^*K) loop effects and not from supplementing them with Isospin Symmetry breaking effects.

Figure 8 emphasizes the important features of our model for the pion form factor. The upper plot shows the function :

$$H(s) = \frac{|F_\pi^e(s)|^2 - |F_\pi^\tau(s)|^2}{|F_\pi^\tau(s)|^2} \quad (59)$$

which summarizes the breaking of Isospin Symmetry all along the physical region. The strong effects at the ω and ϕ mass locations could have been expected. However, one clearly sees a non-zero ‘‘background’’ contribution extending down to threshold (and even below) and beyond the ϕ mass. This simply illustrates that our $\rho - \omega - \phi$ mixing scheme is really invariant mass dependent. This is why it can suppress the unwanted effects exhibited by residuals in more standard approaches (see for instance [26], [35] or [36]).

The lower plot shows instead :

$$H_1(s) = \frac{|F_{\pi,I=1}^e(s)|^2 - |F_\pi^\tau(s)|^2}{|F_\pi^\tau(s)|^2} \quad (60)$$

where $F_{\pi,I=1}^e(s)$ is identified with the ρ^0 part of the pion form factor, as traditionally done. It clearly exhibits that the ρ^0 and ρ^\pm mesons are different kinds of objects in our modelling. We indeed observe³¹ an effect of several percents and functionally s -dependent.

Stated otherwise, the ρ^\pm is indeed a pure isospin 1 meson, while the ρ^0 meson is actually a mixture of isospin 1 (ρ_I^0) and isospin 0 states (ω_I and ϕ_I). A real extraction of the isospin 1 component of $F_\pi^e(s)$ should isolate the isospin 1 part of the ρ , ω , and ϕ amplitudes, which is exactly what our model does, by construction. This allows us to agree with the analysis by K. Maltman [48, 49] concluding that the ρ part of the pion form factor in e^+e^- data does not behave as being isospin 1 ; however, this does not invalidate the e^+e^- data.

Therefore, the single departure from CVC one observes is simply the tiny shift between the ρ^0 and ρ^\pm masses which follows from ALEPH data.

³¹The local minimum just above threshold, which reflects the structure of $F_{\rho\omega}(s)$ we discussed, may be an artifact of the model. Its numerical value at this location makes it however totally invisible.

It also follows from our model that using τ data in order to reconstruct the equivalent e^+e^- spectrum is not straightforward and requires a non-trivial physics input as shown by both plots in Figure 8.

One is tempted to think to extending the model in order to include higher mass vector meson nonets. In order to do it properly within the HLS context, one may change in the covariant derivative (see Eqs. (67)) the (presently) single vector term gV to $\sum g_i V_i$. This induces further transition amplitudes like, for instance, $\rho^0(770) \longleftrightarrow \rho^0(1450)$ which may sharply complicate the model. It is not easy to have a feeling for the numerical magnitude of the inter-nonet transitions as, now, pion and kaon loops contribute. Moreover, the higher mass vector mesons are in the region where all the thresholds of the VP channels are open. Therefore, one may also have to include the corresponding loops explicitly. Whether the problem will not become numerically untractable is therefore unclear.

On the other hand, one has certainly noticed that the problem we have examined is highly non-linear in the parameters. This makes the search for solutions highly dependent on getting a starting point (in the parameter phase-space) reasonably close to the final solution that the fit procedure may succeed. In the present case, it was already a non-trivial (and highly time consuming) task.

13 Conclusion

Within the context of the HLS model, as for other models, at tree level the so-called ideal fields (ρ_I, ω_I, ϕ_I) are mass eigenstates. This simple picture vanishes at one-loop order. In this case, kaon loops generate non-zero amplitudes allowing $\omega_I \longleftrightarrow \phi_I$ transitions. Breaking of Isospin Symmetry in the pseudoscalar sector generates a mass difference between kaons and, besides, the transition amplitudes $\omega_I \longleftrightarrow \rho_I$ and $\phi_I \longleftrightarrow \rho_I$, even if of small magnitude, are no longer vanishing. Additionally, these amplitudes exhibit a dependence on s , the square of the momentum flowing through the vector meson lines. As the physical fields ρ, ω, ϕ are expected to be eigenstates of the squared mass matrix, this unavoidably leads to define them as linear combinations of their ideal partners. However, as the transition amplitudes are s -dependent, it is clear that these combinations should also be s -dependent.

We substantiate these considerations starting from the HLS Lagrangian, modified by including in the squared mass matrix of the neutral vector mesons all self-energies and transition amplitudes. Making the assumption that the physical neutral vector fields should be eigenstates of the loop modified squared mass matrix of the (ideal) neutral vector meson, we solve the eigenvalue problem perturbatively. This leads to physical vector meson fields expressed as linear combinations of their ideal partners with definite s -dependent coefficients, which are actually analytic – or, rather, meromorphic – functions of s . Of course, this algebra leaves unchanged the charged and/or open strangeness sector, as the starting fields acquire a running mass but no transition from one to another meson field.

The main mechanism producing the vector meson field mixing is the occurrence of neutral and charged kaon loops in transitions between the ideal neutral vector meson fields. We have also shown that the anomalous HLS-FKTUY sector provides as supplementing mechanism K^*K loops, occurring in transitions in nice correspondence with kaon loops. Within this

framework, the $\omega\phi$ mixing has been shown to proceed from quantum (loop) effects, while the $\rho^0\omega$ and $\rho^0\phi$ mixings follow from Isospin Symmetry breaking effects and vanish when this symmetry is restored.

This allows to dynamically generate isospin violating couplings $\phi\pi^+\pi^-$ and $\omega\pi^+\pi^-$ at the (modified HLS) Lagrangian level. With this at hand, we have been able to construct the pion form factor expression at one-loop order modified in order to account for Isospin Symmetry breaking through only a dynamically generated $\rho_I - \omega_I - \phi_I$ mixing scheme. *A priori*, this fully affects the e^+e^- physics but, by no way, the τ physics ; in this sector, as a second order refinement, we have been led to accept shifting with respect to each other the Higgs–Kibble masses of the ρ^0 and ρ^\pm mesons. This provides a tiny effect, nevertheless clearly visible in ALEPH τ data, but not obvious in CLEO data. New data expected from B-factories may confirm the need for this mass shift. As a final result, we end up with structureless residuals in the fitted regions, which confirm that our dynamical mixing scheme is appropriate.

The mixing properties have been introduced in the anomalous decay amplitudes $V \rightarrow P\gamma$ and $P \rightarrow V\gamma$. These processes actually represent our main lever arm while defining numerically our isospin breaking scheme.

Beside a good description of the $V \rightarrow P\gamma$ and $P \rightarrow V\gamma$ decay data, this allows a very good simultaneous description of all pion form factor data from the close spacelike region to the ϕ mass, in e^+e^- annihilations as well as in τ decays. The physical ground of this result can be traced back to the fact that the ρ^0 meson is a mixture of isospin 0 and 1 states (as the ϕ and ω meson), in contrast to the ρ^\pm meson which is purely isospin 1. Actually, extracting the isospin 1 part of the pion form factor in e^+e^- annihilations requires to split up the ρ^0 , ϕ and ω contributions. This is done automatically by our model, and one can claim that such a splitting cannot be done without some model.

The net result of this model is to prove that the lineshapes for the pion form factor in e^+e^- annihilations and in τ decays are perfectly consistent with each other, without any further breaking of CVC than a possible tiny $\rho^0 - \rho^\pm$ mass difference. In a further step, one may include the data from KLOE as well as data expected to come from BaBar and Belle concerning the τ spectrum on the one hand and the e^+e^- Initial State Radiation samples on the other hand. However, our comparison of e^+e^- and τ data does not seem to leave room for any kind of new physics.

On the other hand, we have shown that this model allows a good account of all decays of the form $V \rightarrow P\gamma$ and $P \rightarrow V\gamma$. The case of the $\omega \rightarrow \eta\gamma$ and $\omega \rightarrow \pi^0\gamma$ partial widths, where some disagreement is observed with the so-called “fit” values proposed by the PDG, has been discussed and we have argued that the real situation is somewhat unclear.

Our dealing with the pion form factor data has led us to propose improved values for data sharply related with the $e^+e^- \rightarrow \pi^+\pi^-$ annihilation process, namely the $\rho^0 \rightarrow e^+e^-$ and $\omega \rightarrow \pi^+\pi^-$ partial width decays. In both cases, we find that the reference values should be significantly modified and we propose for these new reference data.

Finally, we have briefly commented upon the mass and width of the ρ^0 and ρ^\pm mesons and argued that the best motivated definition should rely on the pole position in the complex s -plane and related definitions.

