FERMILAB-PUB-07/004-T

Topological Physics of Little Higgs Bosons

Christopher T. Hill

and

Richard J. Hill

Fermi National Accelerator Laboratory

P.O. Box 500, Batavia, Illinois 60510, USA

(Dated: 5 January, 2007)

arXiv:hep-ph/0701044v1 7 Jan 2007

Abstract

Topological interactions will generally occur in composite Higgs or Little Higgs theories, extra-dimensional gauge theories in which A_5 plays the role of a Higgs boson, and amongst the pNGB's of technicolor. This phenomena arises from the chiral and anomaly structure of the underlying UV completion theory, and/or through chiral delocalization in higher dimensions. These effects are described by a full Wess-Zumino-Witten term involving gauge fields and pNGB's. We give a general discussion of these interactions, some of which may have novel signatures at future colliders, such as the LHC and ILC.

PACS numbers: 11.30.Rd, 12.60.Rc, 12.60.Fr

I. INTRODUCTION

There is a fundamental topological distinction between an ordinary Higgs boson and a composite [1] or Little Higgs boson [2]. The ordinary Higgs boson is described by a scalar field taking values on a flat, unbounded manifold. The composite or Little Higgs boson, on the other hand, is a (pseudo) Nambu-Goldstone boson (pNGB) of a spontaneously broken chiral symmetry, the analog of a meson such as the kaon (we'll henceforth refer to all such composite Higgs bosons as Little Higgs bosons). It is described by a field that is confined to a compact manifold, of "radius" $1/F$. The Little Higgs is thus an angular variable and it can be translated through $2\pi F$ circumnavigating the manifold. As a consequence, topologically stable configurations can exist that are described by conserved topological currents. These currents arise from the full effective action only if we include a new topological interaction, known as the Wess-Zumino-Witten (WZW) term [3, 4]. This term is the low-energy effective description of anomaly physics in terms of pNGB's and gauge fields [5, 6, 7]. It unifies the topological physics in a chiral Lagrangian, and describes new physical processes that are not expected for the ordinary Higgs boson. It is tied to any particular UV completion model through integer quantities such as the number of "techni-colors" of the constituent "techni-quarks." Thus, the new WZW interactions of Little Higgs bosons probe the underlying UV completion theory, much like the $\pi^0 \to \gamma\gamma$ interaction probes QCD.

To motivate our discussion, we recall the typical interactions contained in the ungauged Wess-Zumino [3] term of QCD,

$$\begin{aligned}\int \mathrm{Tr}(\tilde{\pi} d\tilde{\pi} d\tilde{\pi} d\tilde{\pi}) &= \int d^4x \Big[5K^\dagger \partial_\mu \pi \partial_\nu \pi \partial_\rho \pi \partial_\sigma K - 5\sqrt{3}\eta\, \partial_\mu K^\dagger \partial_\nu K \partial_\rho K^\dagger \partial_\sigma K \\ &\qquad -5\sqrt{3}\eta\, \partial_\mu K^\dagger \partial_\nu \pi \partial_\rho \pi \partial_\sigma K + \ldots \Big] \epsilon^{\mu\nu\rho\sigma} \\ &\propto \int d^4x\, \partial_\mu \pi^+ \partial_\nu \pi^- \partial_\rho \pi^0 K^\dagger \partial_\sigma K \epsilon^{\mu\nu\rho\sigma} + \ldots . \end{aligned} \tag{1}$$

Witten [4] pointed out that this expression arises from a topological construction in $D = 5$. It reflects the nontrivial homotopy group $\pi_5(SU(3)) = \boldsymbol{Z}$ and its coefficient is subject to quantization. In QCD the WZ term locks the parity of the pion to that of spacetime, and it describes allowed strong interactions such as $K\overline{K} \to \pi\pi\pi$, which are otherwise absent in a pure kinetic-term chiral Lagrangian.

In Little Higgs models the Higgs field H is identified with an object like the kaon, $H \sim K$.

Upon electroweak gauging, we thus expect to have a nontrivial WZW term, and by analogy to the QCD chiral Lagrangian, new interactions can arise involving gauge bosons and Little Higgs bosons. In generic schemes we might expect novel interactions like:

$$\int d^4x\, W^+_\mu W^-_\nu Z^0_\rho H^\dagger_i \partial_\sigma H_j \epsilon^{\mu\nu\rho\sigma} \; + ..., \tag{2}$$

with multiple Little Higgs bosons H_i.

The discussion of such topological interactions requires the full gauging of the Wess-Zumino-Witten term. Witten originally initiated the program of brute force gauging of the WZ term, which was developed by a number of subsequent authors [4, 8, 9, 10]. For the cases where it is applicable, we will employ the most transparent of these, the form of Kaymacalan, Rajeev and Schechter (KRS) [8]. The full WZW term of KRS can be seen to descend from the Chern-Simons term and the Bardeen counterterm [7] in a compactified pure Yang-Mills theory of flavor in $D = 5$ [11].

To retain intuition based on the familiar chiral Lagrangian of QCD, we first consider a theory based on $SU(3)_L \times SU(3)_R \times U(1)/SU(3) \times U(1)$, gauging an $SU(2)$ subgroup of $SU(3)_L$. This is a simplified version of technicolor. The isovector techni-pions are eaten to become longitudinal W and Z bosons, and physical techni-kaons ($\sim H$) and a techni-eta remain in the spectrum. The resulting WZW term indeed yields techni-kaon interactions of the form $\epsilon^{\mu\nu\rho\sigma} W^+_\mu W^-_\nu Z^0_\rho H^\dagger \partial_\sigma H$.

We then turn to the simplest Little Higgs theories in which the techni-pions of the previous example do not occur. This ensures that the W and Z bosons remain massless prior to electroweak symmetry breaking. These are generic Kaplan-Schmaltz (KS) models [12]. They can be obtained by two physically distinct approaches. First, we consider reducing the $SU(3)_L \times SU(3)_R \times U(1)/SU(3) \times U(1)$ QCD-like scheme to an $SU(3) \times U(1)/SU(2) \times U(1)$ scheme by decoupling the isovector techni-pions. In particular, we can "eat and decouple" the pions by introducing an $SU(2)$ A_R gauge field in a strong coupling limit. This converts the usual unitary matrix chiral field into a *nonlinear realization* of $SU(3)$ containing only the isodoublet Higgs (kaon) and a singlet (eta). This construction enforces the correct gauge transformation for a nonlinear realization, and dictates the proper form of the covariant derivative. Nonlinear realizations afford an interesting point of departure for the construction of Little Higgs models in general [13]. With the correct gauging established, we can then directly use the KRS form to obtain the full WZW term for $SU(3) \times U(1)/SU(2) \times U(1)$

Little Higgs models.

Alternatively, we may contemplate directly the topological structure of the KS models. The pNGB's for $SU(3)\times U(1)/SU(2)\times U(1)$ are described by a complex triplet scalar field Φ with $\Phi^\dagger\Phi =$ constant. The fields thus live in a space that is topologically equivalent to the five-dimensional unit sphere, denoted S^5. The topological interactions reflect the obvious but nontrivial homotopy group $\pi_5(S^5) = \boldsymbol{Z}$, and are described by the $SU(3)\times U(1)$-invariant form: [23]

$$\omega_{ABCDE} = -\frac{i}{8}\Phi^\dagger\partial_{[A}\Phi\partial_B\Phi^\dagger\partial_C\Phi\partial_D\Phi^\dagger\partial_{E]}\Phi\,, \tag{3}$$

which corresponds to the surface area of S^5 parameterized by NGB's. Under a local gauge transformation, $\delta\omega$ is a total derivative in $D=5$ and can thus be gauged in $D=4$ to yield the WZW term. We describe the intricate procedure of gauging this structure, and discuss the interesting question of equivalence of the two approaches to the gauged topological action for $SU(3)\times U(1)/SU(2)\times U(1)$. We also touch on the related question of UV completions, noting that the latter model can be viewed as arising from an underlying fermion theory in which a triplet of techni-quarks ψ_L condenses with a singlet q_R, while the previous model arises from a condensate of two triplets of techni-quarks ψ_L and ψ_R with the pions eaten and decoupled.

