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Randall-Sundrum with AdS(7)

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Abstract: In the same sense that AdS_5 warped geometries arise naturally from Type IIB string theory with stacks of $D3$ branes, AdS_7 warped geometries arise naturally from M theory with stacks of $M5$ branes. We compactify two spatial dimensions of AdS_7 to get $AdS_5 \times \Sigma^2$, where Σ^2 is *e.g.* a torus T^2 or a sphere S^2. The metric for Σ inherits the same warp factor as appears in the AdS_5. Bulk fields generically have both Kaluza-Klein and winding modes associated with Σ. In the effective 5d action these will contribute exotic new excitations. We analyze the 5d spectrum in detail for the case of a bulk scalar or a graviton in $AdS_5 \times T^2$, in a setup which mimics the first Randall-Sundrum model. The results display several novel features, some of which might be observed in experiments at the LHC. For example, we obtain TeV scale string winding states without lowering the string scale. This is due to the *double warping* which is a generic feature of winding states along compactified AdS directions. Experimental verification of these signatures of AdS_7 could be interpreted as direct evidence for M theory.

Keywords: warped geometry, branes, Kaluza-Klein, M theory.

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1. Introduction

The five-dimensional warped models of Randall and Sundrum [1] can be thought of as stripped down effective field theory approximations to the near horizon geometry [2] of a stack of N parallel $D3$ branes in Type IIB string theory. This near horizon geometry is $AdS_5 \times S^5$, where the Anti-de Sitter radius R_{AdS} and the radius of the sphere are equal, scaling like $N^{1/4}$. Thus for large N the AdS physics involves an energy scale which is parametrically lower than the string (or Planck) scale. So does the physics on S^5, but we also expect this physics to be drastically altered as one breaks supersymmetry, and otherwise attempt to make the compactification more realistic. Since in any event the sphere S^5 is not warped, in seems reasonable to factor out this physics and examine low energy effective 5d field theory on AdS_5. In the Randall-Sundrum (RS) approach, a phenomenological 3-brane called the Planck brane is added to truncate AdS_5 in a way which

excludes the AdS boundary. From the point of view of string theory, this is a boundary condition which replaces the original matching of the near horizon geometry to $M_5 \times S^5$; in the AdS/CFT dual language, the Planck brane is a particular choice of UV cutoff for the 4d CFT coupled to 4d gravity. The first Randall-Sundrum model (RS1) also introduces a second brane, called the TeV brane. This brane truncates the AdS geometry in the other direction, excluding now the AdS horizon. This is a wise idea for a low energy effective field theory approach, since close to the horizon the field theory cutoff necessarily shrinks to zero and stringy effects become important.

In this paper we examine an analogous story which starts with eleven dimensional M theory. The low energy supergravity theory has a solution corresponding to a stack of N parallel $M5$ branes. These $M5$ branes are just the magnetic sources for the antisymmetric tensor gauge field A_{MNP} of 11d supergravity, which also has electric sources called $M2$ branes. The metric part of this solution is [2]:

$$ds^2 = f^{-1/3}\left(\eta_{\mu\nu}dx^\mu dx^\nu + \delta_{ij}dx^i dx^j\right) + f^{2/3}\left(dr^2 + r^2 d\Omega_4^2\right) \ , \tag{1.1}$$

where Minkowski indices μ, ν, run over 0, 1, 2, 3, while i, j run over 4, 5. Also, $d\Omega_4$ is the line element for a 4-sphere, and the function f is given by

$$f(r) = 1 + \frac{\pi N \ell_p^3}{r^3} \ , \tag{1.2}$$

where ℓ_p is the Planck length, in the convention where the $M2$ branes have inverse tension $(2\pi)^2\ell_p^3$.

We are interested in the near horizon geometry of this solution for (moderately) large N. After a change of variables this is:

$$ds^2 = \mathrm{e}^{-2ky}\left(\eta_{\mu\nu}dx^\mu dx^\nu + \delta_{ij}dx^i dx^j\right) + dy^2 + \frac{1}{4k^2}d\Omega_4^2 \ , \tag{1.3}$$

where $k = 1/R_{AdS}$, and $R_{AdS} = 2(\pi N)^{1/3}\ell_p$.