Having shown that e^+e^- data and τ data are perfectly consistent with each other pro-

vided one uses an appropriate model of Isospin Symmetry breaking, we can conclude that there is no reason to question the e^+e^- data. This result is important as these data serve to estimate numerically the hadronic photon vacuum polarization used in order to predict the value of the muon anomalous magnetic moment $g - 2$. Therefore our model indirectly confirms the 3.3σ discrepancy between the BNL direct measurement of the muon anomalous magnetic moment and its theoretical estimate.

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Decay Mode	Fit Value	PDG/Reference
$\rho \rightarrow \pi^0 \gamma [\times 10^4]$	5.17 ± 0.04	6.0 ± 0.8
$\rho \rightarrow \pi^\pm \gamma [\times 10^4]$	5.03 ± 0.03	4.5 ± 0.5
$\rho \rightarrow \eta \gamma [\times 10^4]$	3.05 ± 0.04	2.95 ± 0.30
$\eta' \rightarrow \rho \gamma [\times 10^2]$	33.3 ± 1.0	29.4 ± 0.9
$K^{*\pm} \rightarrow K^\pm \gamma [\times 10^4]$	9.8 ± 0.9	9.9 ± 0.9
$K^{*0} \rightarrow K^0 \gamma [\times 10^3]$	2.26 ± 0.02	2.31 ± 0.20
$\omega \rightarrow \pi^0 \gamma [\times 10^2]$	8.23 ± 0.04	$8.9_{-0.23}^{+0.27} (*)$
$\omega \rightarrow \eta \gamma [\times 10^4]$	6.60 ± 0.09	$4.9 \pm 0.5 (*)$
$\eta' \rightarrow \omega \gamma [\times 10^2]$	3.14 ± 0.10	3.03 ± 0.31
$\phi \rightarrow \pi^0 \gamma [\times 10^3]$	1.24 ± 0.07	1.25 ± 0.07
$\phi \rightarrow \eta \gamma [\times 10^2]$	1.292 ± 0.025	1.301 ± 0.024
$\phi \rightarrow \eta' \gamma [\times 10^4]$	0.60 ± 0.02	0.62 ± 0.07
$\eta \rightarrow \gamma \gamma [\times 10^2]$	35.50 ± 0.56	39.38 ± 0.26
$\eta' \rightarrow \gamma \gamma [\times 10^2]$	2.10 ± 0.06	2.12 ± 0.14
$\rho \rightarrow e^+ e^- [\times 10^5]$	5.56 ± 0.06	$4.70 \pm 0.08 (**)$
$\omega \rightarrow e^+ e^- [\times 10^5]$	7.15 ± 0.13	7.18 ± 0.12
$\phi \rightarrow e^+ e^- [\times 10^4]$	2.98 ± 0.05	2.97 ± 0.04
$\omega \rightarrow \pi^+ \pi^- [\times 10^2]$	1.13 ± 0.08	$1.70 \pm 0.27 (**)$
$g_{\omega\pi^+\pi^-}$ phase [degr]	101.2 ± 1.6	$104.7 \pm 4.1 (**)$ [50]
$\phi \rightarrow \pi^+ \pi^- [\times 10^5]$	7.14 ± 1.7	7.3 ± 1.3
$g_{\phi\pi^+\pi^-}$ phase [degr]	-27.0 ± 0.5	-34 ± 5 [71]
$\phi \rightarrow K^+ K^- [\times 10^2]$	50.3 ± 1.0	$49.2 \pm 0.6 (**)$
$\phi \rightarrow K_S^0 K_L^0 [\times 10^2]$	33.0 ± 0.7	$34.0 \pm 0.5 (**)$

Table 4: Reconstructed Branching fractions for radiative and leptonic decays using any of the various fit strategies. The reported values are the mean value and the rms of the simulated distributions. The last column displays the recommended branching ratios [54]. The symbol (*) indicates data commented upon in the text, (**) indicates data which are not introduced in the fit procedure.

Appendices

A The Full HLS Non–Anomalous Lagrangian

The construction of the non–anomalous Lagrangian of the Hidden Local Symmetry (HLS) Model has been presented in great detail several times by its authors (see for instance [1] or more recently [2]). Let us simply outline the main steps of the construction procedure.

The HLS model allows to produce a theory with vector mesons as gauge bosons of a hidden local symmetry. These acquire a mass because of the spontaneous breakdown of a global chiral symmetry $G_{\text{global}} = U(3)_L \otimes U(3)_R$. The chiral Lagrangian is written :

$$\mathcal{L}_{\text{chiral}} = \frac{f_\pi^2}{4} \text{Tr}[\partial_\mu U \partial^\mu U] \quad (61)$$

where $U(x) = \exp[2iP(x)/f_\pi]$; here f_π is identified with the pion decay constant ($f_\pi = 92.42$ MeV) and P is the matrix of pseudoscalar mesons (the Goldstone bosons associated with the spontaneous breakdown of G_{global}). This matrix :

$$P = P_8 + P_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & & \pi^+ & & K^+ \\ & \pi^- & & & K^0 \\ & & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & & \\ & & & \bar{K}^0 & \\ & K^- & & & -\sqrt{\frac{2}{3}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 \end{pmatrix}, \quad (62)$$

contains a singlet term besides the octet term ; appropriate combinations of η_8 and η_0 correspond to the physical pseudoscalar fields η and η' . Here and throughout this paper we restrict ourselves to three flavours.

However, besides the global symmetry G_{global} , the chiral Lagrangian possesses a local symmetry $H_{\text{local}} = SU(3)_V$ which is included in the HLS approach by splitting up U as :

$$U(x) = \xi_L^\dagger \xi_R \quad (63)$$

where the ξ fields undergo the local transformation. These variables are parametrized as :

$$\xi_{R,L} = e^{i\sigma/f_\sigma} e^{\pm iP/f_\pi} \quad (64)$$

and the scalar field σ is usually eliminated through a gauge choice, and can be considered absorbed into the gauge bosons and removed. However, the decay constant f_σ goes on appearing in the model through the HLS fundamental parameter $a = f_\sigma^2/f_\pi^2$. Using this parametrization Eq. (61) can be rewritten :

$$\mathcal{L}_{\text{chiral}} = -\frac{f_\pi^2}{4} \text{Tr}[(\partial_\mu \xi_L \xi_L^\dagger - \partial_\mu \xi_R \xi_R^\dagger)^2] \quad (65)$$

This Lagrangian can be gauged for electromagnetism, weak interaction and the hidden local symmetry by changing the usual derivatives ∂_μ to covariant derivatives D_μ [1, 2] and one then

gets :

$$\left\{ \begin{array}{l} \mathcal{L}_{HLS} = \mathcal{L}_A + a\mathcal{L}_V \\ \mathcal{L}_A = -\frac{f_\pi^2}{4} \text{Tr}[(D_\mu \xi_L \xi_L^\dagger - D_\mu \xi_R \xi_R^\dagger)^2] \equiv -\frac{f_\pi^2}{4} \text{Tr}[L - R]^2 \\ \mathcal{L}_V = -\frac{f_\pi^2}{4} \text{Tr}[(D_\mu \xi_L \xi_L^\dagger + D_\mu \xi_R \xi_R^\dagger)^2] \equiv -\frac{f_\pi^2}{4} \text{Tr}[L + R]^2 \end{array} \right. \quad (66)$$

using obvious notations.

Now let us turn to the covariant derivatives. These are given by[2] :

$$\left\{ \begin{array}{l} D_\mu \xi_L = \partial_\mu \xi_L - igV_\mu \xi_L + i\xi_L \mathcal{L}_\mu \\ D_\mu \xi_R = \partial_\mu \xi_R - igV_\mu \xi_R + i\xi_R \mathcal{R}_\mu \end{array} \right. \quad (67)$$

(where we have factored out the universal vector meson coupling constant) with :

$$\left\{ \begin{array}{l} \mathcal{L}_\mu = eQA_\mu + \frac{g_2}{\cos \theta_W} (T_z - \sin^2 \theta_W) Z_\mu + \frac{g_2}{\sqrt{2}} (W_\mu^+ T_+ + W_\mu^- T_-) \\ \mathcal{R}_\mu = eQA_\mu - \frac{g_2}{\cos \theta_W} \sin^2 \theta_W Z_\mu \end{array} \right. \quad (68)$$

Eqs. (67, 68) introduce the matrix of vector meson fields (the gauge bosons of the hidden local symmetry) which is :

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} (\rho^I + \omega^I)/\sqrt{2} & \rho^+ & K^{*+} \\ \rho^- & (-\rho^I + \omega^I)/\sqrt{2} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi^I \end{pmatrix} \quad (69)$$

in terms of the so-called ideal field combinations (indicated by the superscript I) for the neutral vector mesons, which should be distinguished from the physical fields introduced in the main text. A_μ is the electromagnetic field and e the unit electric charge, g_2 and θ_W are respectively the gauge weak coupling constant and the weak (Weinberg) angle. Z_μ and W_μ^\pm are, of course the weak gauge boson fields. Q , the quark charge matrix and the T matrices are SU(3) matrices : $Q = 1/3 \text{Diag}(2, -1, -1)$ and $T_z = 1/2 \text{Diag}(1, -1, -1)$, while $T_+ = (T_-)^\dagger$ with :