While we presently sketch how all of this works, the full details will be presented elsewhere [13]. The above constructions provide a building block for many composite Higgs scenarios. Gauging $SU(2)\times U(1)$ for a single complex triplet KS Φ field yields a simple and intriguing model of a composite Higgs boson. Gauging $SU(2)\times U(1)$ for two (or more) Φ_i fields yields a multi-Higgs doublet model, while gauging the full $SU(3)\times U(1)$ for two Φ_i fields describes the Kaplan-Schmaltz Little Higgs model.

When the symmetry breaking pattern respects an internal parity operation the WZW term takes a special form that is identical to that obtained in the QCD chiral Lagrangian. This occurs, for example, in Little Higgs models with "T parity" [14]. As an application, we describe the main results for an $SU(5)/SO(5)$ Little Higgs model [15]. We point out that the WZW term is odd under the internal parity, and describes interactions between a single "T odd" particle and standard model particles.

The WZW term contains interactions that are quantized, subject to Adler-Bardeen non-renormalization [16]. They are suppressed by factors of $1/F$, with $F\sim 1$ TeV for typical Little Higgs models, and occur with loop-order coefficients commensurate with the under-

lying anomalies. Thus, they may be hard to detect. Nonetheless, these interactions can provide a powerful discriminant of underlying short-distance physics, and it is worth understanding what effects can occur and determining whether they are suited to discovery at the next generation of colliders, the LHC and the ILC. Much of this phenomenology is beyond the scope of the present paper, but will be developed elsewhere [13].

II. ILLUSTRATION BASED ON TECHNICOLOR

To illustrate the procedures used in this analysis, we first consider a chiral Lagrangian based on a QCD-like strong gauge group $SU(N_c)$, containing $SU(3)$ flavor triplets of techniquarks, (Ψ_L, Ψ_R), transforming in the fundamental representation with N_c colors. The strong interaction results in a condensate $\langle \psi_L^i \bar{\psi}_R^j \rangle \sim F^3 \delta^{ij}$, leading to an $SU(3)_L \times SU(3)_R \times U(1)/SU(3) \times U(1)$ chiral Lagrangian described by the 3×3 unitary matrix field $U^{ij} \sim \psi_L^i \bar{\psi}_R^j$. We parameterize the field as $U = \exp(2i\tilde{\pi}/F)$, where $\tilde{\pi} = \sum_{a=1}^{8} \pi^a \lambda^a/2$ are Nambu-Goldstone bosons, transforming bilinearly under $SU(3)_L \times SU(3)_R \times U(1)$ as:

$$U \to e^{i\epsilon_L} U e^{-i\epsilon_R} \,. \tag{4}$$

We presently turn off the standard model $U(1)_Y$ coupling constant, although it is straightforward to include it. We thus gauge an $SU(2)$ subgroup of $SU(3)_L$, with covariant derivative:

$$D_\mu U = \partial_\mu U - i A_\mu U \,, \qquad A_\mu = \begin{pmatrix} g W_\mu^a \frac{\sigma^a}{2} & 0 \\ 0 & 0 \end{pmatrix} \,. \tag{5}$$

In general such "left-side" gauging is anomalous, but pure $SU(2)$ gauging is always anomaly free (we either ignore the discrete Witten anomaly presently, or choose N_c even).

The anomaly physics is contained in the gauged WZW term:

$$\Gamma_{WZW} = \Gamma(U, A) \,. \tag{6}$$

This can be obtained directly from the general form of $\Gamma(U, A_L, A_R)$ in eq.(4.18) of KRS [8] (see eq.(19) below), by setting $A_L = A$ and $A_R = 0$, where we use form notation, *e.g.*, $A = A_\mu dx^\mu$, $ABCD = \epsilon^{\mu\nu\rho\sigma} A_\mu B_\nu C_\rho D_\sigma d^4x$, and $dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu dx^\nu$. A chiral current is defined as $\alpha = dUU^\dagger = (2i/F) d\tilde{\pi} + \ldots$. Explicitly, we have:

$$\Gamma(U, A) \;=\; \Gamma_0(U) + \frac{N_c}{48\pi^2} \int_{M^4} \mathrm{Tr} \left[A\alpha^3 - \frac{i}{2}(A\alpha)^2 + i(dAA + AdA)\alpha + A^3\alpha \right] \,, \tag{7}$$

where M^4 denotes four-dimensional spacetime (M^5 below denotes a five-dimensional manifold with spacetime as its boundary). We recall that $\Gamma_0(U)$ is defined as the surface term of a globally chirally invariant operator in $D = 5$:

$$\Gamma_0 = -\frac{iN_c}{240\pi^2}\int_{M^5} \mathrm{Tr}\left(\alpha^5\right) = -\frac{iN_c}{240\pi^2}\int d^5x\, \epsilon^{ABCDE}\ \mathrm{Tr}\left(\alpha_A\alpha_B\alpha_C\alpha_D\alpha_E\right)\ . \tag{8}$$

Γ_0 is not manifestly local in four dimensions, but since $d\,\mathrm{Tr}\,\alpha^5 = \mathrm{Tr}\,\alpha^6 = 0$, it can be written as an expansion in mesons in $D = 4$. Under a left-handed chiral transformation, $\delta U = i\epsilon U$, we have $\delta\alpha = id\epsilon + i[\epsilon, \alpha]$. Using $\alpha^4 = d\alpha^3$, it follows that Γ_0 shifts by an exact differential, a $D = 5$ surface term,

$$\delta\Gamma_0 = \frac{N_c}{48\pi^2}\int_{M^5} d\,\mathrm{Tr}\left(\epsilon\alpha^4\right) = -\frac{N_c}{48\pi^2}\int_{M^4} \mathrm{Tr}\left(d\epsilon\ \alpha^3\right)\ . \tag{9}$$

This shift is compensated by the $D = 4$ term in $\Gamma(U, A)$ involving one gauge field. The residual shift is cancelled by the term with two gauge fields, and so on, leading to eq.(7). [24]

An important comment is in order concerning gauge invariance. The full WZW term of eq.(7) generates a vanishing anomaly in the gauged $SU(2)$ subgroup, but nonvanishing anomalies in the global currents, $a = 4, ..., 8$. That is, under $\delta U = i\epsilon U$ and $\delta A = d\epsilon + i[\epsilon, A]$ we have $\delta\Gamma \propto \mathrm{Tr}[\epsilon(dA\,dA - idA^3/2)]$. This is the left-right symmetric (or "consistent") form of the anomaly and it is not gauge covariant. It is always possible to add Bardeen's counterterm [7] to bring the anomaly into the "covariant" form, $\delta\Gamma \propto 3\,\mathrm{Tr}[\epsilon(dA - iA^2)^2]$. However, the coupling to NGB's is unaffected by the Bardeen counterterm, which is a function only of gauge fields. Expanding eq.(7) to leading order in NGB's, we see that the NGB's participate in an interaction $\propto \mathrm{Tr}[\tilde{\pi}(dA\,dA - idA^3/2)] + ...$ which takes the form of the consistent anomaly, and is superficially non-gauge invariant. How, then, can we see that this gives a bona-fide gauge invariant interaction? The resolution is that, with "left-side" gauging of U, the gauge fields *always acquire a mass.* Since the $SU(2)$ gauge anomalies vanish, we are free to transform to unitary gauge to remove eaten NGB's. The gauge fields can then be expressed in the "Stueckelberg" form, $A' = V^\dagger(A + id)V$, where V is the transformation to unitary gauge that removes the NGB's from U. The Stueckelberg fields are gauge invariant. The residual physical techni-mesons, expressed in the same gauge, couple to A', and the interaction takes the form $\mathrm{Tr}[\tilde{\pi}'(dA'dA' - idA'^3/2)] + ...$, which is manifestly gauge invariant. In summary, "left-side" gauging produces NGB's coupled to the consistent anomaly which is a gauge invariant functional of covariant Stueckelberg fields. [25]