As motivated above we will discard the unwarped sphere S^4 and consider low energy effective 7d field theories built on AdS_7:

$$ds^2 = a^2(y)\left(\eta_{\mu\nu}dx^\mu dx^\nu + \delta_{ij}dx^i dx^j\right) + dy^2 \ , \tag{1.4}$$

where we have introduced the warp factor $a(y) = \exp(-ky)$. Our metric signature convention is $(-+++++++)$. Note that AdS_7 setups are very restricted by the requirement of anomaly cancellation in the 6d boundary gauge theories [3].

2. $\mathbf{AdS_5}$ with extra warpings

To make contact with a Randall-Sundrum type construction, we now compactify two spatial dimensions of the AdS_7. Thus

$$AdS_7 \to AdS_5 \times \Sigma^2 \quad .$$

The simplest choice for Σ^2 is a torus T^2, since in this case we do not need any additional sources in the Einstein equations to get a consistent background solution. The metric becomes:

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + a^2(y)R^2\left(d\theta_1^2 + d\theta_2^2\right) + dy^2 \ , \tag{2.1}$$

where for simplicity we assume a common radius parameter R. Note that the measured radius of the torus includes a warp factor.

Almost as simple is a compactification on a sphere S^2. Since the sphere has curvature, this requires an additional bulk source. As discussed for example in [4], two bulk scalar fields with a simple potential allows a solution of the equations of motion with a "hedgehog" profile for the scalars and the following background metric:

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + a^2(y)R^2 d\Omega_2^2 + dy^2 \ . \tag{2.2}$$

Note that the radius of the sphere includes a warp factor.

Any 7d bulk field can be expanded in a complete orthonormal basis of Kaluza-Klein (KK) modes for Σ^2. These are just sines and cosines for T^2, and spherical harmonics for S^2. Substituting the KK expansion into the 7d action, we can then integrate over Σ^2 to obtain an effective 5d action.

We will do this explicitly for the case of a 7d bulk scalar on $AdS_5 \times T^2$, and then extend the results to discuss the 7d graviton. For the torus with radius R the squared KK momenta are

$$g^{ij}p_i p_j = \frac{n_1^2 + n_2^2}{a^2R^2} \equiv \frac{\vec{n}^2}{a^2R^2} \ , \tag{2.3}$$

where n_1, n_2 are integers.

In the underlying theory there will in general be additional bulk modes arising from $M2$ or $M5$ branes wrapping one or more cycles of the torus. These will be particle-like modes if the the other dimensions of the branes are already wrapped around compact cycles of S^4, or whatever replaces S^4 in a realistic model. For example, an $M5$ brane wrapped on S^4 is a baryonic string [5]-[7], which can in turn wind around cycles of the torus. Such modes can be labelled by their winding numbers, which form a two-vector of integers m^i, $i = 1,2$.

Whereas the KK mode contributions to the world-sheet Hamiltonian of the string are proportional to $\ell_p^2 g^{ij}p_i p_j$, the winding modes contribute

$$\frac{R^2}{\ell_p^2}g_{ij}m^i m^j = \frac{R^2}{\ell_p^2}a^2\vec{m}^2 \quad . \tag{2.4}$$

Note that the parameter ℓ_p above will in general differ from the ℓ_p introduced below (1.2), by a factor of order one which depends upon how strings emerge from the original 11d M theory.

We can write down the equations of motion for the bulk scalar in the 5d effective theory, expanded into KK and winding modes $\phi_{\vec{n}\vec{m}}(x^\mu, y)$:

$$\left[\frac{1}{a^4(y)}\frac{d}{dy}\left(a^6(y)\frac{d}{dy}\right) - \left(p_\mu p^\mu + \frac{\vec{n}^2}{R^2}\right) - a^2(y)m_b^2 - a^4(y)\frac{\vec{m}^2R^2}{\ell_p^4}\right]\phi_{\vec{n}\vec{m}} = 0 \ , \tag{2.5}$$

where m_b is an explicit bulk mass from the 7d theory.

The usual warping of Randall-Sundrum is visible in (2.5) as the fact that the bulk mass-squared term is multiplied by $a^2(y)$ relative to the 4d momentum-squared $p_\mu p^\mu$. We observe as well two new kinds of behavior:

- The KK mode mass-squared contributions from the torus have no warping relative to the 4d momentum-squared $p_\mu p^\mu$.
- The winding mode mass-squared contributions have double warping compared to the ordinary bulk mass term, *i.e.,* they are multiplied by $a^4(y)$ relative to the 4d momentum-squared $p_\mu p^\mu$.