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (70)$$

in terms of elements of the Cabibbo-Kobayashi-Maskawa matrix elements. In this work, the Z_μ terms have not to be considered ; they have been given for completeness. The HLS Lagrangian given above should be completed with the vector meson kinetic energy term[2] but also with the usual free Lagrangian for electromagnetic and weak boson fields. The leptonic sector also has to be added ; it is written as per usual :

$$\mathcal{L}_{\ell,\nu} = \sum_{\ell=(e,\mu,\tau)} \left[q_\ell \ell^- \gamma^\mu \ell^+ A_\mu - \frac{g_2}{2\sqrt{2}} \bar{\nu}_\ell \gamma^\mu (1 - \gamma_5) \ell^- W^+ + \dots \right] \quad (71)$$

From a practical point of view, g_2 defined above is related with the Fermi constant G_F and the W boson mass by :

$$g_2 = 2m_W \sqrt{G_F \sqrt{2}} \quad (72)$$

and it is useful to note that at the τ lepton mass scale one has [54] :

$$g_2 = 0.629 \quad (\text{and } e = 0.30286) .$$

B The HLS Anomalous Sector

QCD admits a non-abelian anomaly which explicitly breaks chiral symmetry. This anomaly is well reproduced by the Wess–Zumino–Witten Lagrangian [90, 91] ; this has been incorporated within the HLS context by Fujiwara, Kugo, Terao, Uehara and Yamawaki along with vector mesons [3, 2]. In this way, it becomes possible to provide a framework which allows one to describe most decays of vector mesons, and especially modes like $\omega \rightarrow \pi^+ \pi^- \pi^0$ and others more relevant in the present context.

Let us briefly outline the derivation and its assumptions which has been presented in comprehensive reviews [7, 92, 2]. The anomalous action can be cast under the form :

$$\Gamma = \Gamma_{\text{WZW}} + \Gamma_{\text{FKTUY}} \quad (73)$$

$$\Gamma_{\text{FKTUY}} = \sum_{i=1}^4 c_i \int d^4x \mathcal{L}_i$$

where Γ_{WZW} is the original WZW Lagrangian. The Lagrangian pieces \mathcal{L}_i where first given in [3] and each of them contains $APPP$ and AAP pieces which would contribute to the anomalous decays, but are cancelled by APV terms. These Lagrangians contain also $VPPP$ and VVP pieces [3, 2]. A priori, the weighting coefficients c_i are arbitrary. However, in order to reconcile this Lagrangian with decay data, especially $\omega \rightarrow \pi^+ \pi^- \pi^0$, FKTUY [3, 2] finally choose the following combination :

$$\mathcal{L}^{\text{FKTUY}} = -\frac{3g^2}{4\pi^2 f_\pi} \epsilon^{\mu\nu\rho\sigma} \text{Tr}[\partial_\mu V_\nu \partial_\rho V_\sigma P] - \frac{1}{2} \mathcal{L}_{\gamma PPP} , \quad (74)$$

which turns out to complement the usual WZW term for γPPP interaction with only a VVP term. In this model, for instance the decay $\pi^0 \rightarrow \gamma\gamma$ occurs solely through $\pi^0 \rightarrow \omega \rho^0$ followed by the (non-anomalous) transitions $\omega \rightarrow \gamma$ and $\rho^0 \rightarrow \gamma$ and the partial width is identical to the Current Algebra prediction reproduced by $\mathcal{L}_{\gamma\gamma P}$:

$$\mathcal{L}_{\gamma\gamma P} = -\frac{N_c e^2}{4\pi^2 f_\pi} \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma \text{Tr}[Q^2 P] \quad (75)$$

The model given by Eq. (74) with :

$$\mathcal{L}_{\gamma PPP} = -\frac{ieN_c}{3\pi^2 f_\pi^3} \epsilon^{\mu\nu\rho\sigma} A_\mu \text{Tr}[Q \partial_\nu P \partial_\rho P \partial_\sigma P] \quad (76)$$

($N_c = 3$) has been also shown [85] to reproduce perfectly the data on $\eta/\eta' \rightarrow \pi^+\pi^-\gamma$, especially the most precise ones [88]. Indeed, with no free parameter, the distortion of the ρ lineshape is accurately accounted for and this can be considered as the signature for the box anomaly in experimental data.

Accounting for the light meson radiative decays of the AVP or AAP forms is then a VVP coupling followed by one or two $V \rightarrow \gamma$ transition(s). From a practical point of view, it has been also shown [9] that the corresponding couplings can be directly derived from the following Lagrangian piece :

$$\mathcal{L} = C\epsilon^{\mu\nu\rho\sigma}\text{Tr}[\partial_\mu(eQA_\nu + gV_\nu)\partial_\rho(eQA_\sigma + gV_\sigma)P] \quad , \quad C = -\frac{3}{4\pi^2 f_\pi} . \quad (77)$$

Let us note that in the meson decays we are interested in, the weak boson sector is irrelevant. Finally, one can find the VVP Lagrangian expanded in [8], more precisely in Appendix 1 and 4 for respectively the fully flavor symmetric case and the $SU(3)/U(3)$ broken case³²

C $SU(3)/U(3)$ Symmetry Breaking of the HLS Model

The HLS Lagrangian we are interested in is the lowest order expansion of Eq. (66) supplemented with Eq. (77). However, in order to use it with most real data, one cannot avoid defining an appropriate symmetry breaking mechanism. Several breaking schemes have been proposed [7, 4, 5, 6] as there is no unique way to implement such a mechanism in the HLS model. We will prefer the method proposed in [6] which looks to be the simplest that automatically fulfills the hermiticity requirement. This symmetry breaking scheme turns out to modify the non-anomalous Lagrangian terms in Eq. (66) to :

$$\left\{ \begin{array}{l} \mathcal{L}_A = -\frac{f_\pi^2}{4}\text{Tr}[(L - R)X_A]^2 \\ \mathcal{L}_V = -\frac{f_\pi^2}{4}\text{Tr}[L + R]X_V^2 \end{array} \right. \quad (78)$$

where the $SU(3)$ symmetry breaking matrices X_A and X_V can be written :

$$\left\{ \begin{array}{l} X_A = \text{Diag}(1, 1, \sqrt{z_A}) \\ X_V = \text{Diag}(1, 1, \sqrt{z_V}) \end{array} \right. . \quad (79)$$

As the parameter $z_A = [f_K/f_\pi]^2$ is fixed here by kaon decay data, it can hardly be considered as a truly free parameter, even if one allows it to vary within errors [54] : $z_A = 1.495 \pm 0.031$. As shown in the Lagrangian pieces given in the main text, the second breaking parameter, z_V , allows one to shift the ϕ meson mass away from those of the ρ and ω mesons ; practically we have more freedom in varying it. The full $SU(3)$ broken HLS Lagrangian produced by this mechanism (without W^\pm interaction terms) has been given in [6]. Using this

³²In the Appendix 4, the two breaking parameters ℓ_W and ℓ_T (denoted z_T in the present paper) have been found in this reference to fulfill $\ell_W\ell_T^2 = 1$.

mechanism, however, the pseudoscalar kinetic energy term of the HLS Lagrangian is no longer canonical and a renormalization of the pseudoscalar fields is required [7] :

$$P' = X_A^{1/2} P X_A^{1/2} \quad (80)$$

where P and P' stand respectively for the bare and renormalized pseudoscalar field matrices. With this redefinition, the kinetic energy term of the SU(3) broken Lagrangian is once again canonical. However, the coupling constants to kaons have to be changed correspondingly by introducing renormalized fields.

With this symmetry breaking mechanism, the realm of practical relevance for the HLS model extends to pions and kaons, as far as pseudoscalar mesons are concerned. In order to bring η and η' mesons into the game, one needs first to define them in terms of the η_8 and η_0 fields, second to extend the breaking scheme from SU(3) to U(3).

We use the one angle traditional mixing expression :

$$\begin{bmatrix} \eta \\ \eta' \end{bmatrix} = \begin{bmatrix} \cos \theta_P & -\sin \theta_P \\ \sin \theta_P & \cos \theta_P \end{bmatrix} \begin{bmatrix} \pi_8 \\ \eta_0 \end{bmatrix} \quad (81)$$

It has been shown [11] that the one–angle description was equivalent at the first two leading orders to the two–angle, two–decay constant description in favor since the Extended Chiral Perturbation Theory (EChPT) [13, 14, 15].