Let us examine some typical physical processes contained in eq.(7). A convenient choice of $SU(3)$ coordinates around $U = 1$ is:

$$U = \begin{pmatrix} e^{i\eta/F}\sqrt{1 - HH^\dagger/F^2}\,\hat{U} & H/F \\ -e^{-i\eta/F}H^\dagger\hat{U}/F & e^{-2i\eta/F}\sqrt{1 - H^\dagger H/F^2} \end{pmatrix}, \tag{10}$$

where $\hat{U} = \exp(2i\hat{\pi})$ is an $SU(2)$ matrix containing the techni-pions, $\hat{\pi} = \sum_{a=1}^{3}\hat{\pi}^a\sigma^a/2$, η is a real phase and H is a complex iso-doublet (techni-kaon). Here $HH^\dagger$ denotes a dyadic product. Under an $SU(2)_L$ transformation, $\epsilon_L = \mathrm{diag}(\hat{\epsilon}, 0)$ and $\epsilon_R = 0$ in eq.(4), we have $H \to e^{i\hat{\epsilon}}H$, $\hat{U} \to e^{i\hat{\epsilon}}\hat{U}$ and $\eta \to \eta$.

We presently focus attention on terms involving H through second order in $1/F$. Thus the chiral current becomes:

$$\alpha = \begin{pmatrix} \frac{-1}{2F^2}H \overleftrightarrow{d} H^\dagger & \frac{1}{F}dH \\ \frac{-1}{F}dH^\dagger & \frac{-1}{2F^2}H^\dagger \overleftrightarrow{d} H \end{pmatrix}, \tag{11}$$

and we find the WZW interactions of eq.(6) take the form:

$$\begin{aligned} \Gamma_{WZW} \;=\; & -\frac{ig^2N_c}{192\pi^2F^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\left(Z^0_\mu\partial_\nu Z^0_\rho + W^-_\mu\partial_\nu W^+_\rho + W^+_\mu\partial_\nu W^-_\rho - \frac{3i}{2}gZ^0_\mu W^+_\nu W^-_\rho\right) \\ & \times\left(H^0 \overleftrightarrow{\partial}_\sigma H^{0*} + H^+ \overleftrightarrow{\partial}_\sigma H^-\right) + \ldots . \end{aligned} \tag{12}$$

The kinetic term allowed by $SU(3)_L \times SU(3)_R \times U(1)$ invariance is $F^2\mathrm{Tr}|D_\mu U|^2$. In this "technicolor" scheme, the W and Z bosons eat the $\hat{\pi}$ degrees of freedom, acquiring a common mass gF. We can transform to unitary gauge, $\hat{U} \to 1$, leaving physical fields η and H, and eq.(12) describes the anomalous interactions of these physical fields with the massive Stueckelberg W and Z gauge bosons. The WZW term for this technicolor scheme contains the interesting physical processes $e^+e^- \to Z^* \to W^+W^-(H^0H^{0*} + H^+H^-)$ and $e^+e^- \to Z^* \to Z^0(H^0H^{0*} + H^+H^-)$, with amplitudes that count the number of underlying techniquark colors, N_c.

Eq.(12) does not describe a Little Higgs theory, but we can deform this technicolor theory to imitate a Little Higgs scheme by restricting the kinetic term to the form $F^2\,\mathrm{Tr}\,|D_\mu UP'|^2$ with the projection matrix $P' = \mathrm{diag}(0, 0, 1)$. Doing so blocks the W and Z from acquiring mass by projecting out their longitudinal coupling with the isovector techni-pions, but unfortunately leaves the $\hat{\pi}$ as nonpropagating auxiliary fields in the theory. We will see subsequently that removing the unphysical techni-pions enforces that U transform as a *non-linear realization* of $SU(3)$. The problem for Little Higgs theories is therefore to construct

the WZW term either by attacking directly and rederiving the full WZW action in terms of a restricted manifold of NGB's, or by adapting the above familiar form of the WZW term to the case of U treated as a nonlinear realization. We turn to this issue in the next sections, and derive the topological physics of bona-fide Little Higgs models.

III. WZW TERM FOR MODELS INVOLVING $SU(3)/SU(2)$

A set of "simple" Little Higgs models are based on $SU(3)\times U(1)/SU(2)\times U(1)$, (or more generally $SU(n)\times U(1)/SU(n-1)\times U(1)$). These were introduced by Kaplan and Schmaltz (KS) [12] . We begin by considering "one half" of such an $SU(3)/SU(2)$ Little Higgs model, described by a single scalar field Φ which transforms as a triplet under $SU(3)$.

We can view the KS model as arising from a UV completion scheme in which Φ is a bound state $\phi^i \sim \psi^i_L \bar{q}_R$, where Ψ_L is a flavor triplet and q_R a singlet. The fermions transform in the fundamental representation of a color group $SU(N_c)$ and we assume that a $(3,1)$ condensate forms with $\langle \Psi_L \bar{q}_R \rangle \sim (0,0,F^3)^T$. We have constructed such UV completions and will discuss their full content elsewhere. The unbroken $SU(2)\times U(1)$ subgroup is anomaly free, with a vector-like $U(1)$ current $\propto \overline{\psi}^3_L \gamma_\mu \psi^3_L + \bar{q}_R \gamma_\mu q_R$. This subgroup can thus be identified with the electroweak gauge group of the standard model. We can obtain the topological interactions by deriving the WZW term directly as a functional of Φ, which amounts to gauging the sphere S^5 defined by $\Phi^\dagger\Phi = 1$.

Alternatively, the KS model can be viewed as arising from a UV completion with an $SU(3)_L \times SU(3)_R$ chiral structure, a $(3,\bar{3})$ condensate of the form $U \sim \psi_L \bar{\psi}_R$. The iso-vector pions are removed by gauging on the right with a strongly coupled iso-vector gauge field A_R. A_R eats the pions, and becomes a functional of H, η and the remaining gauge fields A_L, while U becomes a nonlinear realization with a modified covariant derivative. We begin with the nonlinear realization.

A. Nonlinear Realization

For the low energy effective Kaplan-Schmalz theory we can write $\Phi = U \times \langle \Phi \rangle$, where $\langle \Phi \rangle$ is a constant vector, and U is a unitary matrix function of the five (or $2n+1$) NGB's.

A convenient choice of coordinates for $\langle\Phi\rangle$ and U is

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad U = \exp\left[\frac{i}{F}\begin{pmatrix} \eta I_2/\sqrt{3} & H \\ H^\dagger & -2\eta/\sqrt{3} \end{pmatrix}\right]. \tag{13}$$

H has the usual standard model quantum numbers of the Higgs, and η is a standard model singlet. For simplicity, we will ignore the $U(1)$ factor in the remainder of this section, although it is straightforward to include it.