Changing variables to the conformal coordinate $z = 1/ka(y)$, and suppressing indices, the equation of motion becomes:

$$\phi'' - \frac{5}{z}\phi' + \left[\mu^2 - \frac{\vec{n}^2}{R^2} - \frac{m_b^2}{z^2} - \frac{\vec{m}^2 R^2}{k^4 \ell_p^4}\frac{1}{z^4}\right]\phi = 0 \,, \tag{2.6}$$

where we are now putting the 4d momenta on-shell, with $\mu^2 = -p_\mu p^\mu$ denoting the physical 4d mass. It is the mass spectrum of μ that we want to determine from an analysis of the solutions of (2.6).

A useful rescaling is $\phi(z) = k^3 z^3 h(z)$; then (2.6) becomes:

$$z^2 h'' + z h' + \left[\left(\mu^2 - \frac{\vec{n}^2}{R^2}\right) z^2 - \nu^2 - \frac{\vec{m}^2 R^2}{k^4 \ell_p^4}\frac{1}{z^2}\right] h = 0 \,, \tag{2.7}$$

where $\nu^2 = m_b^2 + 9$. For $\vec{n}^2 = \vec{m}^2 = 0$, this is the equation of motion for a 5d massive bulk scalar in AdS_5, as first analyzed in [8]. In general (2.7) is equivalent to the Mathieu equation [9].

3. 5d Kaluza-Klein mode analysis

We can take (2.7) as the starting point for a 5d Kaluza-Klein mode analysis in AdS_5. To get the analog of scalars in a Randall-Sundrum model, we can impose Neumann boundary conditions at the location of one or two flat branes. These branes have co-dimension one, i.e. they are 5-branes, located at $z = 1/k$ and $z = 1/T$. The parameter T is assumed to be of order the TeV scale.

To analyze graviton modes, we use the fact that in *e.g.* harmonic gauge each component of the graviton obeys (2.7). The appropriate boundary conditions for the graviton modes, after tuning the brane tensions to their conventional RS values, are:

$$\left[z h'(z) + 3h(z)\right]_{z=1/k} = \left[z h'(z) + 3h(z)\right]_{z=1/T} = 0 \,. \tag{3.1}$$

This means that the graviton mode solutions are identical to the $m_b = 0$ scalar solutions, up to an overall factor.

Let us start with the simple case where $\vec{m} = 0$. The solution to (2.7) depends upon the sign of $\mu^2 - \vec{n}^2/R^2$:

$\mu^2 - \frac{\vec{n}^2}{R^2} > 0$**:**

The solution in this case is

$$\phi(z) = \frac{k^3 z^3}{N}\left[J_\nu\Big(z\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big) + bY_\nu\Big(z\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big)\right] . \tag{3.2}$$

Imposing the boundary condition of $\phi'(z) = 0$ at $z = 1/k$ and $z = 1/T$ one gets

$$b = -\frac{3J_\nu\Big(\frac{1}{k}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big) + \frac{1}{k}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\, J'_\nu\Big(\frac{1}{k}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big)}{3Y_\nu\Big(\frac{1}{k}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big) + \frac{1}{k}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\, Y'_\nu\Big(\frac{1}{k}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big)} , \tag{3.3}$$

and

$$b = -\frac{3J_\nu\Big(\frac{1}{T}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big) + \frac{1}{T}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\, J'_\nu\Big(\frac{1}{T}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big)}{3Y_\nu\Big(\frac{1}{T}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big) + \frac{1}{T}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\, Y'_\nu\Big(\frac{1}{T}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}}\Big)} . \tag{3.4}$$

If we only consider the case of light modes where $\mu^2 - \frac{\vec{n}^2}{R^2} \ll k^2$, then the boundary conditions yield

$$3J_\nu(x) + xJ'_\nu(x) = 0 , \tag{3.5}$$

where

$$x = \frac{1}{T}\sqrt{\mu^2 - \frac{\vec{n}^2}{R^2}} . \tag{3.6}$$

The roots of (3.5) determine the mass spectrum. These solutions represent towers of toroidal KK modes on top of the TeV-spaced massive warped KK modes of Randall-Sundrum. The RS spectrum itself is also modified, since *e.g.*, for $m_b = 0$, $\vec{n} = 0$ the relevant Bessel functions are $J_3(\mu z)$ and $Y_3(\mu z)$, rather than $J_2(\mu z)$ and $Y_2(\mu z)$ as in standard RS.