Now the question is how Nonet Symmetry Breaking (NSB) can be incorporated within the (SU(3) broken) HLS Lagrangian already defined. This can be done by means of determinant terms [12] which break the $U_A(1)$ symmetry :

$$\mathcal{L} = \mathcal{L}_{\text{HLS}} + \frac{\mu^2 f_\pi^2}{12} \ln \det U \cdot \ln \det U^\dagger + \lambda \frac{f_\pi^2}{12} \ln \det \partial_\mu U \cdot \ln \det \partial^\mu U^\dagger \quad (82)$$

where U is defined by Eqs.(63) and (64) after removal of the σ field matrix. This can be rewritten more explicitly :

$$\mathcal{L} = \mathcal{L}_{\text{HLS}} + \mathcal{L}'_{\text{HLS}} \equiv \mathcal{L}_{\text{HLS}} + \frac{1}{2} \mu^2 \eta_0^2 + \frac{1}{2} \lambda \partial_\mu \eta_0 \partial^\mu \eta_0 \quad (83)$$

Therefore, in this manner, one provides both a mass to the singlet and a modification of the kinetic singlet term which is thus no longer canonical and, then, calls for a renormalization. The exact renormalization relation is given in [11], where it has also been shown that, at leading order, this transformation is equivalent to using the HLS Lagrangian but replacing Eq. (80) by :

$$P'_8 + x P'_0 = X_A^{1/2} (P_8 + P_0) X_A^{1/2} \quad (84)$$

(with obvious notations). The Nonet Symmetry Breaking (NSB) mechanism introduces a parameter x which can be related [11] with λ by :

$$x = 1 - \frac{\lambda}{2} B^2 \simeq \frac{1}{\sqrt{1 + \lambda B^2}} \implies \lambda \simeq 0.20 - 0.25, \quad (85)$$

($B = (2z_A + 1)/3z_A$) with a precision better than $\simeq 5\%$.

Therefore one has only to equip the SU(3) broken HLS Lagrangian with the U(3) broken renormalization condition given by Eq.(84). Ref. [11] showed that, at leading order in breaking parameters one recovers the ChPT expectations.

In order to achieve this general presentation of the broken HLS model, we recall in a few words the breaking procedure of the anomalous Lagrangian. A priori, the transformation to renormalized fields given by Eq. (84) induces a breaking mechanism into the anomalous HLS Lagrangian given by Eqs. (74) and (76). It has been shown [8, 9] that, alone, this breaking scheme (as well as no breaking at all, both) implies that the coupling constant ratio $G_{K^*0K^0\gamma}/G_{K^*\pm K^\pm\gamma}$ equals 0.5 in sharp disagreement with experimental data [54]. Interestingly, the non-relativistic quark model (NRQM) allows more freedom by exhibiting a dependence of this ratio upon the ratio of quark magnetic moments r [93] :

$$\frac{G_{K^*0K^0\gamma}}{G_{K^*\pm K^\pm\gamma}} = -\frac{1+r}{2-r} \quad (86)$$

In [8, 9], it has been shown that this effect can be obtained by mixing a symmetry breaking scheme proposed by Bramon, Grau and Pancheri [4, 5] with some sort of vector field renormalization. Numerical analysis implies that these two mechanisms are highly correlated with the neat result that the broken VVP Lagrangian in Eq. (77) becomes :

$$\mathcal{L} = C\epsilon^{\mu\nu\rho\sigma}\text{Tr}[X_T\partial_\mu(eQA_\nu + gV_\nu)X_T^{-2}\partial_\rho(eQA_\sigma + gV_\sigma)X_TP]. \quad (87)$$

with P being replaced by renormalized fields using Eq. (84) above. Therefore, the Lagrangian we use in order to account for anomalous decays is³³ Eq. (87) with :

$$X_T = \text{Diag}(1, 1, \sqrt{z_T}) \quad (88)$$

where z_T is a parameter to be fitted. We should stress that this specific breaking, which allows one to recover Eq. (86) leaves all other couplings of physical interest (AVP and AAP) unchanged. One should note that, except for (conceptually unavoidable) mixing angles, the model we use introduces only two parameters z_A and z_T in the anomalous sector, the former being essentially fixed by pure kaon physics. Taking into account that our broken anomalous Lagrangian aims at accounting for 14 decay modes, the parameter freedom is actually very limited.

D Radiative and Leptonic Coupling Constants

In order to simplify the main text we prefer to list here the coupling constants entering decay widths expression which will be treated in this paper. Most of them can be derived trivially from expressions already given in Appendix E in [9].

³³ Eq. (76) might have also to be broken similarly. However, existing data on box anomalies allow access only to the limited sector $\pi^0/\eta/\eta' \rightarrow \pi^+\pi^-\gamma$ not affected by more breaking than reflected by Eq. (84) ; it has been shown [85] that Eq. (76) as it already stands, suffices for satisfactorily accounting for the data. There is therefore no need to go beyond for this term.

D.1 Radiative Decays

Starting from the Lagrangian in Eq. (87), and using the breaking procedure as defined by Eq. (84), one can compute the coupling constants for all radiative decays of relevance. Let us define :

$$G = -\frac{3eg}{8\pi^2 f_\pi} , \quad G' = -\frac{3eg}{8\pi^2 f_K} , \quad Z = \left[\frac{f_\pi}{f_K} \right]^2 = \frac{1}{z_A} , \quad \delta_P = \theta_P - \theta_0 \quad (\tan \theta_0 = 1/\sqrt{2}). \quad (89)$$

Some $VP\gamma$ coupling constants are not affected by Isospin Symmetry breaking :

$$\begin{cases} G_{\rho^\pm \pi^\pm \gamma} = & \frac{1}{3}G \\ G_{K^{*0} K^0 \gamma} = & -\frac{G'}{3} \sqrt{z_T} \left(1 + \frac{1}{z_T}\right) \\ G_{K^{*\pm} K^\pm \gamma} = & \frac{G'}{3} \sqrt{z_T} \left(2 - \frac{1}{z_T}\right) . \end{cases} \quad (90)$$

The $\rho_I P \gamma$ coupling constants are :

$$\begin{cases} G_{\rho_I \pi^0 \gamma} = & \frac{1}{3}G \\ G_{\rho_I \eta \gamma} = & \frac{1}{3}G \left[\sqrt{2}(1-x) \cos \delta_P - (2x+1) \sin \delta_P \right] \\ G_{\rho_I \eta' \gamma} = & \frac{1}{3}G \left[\sqrt{2}(1-x) \sin \delta_P + (2x+1) \cos \delta_P \right] . \end{cases} \quad (91)$$

The other single photon radiative modes provide the following coupling constants :

$$\begin{cases} G_{\omega_I \pi^0 \gamma} = & G \\ G_{\phi_I \pi^0 \gamma} = & 0 \\ G_{\omega_I \eta \gamma} = & \frac{1}{9}G \left[\sqrt{2}(1-x) \cos \delta_P - (1+2x) \sin \delta_P \right] \\ G_{\omega_I \eta' \gamma} = & \frac{1}{9}G \left[(1+2x) \cos \delta_P + \sqrt{2}(1-x) \sin \delta_P \right] \\ G_{\phi_I \eta \gamma} = & \frac{2}{9}G \left[Z(2+x) \cos \delta_P - \sqrt{2}Z(1-x) \sin \delta_P \right] \\ G_{\phi_I \eta' \gamma} = & \frac{2}{9}G \left[\sqrt{2}Z(1-x) \cos \delta_P + Z(2+x) \sin \delta_P \right] . \end{cases} \quad (92)$$

In order to go from ideal field couplings to physical vector field couplings, one has to use linear combinations of the couplings in Eqs. (91-92) weighted by elements of the transformation matrix $R(s)$ given in the body of the paper.

D.2 $P\gamma\gamma$ and $V - \gamma$ Modes

The 2-photon decay modes are not affected by Isospin Symmetry breaking in the vector sector and keep their usual form within the HLS model [8, 9, 11] :

$$\begin{cases} G_{\eta\gamma\gamma} = & -\frac{\alpha_{em}}{\pi\sqrt{3}f_\pi} \left[\frac{5-2Z}{3} \cos\theta_P - \sqrt{2}\frac{5+Z}{3}x \sin\theta_P \right] \\ G_{\eta'\gamma\gamma} = & -\frac{\alpha_{em}}{\pi\sqrt{3}f_\pi} \left[\frac{5-2Z}{3} \sin\theta_P + \sqrt{2}\frac{5+Z}{3}x \cos\theta_P \right] \\ G_{\pi^0\gamma\gamma} = & -\frac{\alpha_{em}}{\pi f_\pi} . \end{cases} \quad (93)$$

As stated in the text, we actually replace this last coupling by using the world average value for f_π as given in the RPP [54].