The critical element of this formalism is that U transforms as a *nonlinear realization* of $SU(3)$ [17]. That is,

$$U \to e^{i\epsilon} U e^{-i\epsilon'}, \tag{14}$$

where $\epsilon \in SU(3)$ and ϵ' is a matrix function of ϵ and U, belonging to the unbroken $SU(2)$ subgroup, the upper left block with our choice of coordinates. Hence, while U contains only the five degrees of freedom described by H and η, we can implement the full $SU(3)$ transformation, albeit nonlinearly.

The key challenge with nonlinear realizations is that the ordinary derivative, dU, is not covariant under global transformations with constant ϵ, owing to the spacetime dependence of $\epsilon'(\epsilon, U(x))$. We therefore need to construct a covariant derivative that involves the gauge field A_L and acts on U so that $DU \to e^{i\epsilon} DU e^{-i\epsilon'}$. Such a covariant derivative can be written as follows:

$$DU = dU - iA_L U + iUA_R, \qquad A_R = PU^\dagger(A_L + id)UP - \frac{1}{n-1}\mathrm{Tr}\left[PU^\dagger(A_L + id)UP\right]. \tag{15}$$

A_R is the projection of $U^\dagger(A_L + id)U$ onto the unbroken subgroup. Here we have defined the projection matrix

$$P = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}. \tag{16}$$

Under the gauge transformation eq.(14), with

$$\delta A_L = d\epsilon + i[\epsilon, A_L], \tag{17}$$

A_R transforms as:

$$\delta A_R = d\epsilon' + i[\epsilon', A_R], \tag{18}$$

ensuring that DU is covariant.

The nonlinear realization eq.(14) embeds the NGB's from $SU(3) \to SU(2)$ inside a larger manifold corresponding to $SU(3)_L \times SU(3)_R \to SU(3)$. We can describe the removal of the extra pionic degrees of freedom in physical terms as follows. We begin by treating A_L and A_R as independent gauge fields, where A_L is a general $SU(3)$ matrix field, and A_R belongs to the unbroken $SU(2)$ subgroup. Correspondingly, the ϵ and ϵ' in the transformation eq.(14) are independent rotations. Now suppose that the A_R kinetic term vanishes, corresponding to very strong coupling, *i.e.*, $(-1/g_R^2)\,\mathrm{Tr}\, F_{R\mu\nu}F_R^{\mu\nu} \to 0$. A_R then becomes an auxiliary field with equation of motion determined by the $SU(3)_L \times SU(3)_R$ chiral-invariant kinetic term, $F^2\,\mathrm{Tr}\,|D_\mu U|^2$. Noting that $PA_RP = A_R$ and $\mathrm{Tr}(A_R) = 0$, we obtain precisely the locking condition eq.(15) as the solution for A_R as a function of A_L and U. This allows us to "eat and decouple" the unwanted isovector NGB's in U. Using the gauging of eq.(15) and expanding the resulting kinetic term in powers of $1/F$ we see that $(F^2/2)|D_\mu U|^2 \to |D_\mu H|^2 + (\partial_\mu \eta)^2 + \dots$, and we are therefore dealing with a Little Higgs theory. Thus, in a sense, a Little Higgs theory is just a technicolor theory with the usual chiral field U replaced by a nonlinear realization.

For any chiral theory based on a unitary matrix U transfoming as $U \to e^{i\epsilon}Ue^{-i\epsilon'}$ and gauge fields A_L and A_R that likewise transform under ϵ and ϵ' as in eqs.(17) and eq.(18), the gauged WZW term is given explicitly by KRS eq.(4.18):

$$\begin{aligned}
\Gamma_{WZW}(U, A_L, A_R) = \Gamma_0(U) \; &+ \frac{N}{48\pi^2}\,\mathrm{Tr}\int_{M^4} \big\{(A_L\alpha^3 + A_R\beta^3) - \frac{i}{2}[(A_L\alpha)^2 - (A_R\beta)^2] \\
&+ i\big[(dA_LA_L + A_LdA_L)\alpha + (dA_RA_R + A_RdA_R)\beta\big] + (A_L^3\alpha + A_R^3\beta) \\
&+ i(A_LUA_RU^\dagger\alpha^2 - A_RU^\dagger A_LU\beta^2) + i(dA_RdU^\dagger A_LU - dA_LdUA_RU^\dagger)\big] \\
&- (dA_LA_L + A_LdA_L)UA_RU^\dagger + (dA_RA_R + A_RdA_R)U^\dagger A_LU \\
&- i(A_LUA_RU^\dagger A_L\alpha + A_RU^\dagger A_LUA_R\beta) + i\big[A_L^3UA_RU^\dagger - A_R^3U^\dagger A_LU - \frac{1}{2}(UA_RU^\dagger A_L)^2\big\}\,,
\end{aligned} \tag{19}$$

where N is an integer; *e.g.*, in the QCD chiral Lagrangian, $N = N_c = 3$ is the number of colors. Here $\alpha = dUU^\dagger$ and $\beta = U^\dagger dU$. The function $\Gamma_0(U)$ is given by eq.(8), which in four dimensions reads

$$\Gamma_0(U) = \frac{2N}{15\pi^2F^5}\int_{M^4} \mathrm{Tr}\left[\tilde{\pi}(d\tilde{\pi})^4\right] + \dots\,. \tag{20}$$

In the present case we need only substitute the representation of U given in eq.(13) and

the locking of A_R to A_L and U given in eq.(15). We can then expand to a given order in $1/F$ to obtain the topological interactions of the mesons and gauge fields. We are presently ignoring $U(1)$ factors, and identify the unbroken $SU(2)$ subgroup with electroweak gauge interactions (we'll also presently ignore the $U(1)_Y$ gauge subgroup):

$$A_L = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} , \tag{21}$$

where $W = gW^a\sigma^a/2$. Defining

$$A_R = A_L + \begin{pmatrix} \hat{A}_R & 0 \\ 0 & 0 \end{pmatrix} , \tag{22}$$

we find to second order in $1/F$:

$$\hat{A}_R = -\frac{1}{2F^2}\left(\{W, HH^\dagger\} - H^\dagger W H\right) + \frac{i}{2F^2}\left(H \overleftrightarrow{d} H^\dagger + \frac{1}{2}H^\dagger \overleftrightarrow{d} H\right) , \tag{23}$$

and

$$\alpha = \begin{pmatrix} \frac{i}{\sqrt{3}F}d\eta - \frac{1}{2F^2}H \overleftrightarrow{d} H^\dagger & \frac{i}{F}\left(1 + \frac{\sqrt{3}i}{2F}\eta\right)dH + \frac{\sqrt{3}}{2F^2}Hd\eta \\ \frac{i}{F}\left(1 - \frac{\sqrt{3}i}{2F}\eta\right)dH^\dagger - \frac{\sqrt{3}}{2F^2}H^\dagger d\eta & -\frac{2i}{\sqrt{3}F}d\eta - \frac{1}{2F^2}H^\dagger \overleftrightarrow{d} H \end{pmatrix} . \tag{24}$$

Here $A \overleftrightarrow{d} B \equiv A \overrightarrow{d} B - A \overleftarrow{d} B$. The expansion for β is given (to all orders in $1/F$) by substituting $H \to -H$ and $\eta \to -\eta$ into $-\alpha$.