Since the toroidal KK modes are not warped, $R \gtrsim 1/T$ corresponds to compactifications large enough that we will see the toroidal KK excitations with spacings comparable to the Randall-Sundrum KK modes. In order not to reintroduce the hierarchy problem, we would then need some mechanism which naturally stabilizes R at inverse TeV values. By contrast, if our torus were compactifying flat rather than AdS directions, then $R \sim 1/k$ would give TeV-spaced KK excitations [10].

$\mu^2 - \frac{\vec{n}^2}{R^2} = 0$**:**

Now the solution to the equation of motion is

$$\phi(z) = \frac{k^3 z^3}{N}\left(z^{-\nu} + bz^{\nu}\right) . \tag{3.7}$$

Similarly, the boundary conditions give us

$$(3-\nu)k^\nu + b(3+\nu)k^{-\nu} = 0 , \tag{3.8}$$

$$(3-\nu)T^\nu + b(3+\nu)T^{-\nu} = 0 . \tag{3.9}$$

This set of equations has no solution except when $\nu = 3$, *i.e.,* when there is a scalar zero mode. The solutions then represent the tower of toroidal KK modes on top of the zero mode.

$\mu^2 - \frac{\vec{n}^2}{R^2} < 0$:

In this case, the solution to the equation of motion is

$$\phi(z) = \frac{k^3 z^3}{N}\left[I_\nu\Big(z\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big) + bK_\nu\Big(z\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big)\right] . \tag{3.10}$$

Imposing the same boundary condition of $\phi'(z) = 0$ again at $z = 1/k$ and $z = 1/T$ one gets

$$b = -\frac{3I_\nu\Big(\frac{1}{k}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big) + \frac{1}{k}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\, I'_\nu\Big(\frac{1}{k}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big)}{3K_\nu\Big(\frac{1}{k}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big) + \frac{1}{k}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\, K'_\nu\Big(\frac{1}{k}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big)} , \tag{3.11}$$

and

$$b = -\frac{3I_\nu\Big(\frac{1}{T}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big) + \frac{1}{T}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\, I'_\nu\Big(\frac{1}{T}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big)}{3K_\nu\Big(\frac{1}{T}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big) + \frac{1}{T}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\, K'_\nu\Big(\frac{1}{T}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2}\Big)} . \tag{3.12}$$

Again if we only consider the case of light modes where $\frac{\vec{n}^2}{R^2} - \mu^2 \ll k^2$, then the boundary conditions yield

$$3I_\nu(x) + xI'_\nu(x) = 0 , \tag{3.13}$$

where

$$x = \frac{1}{T}\sqrt{\frac{\vec{n}^2}{R^2}-\mu^2} . \tag{3.14}$$

However, (3.13) does not have a solution for positive x. Therefore no solutions exist with $\mu^2 - \frac{\vec{n}^2}{R^2} < 0$.

4. 5d winding mode analysis

The most interesting new feature of our analysis is the spectrum of winding modes. For simplicity, we will only consider the simplest case, with $\vec{n} = 0$ and $m_b = 0$. Then (2.7) reduces to:

$$z^2 h'' + zh' + \Big(\mu^2 z^2 - 9 - \frac{\tilde{R}^2}{z^2}\Big)h = 0 , \tag{4.1}$$

where we have introduced the notation $\tilde{R} = R|\vec{m}|/k^2\ell_p^2$. As usual in Randall-Sundrum setups, we will assume that $k\ell_p$ is less than 1 but not dramatically so, *e.g.,* $k\ell_p \simeq 0.1$.

First of all, we want to see if there exists a zero mode. Setting $\mu = 0$ the solution of (4.1) is

$$\phi(z) = \frac{k^3 z^3}{N} \left[I_\nu\Big(\frac{\tilde{R}}{z}\Big) + b K_\nu\Big(\frac{\tilde{R}}{z}\Big) \right] . \tag{4.2}$$

The boundary conditions give:

$$b = -\frac{3I_\nu(k\tilde{R}) - k\tilde{R}\, I'_\nu(k\tilde{R})}{3K_\nu(k\tilde{R}) - k\tilde{R}\, K'_\nu(k\tilde{R})} , \tag{4.3}$$

$$b = -\frac{3I_\nu(T\tilde{R}) - T\tilde{R}\, I'_\nu(T\tilde{R})}{3K_\nu(T\tilde{R}) - T\tilde{R}\, K'_\nu(T\tilde{R})} , \tag{4.4}$$

which cannot both be true at the same time unless $T = k$. Therefore there is no zero mode for the case with winding modes only.