Finally, in the non-anomalous sector, the leptonic decay widths of vector mesons depend on the HLS $V - \gamma$ couplings. For the ideal combinations, we have :

$$\begin{cases} f_{\rho_I\gamma} = & a f_\pi^2 g \\ f_{\omega_I\gamma} = & \frac{f_{\rho_I\gamma}}{3} \\ f_{\phi_I\gamma} = & -\frac{f_{\rho_I\gamma}}{3} \sqrt{2} z_V . \end{cases} \quad (94)$$

It was shown in [11] that the pseudoscalar mixing angle is not a free parameter, but is related with the SU(3) breaking parameter $Z (= 1/z_A)$ and the Nonet Symmetry breaking parameter x by :

$$\tan\theta_P = \sqrt{2} \frac{Z-1}{2Z+1} x \quad (95)$$

with a very good accuracy. This relation is used in our fits as a constraint.

D.3 Partial widths

We list for completeness in this Section the expressions for the partial widths in terms of the coupling constants for the various cases which are examined in the text.

The two-photon partial widths are :

$$\Gamma(P \rightarrow \gamma\gamma) = \frac{m_P^3}{64\pi} |G_{P\gamma\gamma}|^2 , \quad P = \pi^0, \eta, \eta' . \quad (96)$$

The leptonic partial widths are :

$$\Gamma(V \rightarrow e^+e^-) = \frac{4\pi\alpha^2}{3m_V^3} |f_{V\gamma}|^2 . \quad (97)$$

The radiative widths are :

$$\Gamma(V \rightarrow P\gamma) = \frac{1}{96\pi} \left[\frac{m_V^2 - m_P^2}{m_V} \right]^3 |G_{VP\gamma}|^2 , \quad (98)$$

where V is either of ρ^0, ω, ϕ and P is either of π^0, η, η' , and :

$$\Gamma(P \rightarrow V\gamma) = \frac{1}{32\pi} \left[\frac{m_P^2 - m_V^2}{m_P} \right]^3 |G_{VP\gamma}|^2 . \quad (99)$$

The decay width for a vector meson decaying to $V + P$ is :

$$\Gamma(V' \rightarrow VP) = \frac{1}{96\pi} \left[\frac{\sqrt{[m_{V'}^2 - (m_V + m_P)^2][m_{V'}^2 - (m_V - m_P)^2]}}{m_{V'}} \right]^3 |G_{V'VP}|^2 . \quad (100)$$

Finally, the partial width for a vector meson decaying into two pseudoscalar mesons of equal masses is :

$$\Gamma(V \rightarrow PP) = \frac{1}{48\pi} \frac{[m_V^2 - 4m_P^2]^{3/2}}{m_V^2} |G_{VPP}|^2 . \quad (101)$$

E The Loop Functions

The loop functions can be written quite generally as :

$$\Pi(s) = f(s)K(s) + P(s) \quad (102)$$

where $f(s)$ is a polynomial $Q(s)$ divided by some power of s . The degree of the polynomial $P(s)$ is fixed always at second degree and we require $P(0) = 0$. we have :

$$\begin{cases} \text{Im}K(s) = -(s - s_c)^{1/2}(s - s_0)^{1/2} , & (s \geq s_0) \\ K(s) = c_0 + c_1s + c_2s^2 + \frac{s^3}{\pi} \int_{s_0}^{\infty} \frac{\text{Im}K(z)}{z^3(z - s + i\epsilon)} dz \end{cases} \quad (103)$$

where s_0 is the (direct) threshold mass squared, while s_c is the (crossed) threshold mass squared.

E.1 The PP loop

In the case of equal masses, $s_c = 0$ and we have [9] :

$$\Pi(s) = \frac{g_{VPP}^2}{48\pi} \frac{s - s_0}{s} K(s) + P(s) \quad (104)$$

and the solution is

$$\left\{ \begin{array}{l} \Pi(s) = d_0 + d_1 s + Q(s) \\ Q(s) = \frac{g_{VPP}^2}{24\pi^2} [G(s) + s_0] \\ s \leq 0 \quad : \quad G(s) = \frac{1}{2} \frac{(s_0 - s)^{3/2}}{(-s)^{1/2}} \ln \frac{(s_0 - s)^{1/2} - (-s)^{1/2}}{(s_0 - s)^{1/2} + (-s)^{1/2}} \\ 0 \leq s \leq s_0 \quad : \quad G(s) = -\frac{(s_0 - s)^{3/2}}{s^{1/2}} \arctan \sqrt{\frac{s}{s_0 - s}} \\ s \geq s_0 \quad : \quad G(s) = -\frac{1}{2} \frac{(s - s_0)^{3/2}}{s^{1/2}} \left[\ln \frac{s^{1/2} - (s - s_0)^{1/2}}{s^{1/2} + (s - s_0)^{1/2}} \right] \\ \quad \quad \quad - \frac{i\pi (s - s_0)^{3/2}}{2 s^{1/2}} \end{array} \right. \quad (105)$$

The behavior of $\Pi(s)$ near $s = 0$ is simply $\mathcal{O}(s)$, and $Q(s)$ behaves like $\mathcal{O}(s)$ near the origin. This result coincides with the one of [52, 9]. By performing more subtractions, one could choose to fix externally the actual s^n behavior of the loop near the origin.

E.2 The PP' Loop

In this case, we have :

$$\Pi(s) = \frac{g_{VPP}^2}{48\pi} \frac{(s - s_0)(s - s_c)}{s^2} K(s) + P(s) \quad (106)$$

Let us define

$$\left\{ \begin{array}{l} s \leq s_c \quad : \quad \varphi(s) = \frac{1}{\pi} (s_0 - s)^{1/2} (s_c - s)^{1/2} \ln \frac{(s_0 - s)^{1/2} - (s_c - s)^{1/2}}{(s_0 - s)^{1/2} + (s_c - s)^{1/2}} \\ s_c \leq s \leq s_0 \quad : \quad \varphi(s) = \frac{2}{\pi} (s_0 - s)^{1/2} (s - s_c)^{1/2} \arctan \sqrt{\frac{s - s_c}{s_0 - s}} \\ s \geq s_0 \quad : \quad \varphi(s) = \frac{-1}{\pi} (s - s_0)^{1/2} (s - s_c)^{1/2} \left[\ln \frac{(s - s_c)^{1/2} - (s - s_0)^{1/2}}{(s - s_c)^{1/2} + (s - s_0)^{1/2}} \right] \\ \quad \quad \quad - i (s - s_0)^{1/2} (s - s_c)^{1/2} \end{array} \right. \quad (107)$$

The solution for K is obtained by subtracting a polynomial in such a way that the behaviour of Eq. (106) is not singular at origin :

$$K(s) = \varphi(s) - [c_0 + c_1 s + c_2 s^2] \quad (108)$$

with ($s_0 = m_0^2$ and $s_c = m_c^2$):

$$\begin{cases} c_0 = \frac{m_0 m_c}{\pi} \ln \frac{m_0 - m_c}{m_0 + m_c} \\ c_1 = -\frac{1}{2\pi} \ln \frac{m_0 - m_c}{m_0 + m_c} \frac{(m_0 - m_c)^2}{m_0 m_c} \\ c_2 = -\frac{1}{8\pi} \left[\frac{(m_0^2 - m_c^2)^2}{m_0^3 m_c^3} \ln \frac{m_0 - m_c}{m_0 + m_c} + 2 \frac{m_0^2 + m_c^2}{m_0^2 m_c^2} \right] \end{cases} \quad (109)$$

The exact behaviour for $\Pi(s)$ at origin is then determined by the choice of $P(s)$.

E.3 The PP' Loop In the Complex s -Plane

The expressions in the two Subsections above give the value of the loop functions for any real values of s . It is interesting to know how these functions extend into the complex s -plane, *i.e.* for complex values of s . It is actually in this manner that the expressions above have been derived. One can check that :

$$\varphi(z) = -\frac{i}{\pi} (z - s_c)^{1/2} (s_0 - z)^{1/2} \ln \frac{(s_0 - z)^{1/2} + i(z - s_c)^{1/2}}{(s_0 - z)^{1/2} - i(z - s_c)^{1/2}} \quad (110)$$

is – up to a polynomial with real coefficients – the (single) analytic function of z real for $s_c < z < s_0$, having as imaginary part for real $s > s_0$, $\text{Im}K(s)$, given by Eq. (103). The most general solution to Eq. (103), is then written:

$$K(z) = \varphi(z) + P_n(z) \quad (111)$$

with a polynomial, $P_n(z)$, with real coefficients chosen in such a way that the behaviour at $z = 0$ is the required one. The other coefficients have to be fixed by other external (renormalization) conditions. Eq. (110) gives the loop function on the so-called physical sheet of the Riemann surface. The expression for $\varphi(z)$ on the unphysical sheet close to the physical region $s > s_0$ is obtained by a winding of 2π radians around the threshold (branch-point) $s = s_0$.