The leading WZW interactions involving W and NGB's appear at order $1/F$:

$$\Gamma_{WZW} = \frac{-N}{8\pi^2\sqrt{3}F}\int_{M^4} \eta \mathrm{Tr}(F_W^2) + \dots , \tag{25}$$

where $F_W = dW - iW^2$. As the result of $SU(2)$ matrix identities, no additional interactions appear through order $1/F^2$ from the action in eq.(19) involving just W and H. In more general gauging with $U(1)_Y$, and generalizations to multiple chiral fields $\Phi_i \sim U_i P'$ there can occur in principle at this order other interesting operators such as $H^\dagger F_W H F_Y$ which lead to processes such as $e^+e^- \to (\gamma^*, Z^*) \to (h^0, A^0) + (Z, \gamma, W^+W^-)$ (A^0 is a CP-odd Higgs boson). There also appear operators at this order that are separately gauge-invariant and chirally invariant:

$$\Gamma_{GI} = \int_{M^4} ir\mathrm{Tr}(F_L U F_R U^\dagger) = \int_{M^4} -\frac{2ir}{F^2}H^\dagger F_W^2 H + \dots , \tag{26}$$

with a coefficient r that is not quantized, and is sensitive to the details of the underlying UV completion theory. Multi-Higgs topological interactions of W and H occur at order $1/F^4$:

$$\Gamma_{WZW} \supset \frac{N}{96\pi^2 F^4} \int_{M^4} H^\dagger (DH)\, H^\dagger F_W (DH) + (DH^\dagger) H\, (DH^\dagger) F_W H\,. \tag{27}$$

This operator can in principle lead to processes such as: $e^+e^- \to (\gamma^*, Z^*) \to h^0 + (Z, \gamma)$ and will generalize in other schemes [13].

Additional quantized, topological, interactions will occur when the mesons couple to gauge fields for broken symmetry generators. This happens in Little Higgs models, where anomaly cancellation occurs between different sectors that are connected only by weak gauge interactions. Before discussing this phenomenon in more detail, we describe in the next section an alternative, and more direct approach to the WZW term for $SU(3)/SU(2)$.

Finally, let us remark that the construction here based on $SU(3)_L \times SU(3)_R/SU(3)$ relied on being able to gauge the $SU(2)$ subgroup of $SU(3)_R$ and remove the unwanted pion degrees of freedom. [26] In the more general case of $SU(n)/SU(n-1)$ based on $SU(n) \times SU(n)/SU(n)$, the unbroken subgroup is anomalous, and so cannot be gauged without adding more structure. [27] For example, if the anomaly of colored fermions is cancelled by massless right-handed leptons,

$$\Delta\mathcal{L}_{\rm lepton} = \bar{\ell}_R (i\partial\!\!\!/ + A\!\!\!/_R)\ell_R\,, \tag{28}$$

then A_R in eq.(15) is modified to

$$A^a_{R\mu} \to A^a_{R\mu} - \frac{1}{F^2}\bar{\ell}_R t^a \gamma_\mu \ell_R\,. \tag{29}$$

Substituting this new locking condition for A_R as a function of A_L and U into eq.(19) and setting $A_L = 0$ yields an ungauged action involving the leptons and the remaining mesons. The corresponding gauged action can then be constructed by "brute-force" gauging, as described after eq.(9).

B. Gauging the Five Sphere

A more direct method for obtaining the WZW term for KS models is to notice that when the meson fields are described by a vector $\Phi = (\phi^1, \phi^2, \phi^3)^T$ with

$$\Phi^\dagger\Phi = (\mathrm{Re}\,\phi^1)^2 + (\mathrm{Im}\,\phi^1)^2 + (\mathrm{Re}\,\phi^2)^2 + (\mathrm{Im}\,\phi^2)^2 + (\mathrm{Re}\,\phi^3)^2 + (\mathrm{Im}\,\phi^3)^2 = 1\,, \tag{30}$$

the field lives on a manifold with very simple global topological structure, the five-dimensional sphere, S^5. The relevant topological facts are that $\pi_4(S^5) = 0$, guaranteeing that the construction of a WZW term is possible, and $\pi_5(S^5) = \boldsymbol{Z}$, guaranteeing that the result is nontrivial. There is a unique five-form on the five-sphere that is invariant under global $SU(3) \times U(1)$ rotations, namely the volume element of the sphere,

$$\omega = -\frac{i}{8}\Phi^\dagger d\Phi d\Phi^\dagger d\Phi d\Phi^\dagger d\Phi \,. \tag{31}$$

In analogy with the construction of $\Gamma_0(U)$ in eq.(8), we consider the topological action

$$\Gamma_0 = \frac{N}{\pi^2}\int_{M^5} \omega \,. \tag{32}$$

Since any five-form on a five-dimensional manifold is closed, $d\omega = 0$, this action can be written as an expansion in Goldstone bosons in $D = 4$. Using that the area of the five-sphere is π^3, the coefficient satisfies the quantization condition displayed in eq.(32), with N an *even* integer.

Gauging the topological action is more tedious than in the familiar $SU(N) \times SU(N)/SU(N)$ case. At a practical level, the gauging begins by noticing that the variation of the ungauged action yields

$$\delta\Gamma_0 = \frac{N}{8\pi^2}\int_{M^4} \left(\Phi^\dagger d\epsilon d\Phi + d\Phi^\dagger d\epsilon\Phi - d\epsilon_0\Phi^\dagger d\Phi\right) d\Phi^\dagger d\Phi \,, \tag{33}$$

where ϵ and ϵ_0 are the $SU(3)$ and $U(1)$ components in the variation of Φ, respectively:

$$\Phi \to e^{i(\epsilon+\epsilon_0)}\Phi \,. \tag{34}$$

Eq.(33) has made use of the fact that for fields confined to the five-sphere, [28]

$$\begin{gathered} d(\Phi^\dagger\lambda^a\Phi)(d\Phi^\dagger d\Phi)^2 = 0 \,, \\ \left[\Phi^\dagger\lambda^a\Phi d\Phi^\dagger d\Phi - 2d(\Phi^\dagger\lambda^a\Phi)\Phi^\dagger d\Phi + 2d\Phi^\dagger\lambda^a d\Phi\right] d\Phi^\dagger d\Phi = 0 \,. \end{gathered} \tag{35}$$

The variation eq.(33) is compensated by the term with one gauge field,

$$\Gamma_1 = \frac{N}{8\pi^2}\int_{M^4} \left(A_0\Phi^\dagger d\Phi - \Phi^\dagger A d\Phi - d\Phi^\dagger A\Phi\right) d\Phi^\dagger d\Phi \,. \tag{36}$$

The residual shift is cancelled by a term with two gauge fields, and so on. Explicit expressions for the full gauged action will be presented elsewhere [13]. Important aspects of the analysis are the non-uniqueness of the gauged WZW term, and a restrictive interpretation in terms of underlying fermions, which we turn to presently.

The complete result for the gauged WZW term leads to an anomalous gauge variation:

$$\delta\Gamma_{WZW} = -\frac{N}{24\pi^2}\int_{M^4} \mathrm{Tr}\left\{\left(\epsilon - \frac{\epsilon_0}{2}\right)\left[\left(dA - \frac{1}{2}dA_0\right)^2 - \frac{i}{2}d\left(A - \frac{1}{2}A_0\right)^3\right]\right\} + \frac{27}{8}\epsilon_0(dA_0)^2\,. \tag{37}$$

Although the construction has made no mention of fermions, if interpreted in terms of an underlying fermion theory, this variation corresponds precisely to a triplet of left-handed fermions and a single right-handed fermion, transforming under $\epsilon \in SU(3)$ and $\epsilon_0 \in U(1)$ as

$$\Psi_L \to e^{i\epsilon - \frac{i}{2}\epsilon_0}\Psi_L\,, \quad q_R \to e^{-\frac{3i}{2}\epsilon_0} q_R\,. \tag{38}$$

Note that this is precisely the combination of $U(1)$ transformations that is not broken by color anomalies. The quantized coefficient corresponds to an *even* number of colors $N = N_c$ in the color group.