For the massive winding modes, the solutions can be found numerically. The details of the analysis are provided in the appendix; here we summarize the results.

$\mathbf{R \sim 1/k}$

This corresponds to the smallest compactification radius, where we expect the winding modes to be the lightest. Like the KK modes, these should appear as additional excitations on top of the TeV-spaced warped KK modes of Randall-Sundrum. We find that the RS zero mode does not have any winding excitations. The winding excitations of the lowest lying massive RS modes have masses given by $\mu = \mu_1 - 0.40\,\tilde{R}^2 T^3$ and $\mu = \mu_2 + 1.46\,\tilde{R}^2 T^3$, where $\mu_1 = 5.13\,T$ and $\mu_2 = 8.42\,T$ are the masses of the first two massive RS modes. Notice that the mass splittings are very small, of order $10^4 \times T^3/k^2 \sim 10^{-12}$ eV.

$\mathbf{R \sim 1/T}$

This corresponds to a large compactification radius, thus were it not for warping we would expect the winding excitations to be extremely heavy. However, because of the double warping of the winding term in (2.7), we find that in this case the winding excitations are actually of order T. Once again the RS zero mode does not have any winding excitations. The winding excitations of the lowest lying massive RS mode have masses as shown in Figure 1. Note that the spacing of the winding modes can be considerably less than T if R is small, *e.g.,* for $RT \sim 0.01$ and $k\ell_p \simeq 0.1$ the spacing is a fraction of T.

5. Double warping versus T-duality

Our results for the spectrum of KK and winding modes can be understood qualitatively as a combination of warping with the usual T-duality relation between the KK and winding spectra. For simplicity we will not distinguish between k and ℓ_p in this discussion.

When the compactification radius R is of order $1/T$, the toroidal KK modes are or order T, with no additional warping. The T-dual winding modes are of order k^2/T, multiplied by a double warp factor T^2/k^2; thus the winding modes are of order $k^2/T \times T^2/k^2 \sim T$, as reported above.

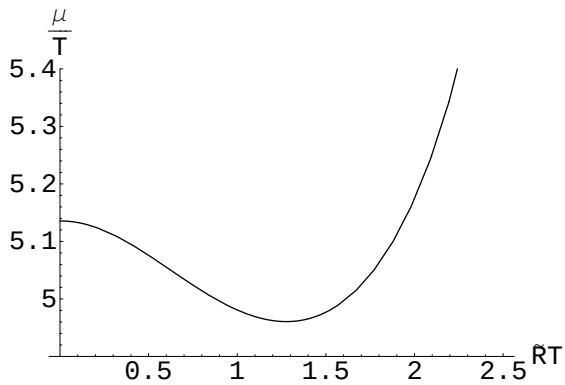

Figure 1: Winding excitations of the lightest RS massive mode, for $\tilde{R}$ of order $1/T$.

When the compactification radius R is of order $1/k$, the toroidal KK modes are of order k, with no additional warping. The T-dual winding modes are of order k as well, but again multiplied by a double warp factor T^2/k^2. The winding excitations are on top of the massive warped KK excitations, which are of order T. Since the masses add in quadrature, we expect the splittings of the winding mode excitations to be of order

$$\sqrt{T^2 + \frac{T^4}{k^2}} - T \sim \frac{T^3}{k^2} \ , \tag{5.1}$$

as reported above.

6. LHC discovery opportunities

The $AdS_7 \to AdS_5 \times \Sigma^2$ scenarios presented here are as well-motivated as the AdS_5 effective theories that are the basis for standard Randall-Sundrum. Both are stripped down versions of physics which could actually emerge from string theory in a robust way, albeit decorated with many stringy complications.

The spectrum of bulk graviton and scalar modes in our AdS_7 models can differ from standard RS1 in several ways. First of all, because we start with AdS_7 instead of AdS_5, the spacing of the massive graviton modes is determined by the zeroes of the Bessel function $J_2(\mu T)$, rather than $J_1(\mu T)$. The smoking gun signature of RS1 at the LHC is [11] a tower of resonances with masses 1, 1.83, 2.66, 3.48, ..., in units of the first heavy resonance. For our models, the prediction is a tower of resonances with masses 1, 1.64, 2.26, 2.88, ..., in units of the first heavy resonance.