E.4 The Leptonic Loop

In order to compute the photon vacuum polarization, one needs to have at one's disposal the analytic expression of the $\ell^+\ell^-$ loop. More precisely, one needs the ratio of this loop (lepton contribution to the photon self-energy) divided by s , the off-shell photon invariant mass. This can easily be derived from the function $\Pi^0(s)$ given³⁴ in [94] or computed from detailed information from [95] :

$$\Pi(s) = \frac{\alpha}{4\pi} \left[\frac{20}{9} + \frac{4}{3z} - \frac{4(1-z)(1+2z)}{3z} G(z) \right] \quad (112)$$

³⁴In this reference, the computed loop is actually the quark-antiquark one and therefore the color factor 3 has to be removed.

where :

$$G(z) = \frac{2u \ln u}{u^2 - 1}, \quad u = \frac{\sqrt{1 - 1/z} - 1}{\sqrt{1 - 1/z} + 1}, \quad z = \frac{s}{4m_\ell^2} \quad (113)$$

and m_ℓ is the lepton mass and α is the fine structure constant.

However, for explicit computation in a minimization code, one needs the explicit expression along the real axis. This is :

$$\left\{ \begin{array}{l} \Pi(z) = \frac{\alpha}{4\pi} \left[\frac{20}{9} + \frac{4}{3z} + Q(z) \right] \\ z \leq 0 : \quad Q(z) = -\frac{2}{3} \frac{1+2z}{z^2} [-z(1-z)]^{1/2} \ln \frac{(1-z)^{1/2} - (-z)^{1/2}}{(1-z)^{1/2} + (-z)^{1/2}} \\ 0 \leq z \leq 1 : \quad Q(z) = -\frac{4}{3} \frac{1+2z}{z^2} [z(1-z)]^{1/2} \arctan \sqrt{\frac{z}{1-z}} \\ z \geq 1 : \quad Q(z) = \frac{2}{3} \frac{1+2z}{z^2} [z(z-1)]^{1/2} \left[\ln \frac{z^{1/2} - (z-1)^{1/2}}{z^{1/2} + (z-1)^{1/2}} + i\pi \right] \end{array} \right. \quad (114)$$

This function will be summed up with the (parametrized) hadronic vacuum polarization provided to us by M. Davier [68] and H.Burkhardt [69] for, respectively the region above and below the 2-pion threshold. The function $\Pi(z)$ just defined is analytic and vanishes at $z = 0$. The term of order α^2 can be derived from the function $\Pi^{(1)}$ given in [94] but is difficult to handle in fitting procedures. Other expressions for the function we use can be found in [62, 96]. Finally, even if possible in principle, we do not have the freedom of subtracting more the function $\Pi(z)$ as conditions at the Z boson mass for the full photon vacuum polarization seem to fix it to be zero [65, 66, 67].

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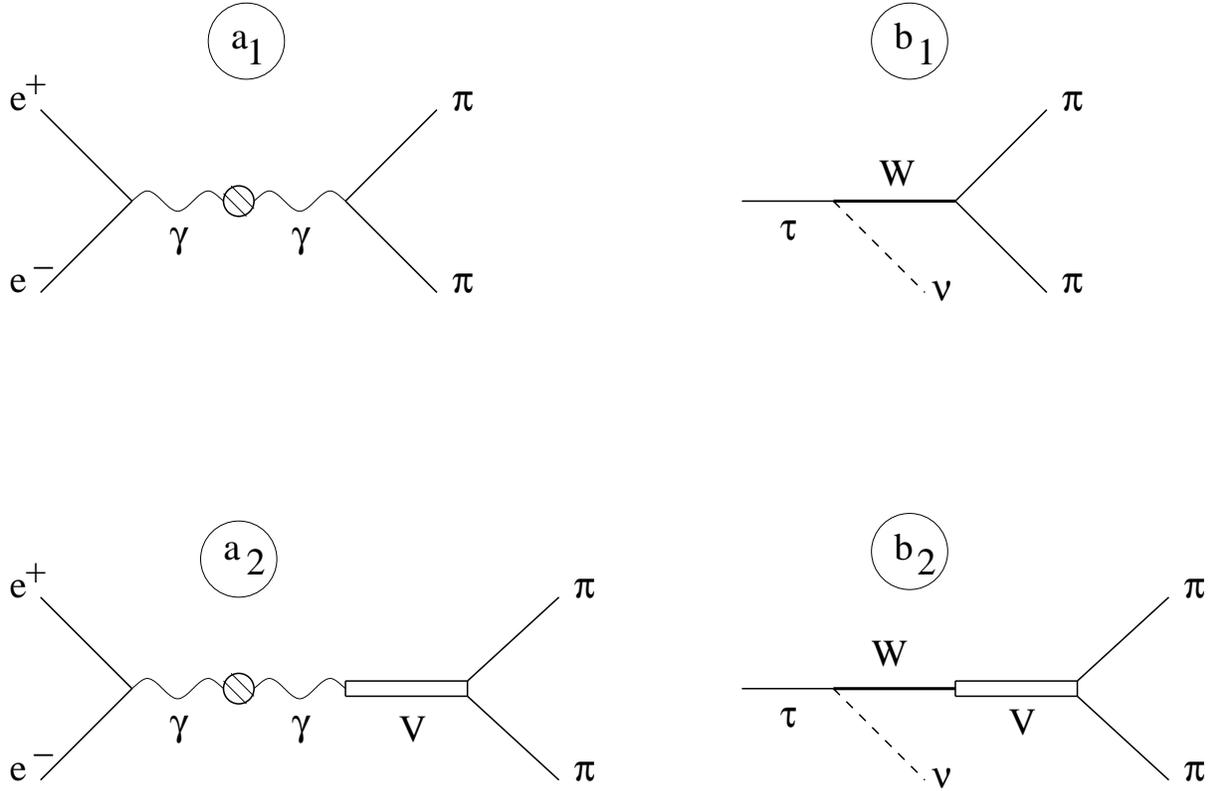


Figure 1: Schematic representation of the Feynman diagrams contributing to the pion form factor. Left plots (referred to as a_1 and a_2) sketch the case of the pion form factor in e^+e^- annihilations, while right plots (referred to as b_1 and b_2) figure out the τ decay. The upper plots show the non-resonant HLS specific diagrams, the lower plots describe the resonance contributions. The shaded blobs represent the photon vacuum polarization. the γV and WV transitions are dressed by s -dependent terms.

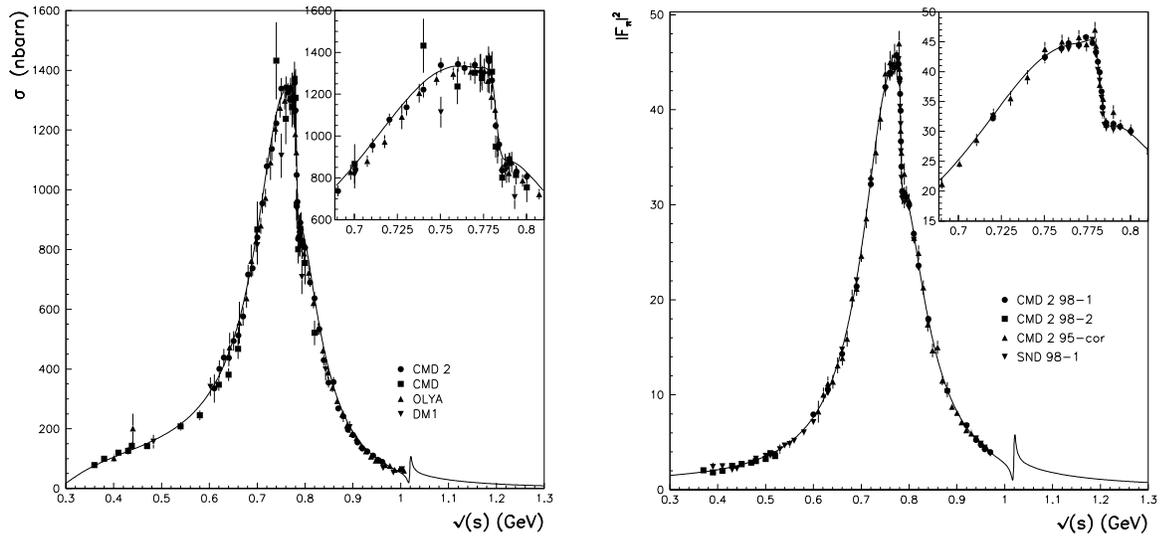


Figure 2: Data and fits for the pion form factor in e^+e^- timelike data. Left figure gives the fit cross section with, superimposed, the data from the Olya and CMD Collaborations [75], the data set [76] from the DM1 Collaboration and the first (corrected) data from CMD2 [27]. Right figure shows the form factor curve with superimposed all data sets collected recently at Novosibirsk [27, 28, 29, 30]. The ϕ region is commented upon in the body of the text.