As mentioned above, the unbroken $SU(2) \times U(1)$ subgroup is anomaly free, and can be identified with the electroweak gauge group of the standard model. [29] The symmetry breaking corresponds to a nonzero VEV $\langle \Psi_L \bar{q}_R \rangle \sim (0, 0, F^3)^T$. For simplicity, we ignore the $U(1)$ factor in the remainder of this section, and use the choice of coordinates of eq.(13). The leading WZW interactions are identical to eq.(25). There are again operators that are invariant under local $SU(3) \times U(1)$ transformations. Suppressing $U(1)_Y$,

$$\Gamma_{GI} = \int_{M^4} c\,\Phi^\dagger (dA - iA^2)^2 \Phi + \ldots\,, \tag{39}$$

where c is a number sensitive to the UV completion theory, and the ellipsis denotes additional terms such as $[\Phi^\dagger(dA - iA^2)\Phi]^2$ relevant at $\mathcal{O}(1/F^4)$. When viewed as deriving from an $SU(3)_L \times SU(3)_R/SU(3)$ theory, parity arguments can be invoked to single out a unique choice of these gauge-invariant operators [8], *e.g.* $r = 0$ in eq.(26), leading to particular values of the coefficients in eq.(39). Sensitivity to the global structure of the field space of NGB's, equivalently, sensitivity to the UV completion theory, is reflected in the undetermined coefficients in eqs.(26) and (39).

It is interesting to investigate further what novel features of $SU(3)/SU(2)$ allow an underlying fermion theory to produce a low-energy theory containing just scalar NGB's and gauge fields, and in particular why a similar construction is not possible for general $SU(n)/SU(n-1)$. At the meson level, the nonlinear realization construction required an anomaly-free gauging of A_R to "eat and decouple" the extra pions; this singled out $SU(2)$

as the unbroken symmetry group (with an even number of colors in the strong color group). The direct construction required the form ω in eq.(31) to be closed, $d\omega = 0$, which is true for S^5, but not for general S^{2n-1} (notice that $\pi_5(S^5) = \boldsymbol{Z}$, but $\pi_5(S^{2n-1}) = 0$ for $n > 3$). At the fermion level, we expect in general that the condensate $\langle \Psi_L \bar{q}_R \rangle \sim (0, 0, \ldots, 0, F^3)^T$ will leave $n-1$ massless fermions ψ_L^i, $i = 1 \ldots (n-1)$, in the low-energy spectrum. For $n > 3$, these fermions enable the low-energy theory to reproduce the full nonabelian anomaly of the underlying UV theory. For the special case of $n = 3$, such fermions are not mandated by anomaly matching. For example, when $N_c = 2$, we observe that it is possible to write the operator

$$\epsilon_{ijk} \epsilon^{ab} \epsilon^{\alpha\beta} \psi^i_{a\alpha} \psi^j_{b\beta} \, \epsilon^{cd} \epsilon^{\gamma\delta} \psi^k_{c\gamma} q_{d\delta} + h.c. \,, \tag{40}$$

where $i, j, \cdots = 1..3$ are flavor indices, $a, b, \cdots = 1..2$ are color indices, and $\alpha, \beta, \cdots = 1..2$ are Lorentz indices in the $(1/2, 0)$ representation of the Lorentz group. The operator eq.(40) is invariant under Lorentz and color $SU(N_c = 2)$ transformations, and also under the flavor $SU(3) \times U(1)$ transformation in eq.(38). When $\Psi_L \bar{q}_R$ develops a VEV, the operator becomes a (Majorana) mass term for the remaining ψ_L^i fermions, removing these degrees of freedom from the low-energy spectrum.

IV. LITTLE HIGGS MODELS

Having constructed the full WZW term, we consider applications to realistic Higgs models. As mentioned in the introduction, the $SU(3)/SU(2)$ symmetry breaking pattern can be applied to a number of scenarios; we focus on the KS model here, concentrating attention on predictions that are independent of the undetermined coefficients in eqs.(26) and (39). We recall that this model in its simplest form consists of two Φ_i fields, with aligned vacuum expectation values. The gauge fields, suppressing $U(1)$ factors, take the form

$$A = \begin{pmatrix} W & C \\ C^\dagger & 0 \end{pmatrix} . \tag{41}$$

Concentrating on interactions involving H, (neglecting η), the kinetic terms are

$$F^2 |D_\mu \Phi_i|^2 = |D_\mu H_i|^2 - F C^\dagger_\mu D_\mu H_i - F (D_\mu H_i^\dagger) C_\mu + F^2 C^\dagger_\mu C_\mu + \ldots\ldots \,, \tag{42}$$

where $i = 1, 2$ and $D_\mu H = (\partial_\mu - i W_\mu) H$. Besides the usual kinetic term for H, this expression contains an F-scale mass for C. The NGB's from the symmetry breaking at scale

F are eaten by the C bosons, and the physical Higgs fields appear as, $H_1 = H$, $H_2 = -H$ (similarly, $\eta_1 = \eta$, $\eta_2 = -\eta$). Terms containing an odd number of meson fields thus cancel in the sum,

$$\mathcal{L}_K = (F^2/2)|D_\mu \Phi_1|^2 + (F^2/2)|D_\mu \Phi_2|^2 = |D_\mu H|^2 + F^2 C_\mu^\dagger C_\mu + \dots \,. \tag{43}$$

The same cancellation will occur with the subleading even-parity "Gasser-Leutwyler" operators [18] (operators not containing an epsilon symbol $\epsilon^{\mu\nu\rho\sigma}$) if the strong-interaction physics is the same in both sectors. In order that anomalies cancel between the Φ_1 and Φ_2 sectors (at the fermion level, $\Phi_1 \sim \Psi_L \bar{q}_R$, $\Phi_2 \sim \Psi_R \bar{q}_L$), the opposite cancellation must occur with odd parity operators such as the WZW term (operators containing an epsilon symbol)—surviving interactions involve an *odd* number of meson fields. Thus for example, the interaction eq.(25) involving η will survive (with a factor of 2 from the sum of the two sectors). The leading terms involving H also occur at order $1/F$, and can be obtained either from the nonlinear realization approach, or from the direct gauging of S^5:

$$\Gamma_{WZW} \supset \frac{N}{16\pi^2 F} \int_{M^4} \left[(DH^\dagger) F_W C - C^\dagger F_W DH \right] \,. \tag{44}$$

This expression is for one sector (a factor of two will appear in the sum of the two sectors). It is manifestly gauge-invariant under electroweak $SU(2)$. The $SU(3)$-invariant odd-parity Gasser-Leutwyler operators in eqs.(26) or (39) contribute only to the orthogonal combination, eq.(44) with the relative minus sign replaced by a plus sign. The new interaction eq.(44) would contribute to the process $e^+e^- \to Z^* \to h^0 C^0$ (note that this is the analog in QCD of the process $e^+e^- \to \rho \to KK^*$).