LHC experiments will also be sensitive to the extra toroidal KK states, if R is of order a TeV. The spectrum of these additional excitations is evenly spaced, distinguishing it from extra toroidal dimensions not associated with AdS directions [10].

LHC experiments may be sensitive to our extra winding states, depending upon which bulk modes have winding excitations, and how they couple to Standard Model particles. For R of order $1/T$, the winding states have discrete mass spacings or order T. For R of order $1/k$, the winding states are essentially a continuum on top of a mass gap of $5.13\,T$.

This is somewhat reminiscent of LR models [12], but clearly distinct since in that case there is no mass gap in the continuum KK spectrum.

Light winding states have been discussed previously in the literature [13] in the context of ADD models [14] with a string scale on the order of a TeV [15]. Note that here we have obtained TeV scale string winding states without lowering the string scale, and without large extra dimensions. This is due to the double warping which is a generic feature of winding states along compactified AdS directions.

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APPENDIX

A. Generic solution

In this appendix we perform a general analysis of the solutions to (2.7) for $m_b = 0$. The equation is:

$$z^2 h'' + z h' + \Big(\mu^2 z^2 - 9 - \frac{\tilde{R}^2}{z^2}\Big) h = 0 \ . \tag{A.1}$$

Since omitting either the μ term or the $\tilde{R}$ term gives Bessel's equation, the solution to (A.1) should be expressible as a double expansion in Bessel functions. Let us test to see if

$$h = \sum C_n J_n(\mu z) I_{n-3}\Big(\frac{\tilde{R}}{z}\Big) \tag{A.2}$$

is indeed a solution, for some set of constants C_n.

With this ansatz we have

$$\begin{aligned} & z^2 h'' + z h' + \Big(\mu^2 z^2 - 9 - \frac{\tilde{R}^2}{z^2}\Big) h \\ & = \sum C_n \left[2n(n-3) J_n(\mu z) I_{n-3}\Big(\frac{\tilde{R}}{z}\Big) - 2xy J_n'(x)|_{x=\mu z} I_{n-3}'(y)\big|_{y=\frac{\tilde{R}}{z}} \right] \ . \end{aligned} \tag{A.3}$$

Using

$$\begin{aligned} I_\nu'(x) &= I_{\nu-1}(x) - \frac{\nu}{x} I_\nu(x) = I_{\nu+1}(x) + \frac{\nu}{x} I_\nu(x) \ , \\ J_\nu'(x) &= J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) \ , \end{aligned} \tag{A.4}$$

it is easy to get

$$\begin{aligned} & xy J_n'(x) I_{n-3}'(y) \\ & = -n(n-3) J_n(x) I_{n-3} + \frac{1}{2} xy \left[J_{n-1}(x) I_{n-4}(y) - J_{n+1}(x) I_{n-2}(y) \right] \ . \end{aligned} \tag{A.5}$$

Then (A.3) reduces to

$$\begin{aligned} & z^2 h'' + z h' + \Big(\mu^2 z^2 - 9 - \frac{\tilde{R}^2}{z^2}\Big) h \\ & = \sum \Big[4n(n-3) C_n + \mu \tilde{R} C_{n-1} - \mu \tilde{R} C_{n+1} \Big] J_n(\mu z) I_{n-3}\Big(\frac{\tilde{R}}{z}\Big) \ . \end{aligned} \tag{A.6}$$

Thus if

$$4n(n-3) C_n + \mu \tilde{R} C_{n-1} - \mu \tilde{R} C_{n+1} = 0 \tag{A.7}$$

then (A.2) is indeed a solution to (A.1).

Since (A.7) is a three term recursion relation, it gives us two independent choices of initial values. For instance, one can choose one set of C_n's such that $C_1 = 1, C_2 = 0$ and

another set with $C_1 = 0, C_2 = 1$. We will verify below that these choices give linearly independent solutions.