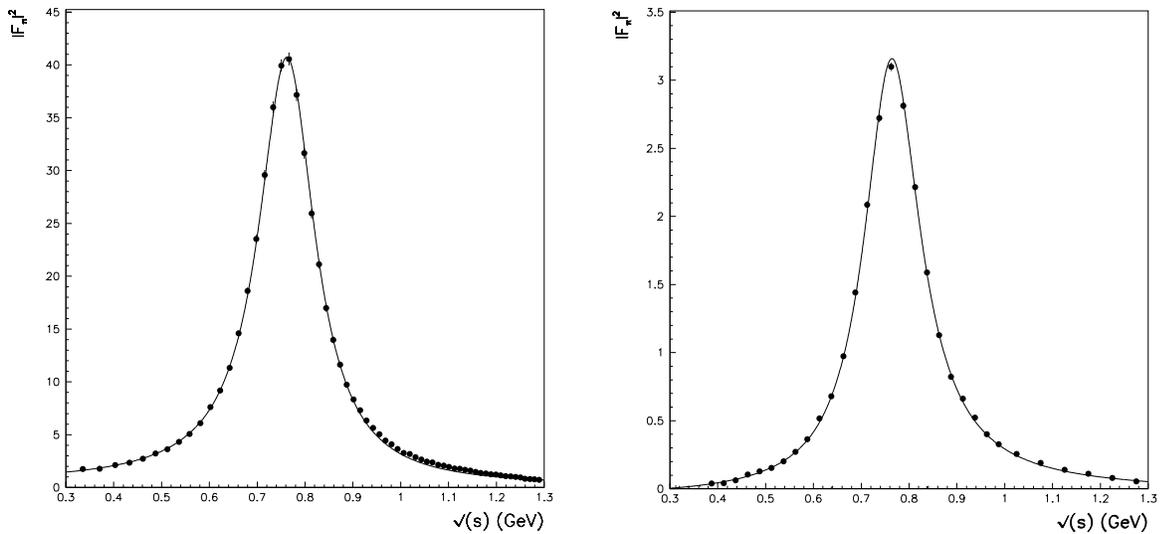


Figure 3: Data and fits for the pion form factor in τ decay. Left figure shows the case for ALEPH data [31], right figure shows the case for CLEO data [33].

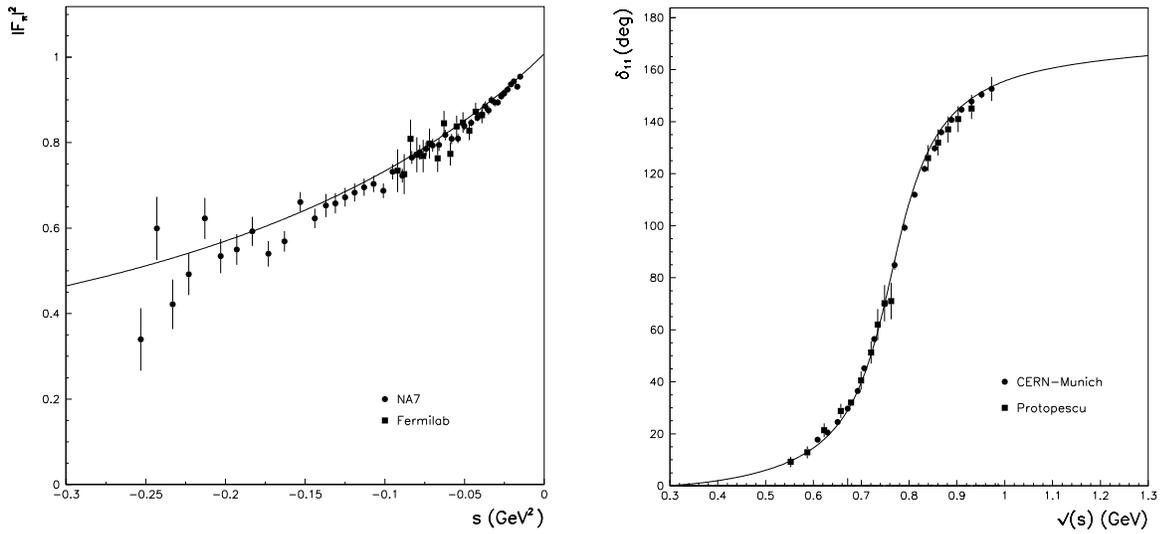


Figure 4: Left Figure shows the fit in the spacelike region close to $s = 0$ together with the data from NA7 [59] and Fermilab [80] ; the fit scale factors (1.008 and 1.006 respectively) have been applied. Rightside figure shows the *prediction* for the P_{11} phase shift with the Cern-Munich data [82] and the data from [83] superimposed.

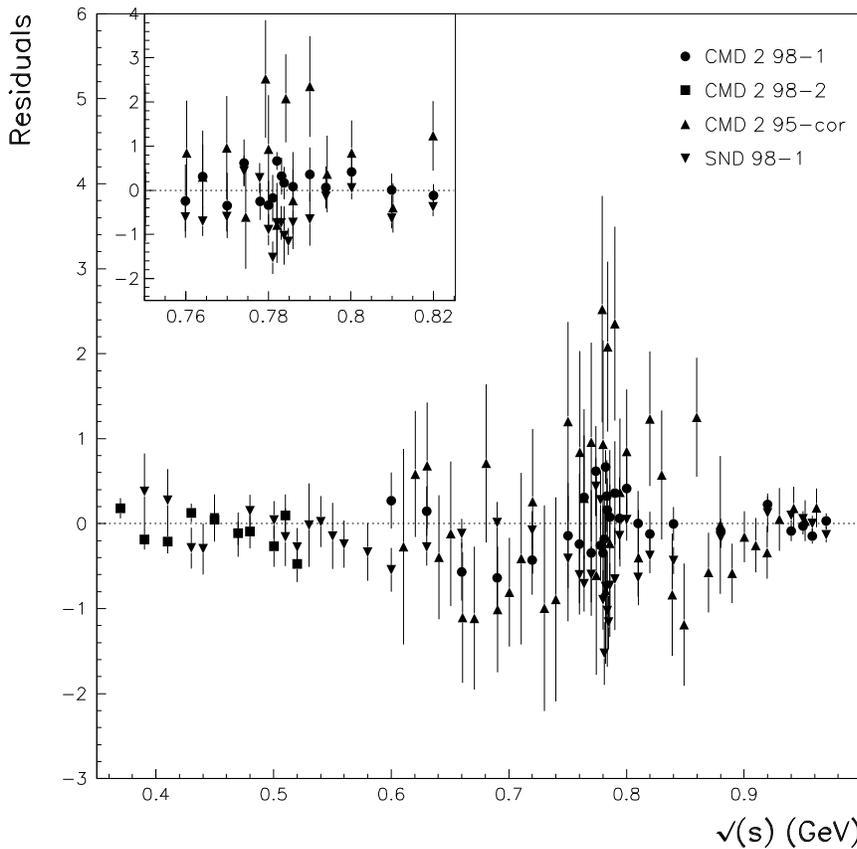


Figure 5: Residual distribution for all the e^+e^- new timelike data over the whole invariant mass interval. The inset magnifies the ρ peak invariant mass region.