V. WZW TERM FOR MODELS WITH AN INTERNAL PARITY

The Lagrangian eq.(19) can be used to describe general symmetry breaking patterns, via reduction to nonlinearly realized symmetries acting on submanifolds of a larger space. For example, we obtained the WZW term for $SU(3)/SU(2)$ by embedding the NGB's inside a full $SU(3) \times SU(3)/SU(3)$ multiplet. A further simplification occurs for general models in which the symmetry breaking pattern respects an internal parity operation, and eq.(19) applies also to these cases. The $SU(N)_L \times SU(N)_R/SU(N)$ QCD chiral Lagrangian is one example. Another example is the class of Little Higgs models containing an internal parity. We examine here the structure of the WZW term for this case.

We recall that by suitable choice of coordinates, the action of an element of the full symmetry group can be defined to act, at least locally, on the Nambu Goldstone bosons as [17]

$$e^{i\pi} \to e^{i\pi'} = e^{i\epsilon} e^{i\pi} e^{-i\epsilon'(\epsilon,\pi)} \,. \tag{45}$$

Here $\pi = \sum_a \pi^a t_A^a$ parameterizes the spontaneously broken "axial" symmetry generators, and $\epsilon' = \sum_a \epsilon'^a t_V^a$ is the combination of unbroken "vector" symmetry generators which ensures that $e^{i\epsilon} e^{i\pi} e^{-i\epsilon'}$ can be expressed as $e^{i\pi'}$ for some π'. Now suppose that the transformation:

$$t_V^a \to R(t_V^a) = t_V^a \,, \quad t_A^a \to R(t_A^a) = -t_A^a \tag{46}$$

preserves the group structure, i.e. $[t_V, t_V] \sim t_V$, $[t_A, t_A] \sim t_V$ and $[t_A, t_V] \sim t_A$. Then by multiplying eq.(45) on the right by the result obtained after acting with R and taking the inverse, the quantity $\Sigma \equiv e^{2i\pi}$ is seen to obey the *linear* transformation law

$$\Sigma \to e^{i\epsilon}\, \Sigma\, e^{-iR(\epsilon)} \,. \tag{47}$$

In a more familiar notation, we may write $\epsilon \equiv \epsilon_V - \epsilon_A$ and $\epsilon_V \equiv \epsilon_V^a t_V^a$, $\epsilon_A \equiv \epsilon_A^a t_A^a$. Then

$$\Sigma \to e^{i\epsilon_L} \Sigma e^{-i\epsilon_R} \,, \tag{48}$$

where $\epsilon_{L,R} \equiv \epsilon_V \mp \epsilon_A$. This generalizes eq.(4) to the case where the elements of Σ do not span a full group manifold.

The choice of variables eq.(48) allows us to immediately write down the topological interactions in the form of a WZW term for models with an internal parity operation. The chiral current is defined as $\alpha = d\Sigma\, \Sigma^\dagger$, and obeys $d\alpha = \alpha^2$. The result is simply eq.(7) with $U \to \Sigma$. The anomalous gauge variation of the resulting WZW action is

$$\begin{aligned} \delta\Gamma_{WZW} = -\frac{N}{24\pi^2} \int_{M^4} \mathrm{Tr} \Big\{ & (\epsilon_V - \epsilon_A) \left[(dA_V - dA_A)^2 - \frac{i}{2} d(A_V - A_A)^3 \right] \\ & - (\epsilon_V + \epsilon_A) \left[(dA_V + dA_A)^2 - \frac{i}{2} d(A_V + A_A)^3 \right] \Big\} \,. \end{aligned} \tag{49}$$

As an example, we consider the interactions arising when the Higgs is identified with a Nambu Goldstone boson of the symmetry breaking $SU(5) \to SO(5)$. This pattern of symmetry breaking has been incorporated into a Little Higgs model by Arkani-Hamed, Cohen, Katz and Nelson [15]. Let Φ be a two-index symmetric tensor representation of

$SU(5)$, developing the VEV

$$\langle\Phi\rangle \equiv \Omega = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} . \tag{50}$$

The fourteen NGB's corresponding to broken symmetry generators are

$$\pi = \pi^a t_A^a = \begin{pmatrix} \chi^T + \frac{1}{2}\eta & H^* & \phi^\dagger \\ H^T & -2\eta & H^\dagger \\ \phi & H & \chi + \frac{1}{2}\eta \end{pmatrix} , \tag{51}$$

where χ is a Hermitian, traceless 2×2 matrix, η is a real singlet, H is a complex doublet and ϕ is a symmetric 2×2 matrix. The ten unbroken symmetry generators correspond to

$$A = A^a t_V^a = \begin{pmatrix} -W^T - \frac{g_1}{2}B & C & D\sigma^2 \\ C^\dagger & 0 & -C^T \\ D^*\sigma^2 & -C^* & W + \frac{g_1}{2}B \end{pmatrix} , \tag{52}$$

where $W = \sum_{a=1}^{3} gW^a\sigma^a/2$ is a Hermitian, traceless 2×2 matrix, B is a real singlet, C is a complex doublet, and D is a complex singlet. The model gauges two $SU(2) \times U(1)$ groups, one corresponding to the standard model W and B in eq.(52), and another corresponding to heavy partners W' and B' that will eat χ and η in eq.(51). This implements the notion of "collective symmetry breaking" to stabilize the Higgs mass.

The WZW term can be evaluated straightforwardly. Its existence is related to the non-trivial homotopy group $\pi_5(SU(5)/O(5)) = \boldsymbol{Z}$. Since the full WZW term is odd under the internal parity, and invariant under weak isospin, interactions involving only standard model fields W, B, H are forbidden (the parity operation takes $W, B \to +W, B$ and $H \to -H$). Interactions do occur involving heavy partners W', B' of the standard model gauge bosons, or the Goldstone boson field ϕ. For example, interactions involving the heavy hypercharge field B' are

$$\Gamma_{WZW} \supset \frac{N}{4\pi^2 F^2} \int_{M^4} (v+h^0)^2 B' \big[g^2(W^+dW^- + W^-dW^+ + W^3dW^3) - gg_1(W^3dB + BdW^3) + g_1^2 BdB - ig^2 W^+W^-(3gW^3 - g_1 B) \big] . \tag{53}$$

Here B' is written in unitary gauge, having eaten the η meson. [30] Similarly, W and B are written in unitary gauge, having eaten the Goldstone bosons inside of H. In this gauge we write $H \sim (0, v + h^0)^T/\sqrt{2}$. This WZW term describes "T parity"-violating interactions, *e.g.* decays of the single "T-odd" field B' into standard model fields.

From eq.(49), it is straightforward to see that the anomalous gauge variation of the WZW action is precisely that of a fermion theory with $2N$ left-handed fermions transforming in the fundamental representation of $SU(5)$: [31]

$$\Psi_L \to e^{i(\epsilon_V + \epsilon_A)} \Psi_L \,. \tag{54}$$

In a composite theory of underlying fermions, the symmetry breaking corresponds to a VEV for the operator

$$\epsilon^{\alpha\beta} \psi_\alpha^{ai} \psi_\beta^{aj} \sim F^3 \Omega^{ij} \,, \tag{55}$$

where $i, j = 1..5$ are flavor indices, Ω is a symmetric matrix as in (50), $a = 1..2N$ is a summed color index, and $\alpha, \beta = 1..2$ are Lorentz indices. Such a scheme is possible for fermions transforming in a real representation of the color group—*e.g.*, in the adjoint representation of $SU(N_c)$ for an odd number of colors, $2N = N_c^2 - 1$, or in the fundamental representation of $SO(N_c)$ for an even number of colors, $2N = N_c$.

VI. SUMMARY

We have summarized the first steps toward the theory of topological interactions of Higgs bosons. Such interactions are present when these fields occur as composite pNGB's, or more generally in theories of extra dimensions.