The analysis simplifies for certain ranges of parameters. Let us first consider the case $\mu\tilde{R} \ll 1$. Then the $C_1 = 1, C_2 = 0$ set, to leading order of $\mu\tilde{R}$, is given by:

$$\begin{aligned}
C_0 &= 8(\mu\tilde{R})^{-1} \\
C_1 &= 1 \\
C_2 &= 0 \\
C_3 &= 1 \\
C_4 &= 0 \\
C_5 &= 1 \\
C_n &= \frac{(n-1)!(n-4)!}{3} 2^{2n-13} (\mu\tilde{R})^{5-n}, \quad n > 5 \\
C_{-1} &= 1 \\
C_{-2} &= -8(\mu\tilde{R})^{-1} \\
C_{-n} &= \frac{(n-1)!(n+2)!}{3} (-1)^{n-1} 2^{2n-4} (\mu\tilde{R})^{1-n}, \quad n > 2 \, .
\end{aligned} \tag{A.8}$$

We let h_1 denote the solution h associated with this set of coefficients. In the same way, let h_2 be the solution with $C_1 = 0, C_2 = 1$. The detailed C_n's are:

$$\begin{aligned}
C_0 &= 1 \\
C_1 &= 0 \\
C_2 &= 1 \\
C_3 &= -8(\mu\tilde{R})^{-1} \\
C_4 &= 1 \\
C_5 &= 8(\mu\tilde{R})^{-1} \\
C_n &= \frac{(n-1)!(n-4)!}{3} 2^{2n-10} (\mu\tilde{R})^{4-n}, \quad n > 5 \\
C_{-1} &= 0 \\
C_{-2} &= 1 \\
C_{-n} &= \frac{(n-1)!(n+2)!}{3} (-1)^{n} 2^{2n-7} (\mu\tilde{R})^{2-n}, \quad n > 2 \, .
\end{aligned} \tag{A.9}$$

Since $\mu\tilde{R} \ll 1$, one can choose a region such that $\mu z \ll 1$ and $\tilde{R}/z \ll 1$. In this region, one can approximate the Bessel functions by powers of their arguments. For h_1, the leading terms for each n are as follows:

	J_n	I_{n-3}	C_n	$J * I * C$
$-n$	$(\mu z)^n$	$(\tilde{R}/z)^{n+3}$	$(\mu\tilde{R})^{1-n}$	$\frac{\mu\tilde{R}^4}{z^3}$
$n=0$	1	$(\tilde{R}/z)^3$	$(\mu\tilde{R})^{-1}$	$\frac{\tilde{R}^2}{\mu z^3}$
$n=1$	μz	$(\tilde{R}/z)^2$	1	$\frac{\mu\tilde{R}^2}{z}$
$n=2$	$(\mu z)^2$	$\tilde{R}/z$	0	0
$n=3$	$(\mu z)^3$	1	1	$\mu^3 z^3$
$n=4$	$(\mu z)^4$	$\tilde{R}/z$	0	0
$n \geq 5$	$(\mu z)^n$	$(\tilde{R}/z)^{n-3}$	$(\mu\tilde{R})^{5-n}$	$\mu^5 \tilde{R}^2 z^3$

where we have suppressed the numerical coefficients in front of these terms. With this choice of parameters, one then finds that the leading contribution for h_1 comes from $n = 0$ and $n = 3$: the contributions are $\tilde{R}^2/\mu z^3$ and $\mu^3 z^3$.

Using the same method one finds that the leading contribution for h_2 comes from $n = 0$ and $n = 3$ as well. This time they are $\tilde{R}^3/z^3$ and $\mu^4\tilde{R}z^3$.

To check linear independence of the solutions, we can further specify the region for z such that $\mu/\tilde{R}^2 \gg 1/z^3 \gg \mu^2/\tilde{R}$. One simple choice is to have z be of order $\sqrt{\mu\tilde{R}}$. With that choice of z, $\tilde{R}^2/\mu z^3 \gg \mu^3 z^3$ and $\mu^4\tilde{R}z^3 \gg \tilde{R}^3/z^3$. Then to leading order one has

$$h_1 \sim \frac{1}{z^3} \,, \tag{A.10}$$

$$h_2 \sim z^3 \,. \tag{A.11}$$

Obviously these are linearly independent.

We conclude that the generic solution for (A.1) is

$$h = A_1 h_1 + A_2 h_2 \,. \tag{A.12}$$

In order to compute the spectrum, we need to apply boundary conditions at both $z = 1/k$ and $z = 1/T$:

$$\frac{\partial}{\partial z} z^3 h = 0 \,. \tag{A.13}$$

These two equations will determine the mass eigenvalues μ. We will only focus on light modes with $\mu \ll k$. Since different approximations are used depending on the choice of parameters, we divide the discussion into cases.