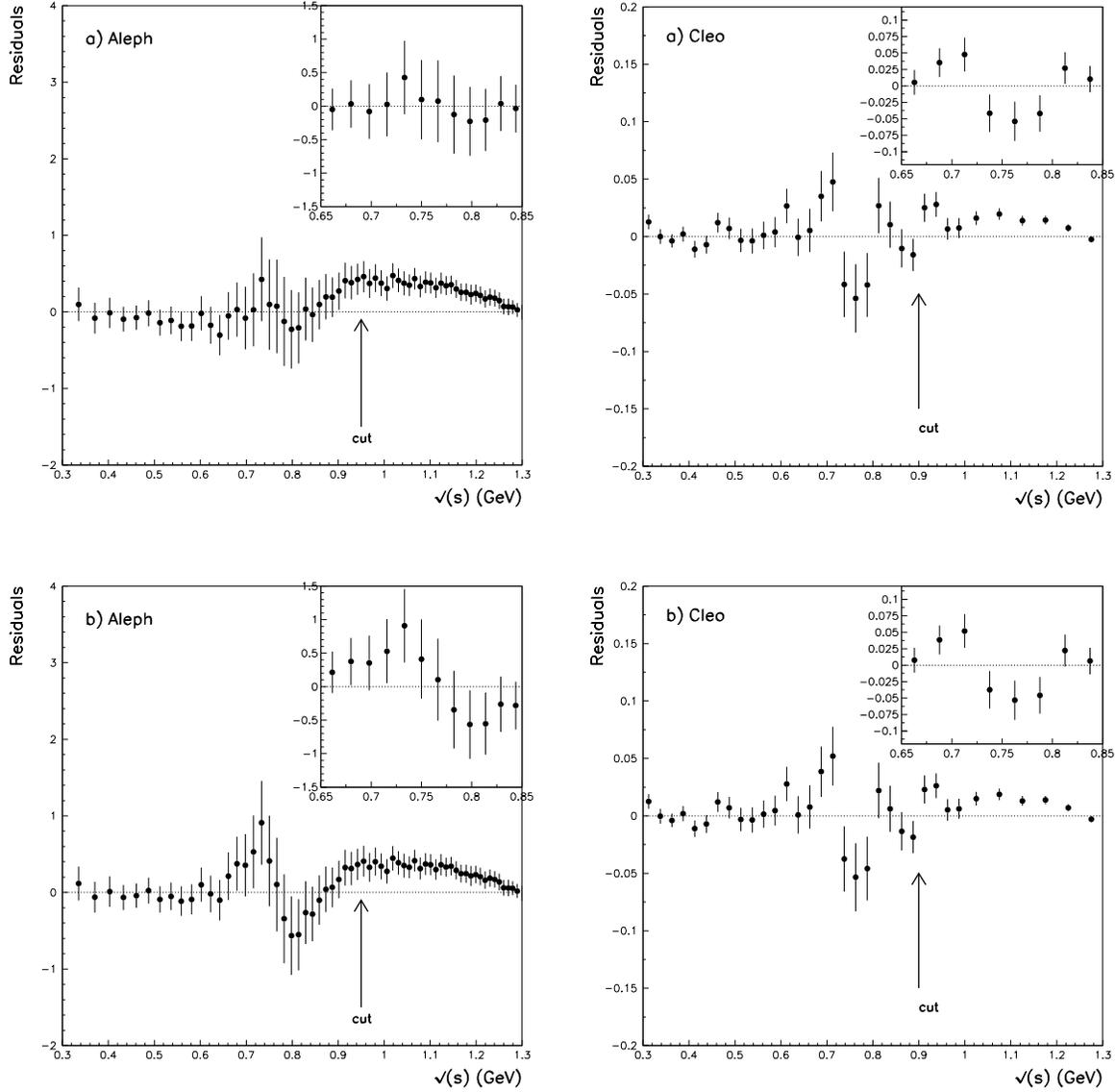


Figure 6: Distribution of fit residuals for the τ data. The upper two plots give the residuals for ALEPH and CLEO data within the model presented. The lower plots shows the τ data residuals when fitting without a $\rho^0 - \rho^\pm$ mass shift (*i.e.* $\delta m^2 \equiv 0$ is required) ; one should note the vanishing of the structure around the ρ peak in ALEPH data produced by the $\rho^0 - \rho^\pm$ mass shift. One should also note that the CLEO residual is not modified.

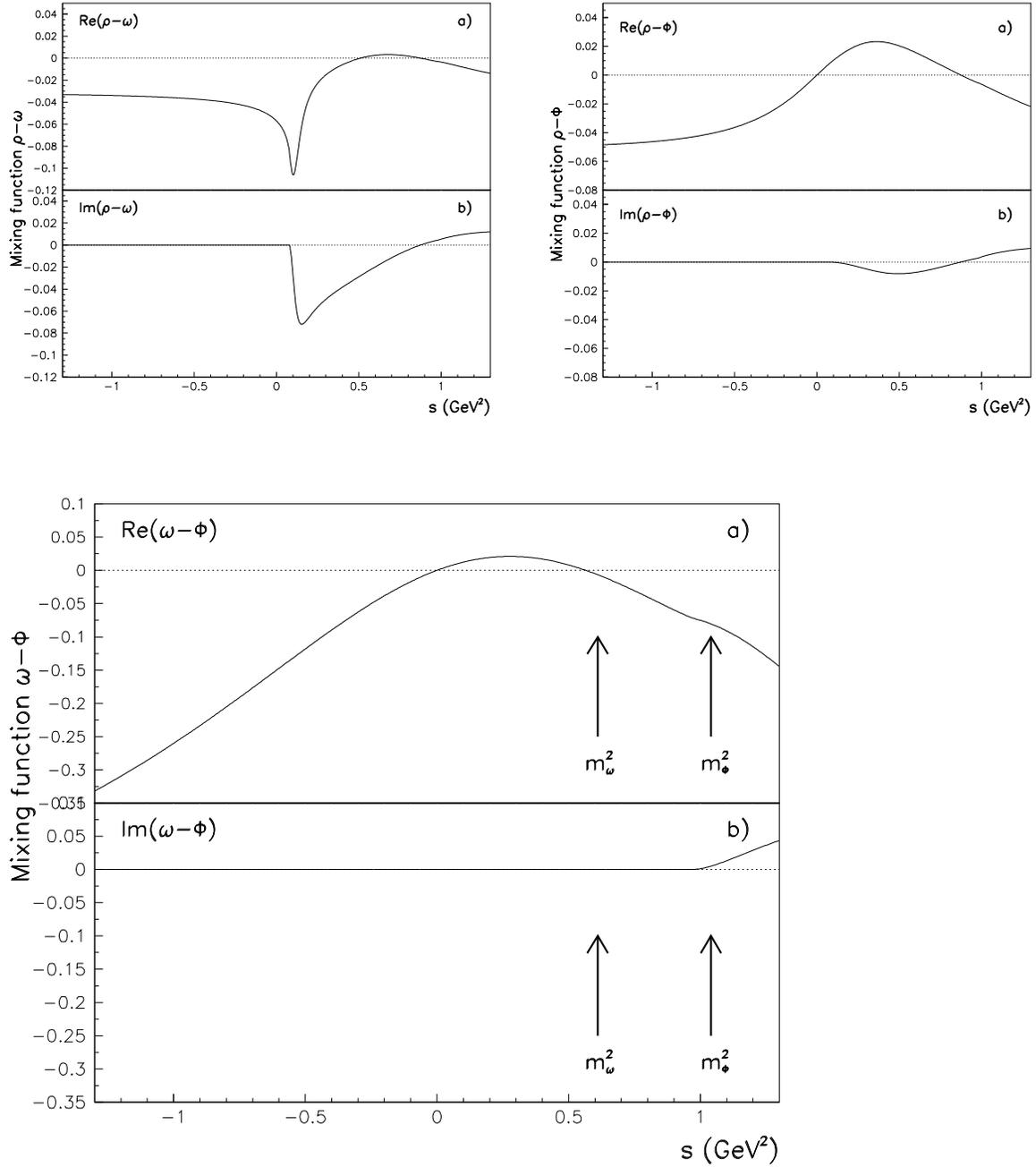


Figure 7: Matrix elements producing the neutral vector meson mixing. The functions shown are those given in Eqs. (58) with their name recalled in each Figure. The upper part of each plot gives the real part of the function, the lower part, its imaginary part.

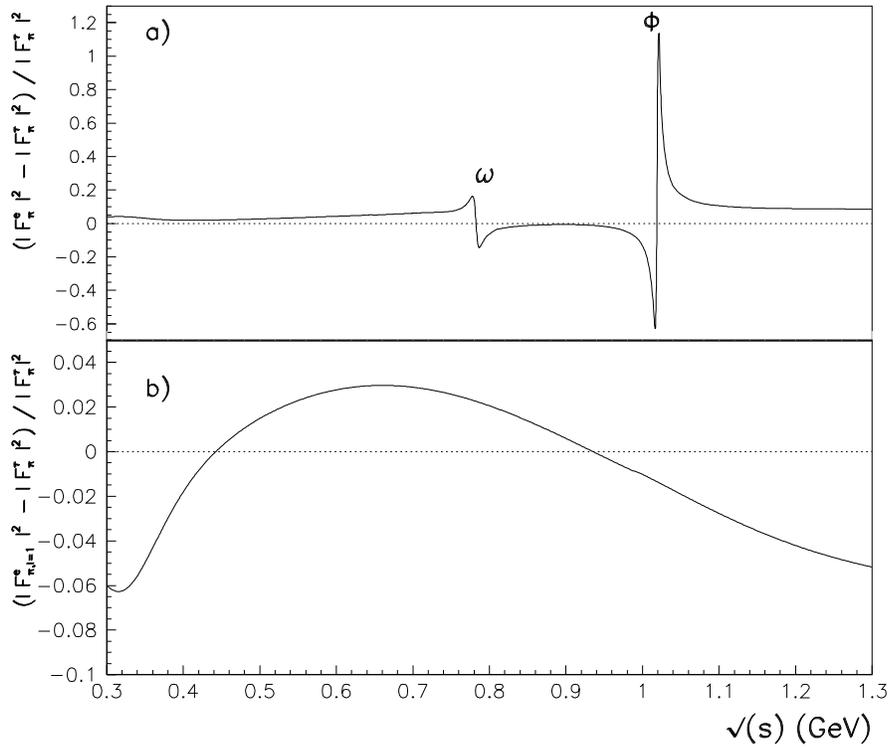


Figure 8: Isospin symmetry breaking effects following from the $\rho^0 - \omega - \phi$ mixing scheme. The upper plot shows the difference between $|F_\pi^e(s)|^2$ and $|F_\pi^\tau(s)|^2$ normalized to the latter. The lower plot instead shows the ρ part of $|F_\pi^e(s)|^2$.