While the analysis involves novel constructions based on topological features of the NGB field manifold, we emphasize that in the spirit of effective field theory, it is simply a mistake to omit such interactions. As in the familiar case of the QCD chiral Lagrangian, the WZW term is a remnant of underlying UV physics, and modifies the predictions of the low-energy theory. For the case of technicolor, we found the interactions eq.(12) amongst the pNGB's and gauge bosons. For cases of interest to Little Higgs theories, we derived the gauged WZW term by two methods, via nonlinear realization of the symmetry group on a restricted submanifold, or via a direct construction beginning with a topological action. Application to specific models give predictions such as eqs.(25) and (44), which are analogs of $\pi^0 \to \gamma\gamma$.

These interactions directly probe the structure of the underlying fermion UV completion theory, *e.g.*, allowing one to count the number of colors in an underlying strong gauge group.

We also pointed out the simple structure of the gauged WZW term for symmetry-breaking patterns that respect an internal parity. This includes the case of the QCD chiral Lagrangian, and the case of Little Higgs models with T parity. If such models arise from composite fermions, we see that T parity cannot be an exact symmetry. As an example, eq.(53) is an interaction between the single "T odd" partner of the hypercharge gauge boson and "T even" standard model particles.

Much work remains in order to explore the phenomenological implications of WZW interactions for Little Higgs bosons, and to identify the most promising signatures at future colliders such as the LHC, ILC, and even beyond to CLIC or a muon collider.

Acknowledgements

We thank W. Bardeen for helpful discussions. This work was hosted at Fermilab in the Fermilab Theoretical Physics Department operated by Fermi Research Alliance, LLC under Contract No. DE-AC02-07CH11359 with the United States Department of Energy.

---

[1] D. B. Kaplan and H. Georgi, Phys. Lett. B **136**, 183 (1984);
D. B. Kaplan, H. Georgi and S. Dimopoulos, Phys. Lett. B **136**, 187 (1984);
H. Georgi, D. B. Kaplan and P. Galison, Phys. Lett. B **143**, 152 (1984);
H. Georgi and D. B. Kaplan, Phys. Lett. B **145**, 216 (1984);
M. J. Dugan, H. Georgi and D. B. Kaplan, Nucl. Phys. B **254**, 299 (1985).

[2] N. Arkani-Hamed, A. G. Cohen and H. Georgi, Phys. Lett. B **513**, 232 (2001) [hep-ph/0105239];
N. Arkani-Hamed, A. G. Cohen, E. Katz, A. E. Nelson, T. Gregoire and J. G. Wacker, JHEP **0208**, 021 (2002) [hep-ph/0206020].

[3] J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971).

[4] E. Witten, Nucl. Phys. B **223**, 422 (1983).

[5] J. S. Bell and R. Jackiw, Nuovo Cim. A **60**, 47 (1969).

[6] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[7] W. A. Bardeen, Phys. Rev. **184**, 1848 (1969).

[8] O. Kaymakcalan, S. Rajeev and J. Schechter, Phys. Rev. D **30**, 594 (1984).

[9] A. Manohar and G. W. Moore, Nucl. Phys. B **243**, 55 (1984).

[10] H. Kawai and S. H. H. Tye, Phys. Lett. B **140**, 403 (1984).

[11] C. T. Hill, Phys. Rev. D **73**, 126009 (2006) [hep-th/0603060];
C. T. Hill, Phys. Rev. D **73**, 085001 (2006) [hep-th/0601154];
C. T. Hill and C. K. Zachos, Phys. Rev. D **71**, 046002 (2005) [hep-th/0411157].

[12] D. E. Kaplan and M. Schmaltz, JHEP **0310**, 039 (2003) [hep-ph/0302049].

[13] C. T. Hill and R. J. Hill, in preparation.

[14] H. C. Cheng and I. Low, JHEP **0309**, 051 (2003) [hep-ph/0308199].

[15] N. Arkani-Hamed, A. G. Cohen, E. Katz and A. E. Nelson, JHEP **0207**, 034 (2002) [hep-ph/0206021].

[16] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969).

[17] S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2239 (1969);
C. G. . Callan, S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2247 (1969).

[18] J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985);
A. J. Buras, A. Poschenrieder, S. Uhlig and W. A. Bardeen, hep-ph/0607189.

[19] E. Witten, Phys. Lett. B **117**, 324 (1982).

[20] E. D'Hoker and S. Weinberg, Phys. Rev. D **50**, 6050 (1994) [hep-ph/9409402].

[21] C. M. Hull and B. J. Spence, Nucl. Phys. B **353**, 379 (1991).

[22] C. S. Chu, P. M. Ho and B. Zumino, Nucl. Phys. B **475**, 484 (1996) [hep-th/9602093].

[23] Note that each term in eq.(1) involves five "coordinates" that locally parameterize an S^5. The term $\sim \eta dK^+ dK^- dK^0 d\bar{K}^0$ corresponds to the submanifold of $SU(3)$ that is identified with Φ.

[24] Alternatively, eq.(7) and eq.(19) can be derived from the Yang-Mills Chern-Simons term and the Bardeen countertyerm in a $D = 5$ Yang-Mills theory compactified on S^1 where A_5 is identified with the mesons [11].

[25] For illustration, consider the hypothetical situation in which the usual Higgs mechanism does not operate, but where the left-handed quarks of some number of Standard Model generations, and hence the QCD chiral Lagrangian, is gauged with the usual $SU(2)$ weak interactions. As

is well known, the W and Z bosons then eat a linear combination of the pions and acquire mass. After transforming to unitary gauge, the remaining pseudoscalar mesons of QCD couple to the W boson through the consistent, not the covariant, anomaly.

[26] The global structure of the NGB manifold in this case (a submanifold of $SU(3)$) allows an arbitrary integer coefficient, N, for the WZW term. However, the natural UV theory in this physical picture has A_R coupled to a right-handed fermion doublet. For this theory to be free of discrete anomalies [19] there should be an even number of such doublets, and therefore an even number of colors. This is related to a factor of 2 advocated by the authors of [20] in cases where the unbroken subgroup H, *e.g.* $H = SU(2)$, has $\pi_4(H) = \boldsymbol{Z}_2$. With the global structure defined by the direct construction based on S^5, a factor of 2 is enforced automatically.

[27] More precisely, a gauged WZW term can be written down for these cases, but the variation of the action results in an anomaly expression containing mesons, so that the theory does not have a simple UV completion in terms of constituent fermions.

[28] For a geometrical description of such identities, see [21].

[29] Having identified an anomaly-free embedding of the unbroken subgroup, and the anomalous gauge variation of the full action, we can work in the opposite direction to reconstruct the gauged WZW term by integration [22]. The non-uniqueness of the WZW term will be reflected in a choice of boundary condition for this construction.

[30] Since the B' symmetry is anomalous, other structure, *e.g.* leptons, must be present to cancel this gauge anomaly.

[31] As noted in [22], eq.(49) is the gauge variation of a Lagrangian with *two* sets of Weyl fermions, Ψ_L and Ψ_R, coupled respectively to the restricted set of gauge fields $A_V - A_A$ and $A_V + A_A$. However, eq.(49) would describe only a subset of the anomalies for such a fermion theory. Using trace identities arising from the anomaly-free embedding of $SO(5)$ inside of $SU(5)$ (*e.g.*, $\mathrm{Tr}[\epsilon_V (dA_V)^2] = 0$, $\mathrm{Tr}[\epsilon_A (dA_A dA_V + dA_V dA_A)] = 0$, $\ldots$), (49) actually gives precisely the nonabelian anomaly for *one* set of Weyl fermions.