B. $\tilde{\mathbf{R}} \sim \mathbf{1/k}$

We start with $\tilde{R} \sim 1/k$. Then again we get $\mu\tilde{R} \ll 1$. Therefore, (A.8) and (A.9) could still be used as approximate definitions for the two solutions. Also because we are only interested in $\mu \ll k$, at $z = 1/k$ we have $\mu z \ll 1$ and thus at $z = 1/k$

$$J_n(\mu z) \approx \left(\frac{\mu}{k}\right)^n \frac{1}{2^n}\frac{1}{n!} \,, \tag{B.1}$$

and

$$J_{-n}(\mu z) \approx \left(\frac{\mu}{k}\right)^n \left(-\frac{1}{2}\right)^n \frac{1}{n!} \,. \tag{B.2}$$

The I_n's are all of order one since $m/z \sim 1$.

We can then list the contributions for each solution and keep only the leading contribution; we find that both of those come from the term with $n = 0$. The boundary condition at $z = 1/k$ implies

$$A_1 = -\frac{1}{8}\mu\tilde{R}A_2 \,. \tag{B.3}$$

Inserting this into (A.8) and (A.9), we get a combination of C_n's for h:

$$
\begin{aligned}
C_0 &\sim (\mu\tilde{R})^2/16 \\
C_1 &= -\mu\tilde{R}/8 \\
C_2 &= 1 \\
C_3 &= -8(\mu\tilde{R})^{-1} \\
C_4 &= 1 \\
C_5 &= 8(\mu\tilde{R})^{-1} \\
C_n &= \frac{(n-1)!(n-4)!}{3} 2^{2n-10}(\mu\tilde{R})^{4-n}, \quad n > 5 \\
C_{-1} &= -\mu\tilde{R}/8 \\
C_{-2} &= 2 \\
C_{-n} &= \frac{(n-1)!(n+2)!}{3}(-1)^n 2^{2n-6}(\mu\tilde{R})^{2-n}, \quad n > 2 \ .
\end{aligned} \tag{B.4}
$$

The coefficients of $(\mu\tilde{R})^2$ comes from the second order contribution and depend on the specific choice of $\tilde{R}$.

B.1 $\mu \ll T$

We first check for the existence of very light modes: $\mu \ll T$. At $z = 1/T$, (B.1) and (B.2) still apply. Furthermore

$$
I_n\Big(\frac{\tilde{R}}{z}\Big) \approx (\tilde{R}T)^{|n|}\frac{1}{2^{|n|}|n|!} \ . \tag{B.5}
$$

We now look at h and determine the leading contributions:

	J_n	I_{n-3}	C_n	$J*I*C$
$-n$	$(\mu/T)^n$	$(T/k)^{n+3}$	$(\mu/k)^{2-n}$	$\frac{\mu^2T^3}{k^5}$
$n=0$	1	$(T/k)^3$	$(\mu/k)^2$	$\frac{\mu^2T^3}{k^5}$
$n=1$	μ/T	$(T/k)^2$	μ/k	$\frac{\mu^2T}{k^3}$
$n=2$	$(\mu/T)^2$	T/k	1	$\frac{\mu^2}{kT}$
$n=3$	$(\mu/T)^3$	1	k/μ	$\frac{\mu^2k}{T^3}$
$n\geq 4$	$(\mu/T)^n$	$(T/k)^{n-3}$	$(\mu/k)^{4-n}$	$\frac{\mu^4}{T^3k}$

We see that the leading contribution of $\partial_z(z^3h)$ comes from $n = 3$, with a non-zero coefficient and nothing else to cancel it. So there are actually no very light modes.

B.2 $\mu \sim T$

Let us now look for modes with $\mu \sim T$. In this case, at $z = 1/T$, $\mu z \sim 1$. So all the J_n's are of order one, but (B.5) still applies. As before, we look for the leading terms:

	I_{n-3}	C_n	$J*I*C$
$-n$	$(T/k)^{n+3}$	$(\mu/k)^{2-n}$	$\frac{\mu^2T^3}{k^5}(\frac{T}{\mu})^n$
$n=0$	$(T/k)^3$	$(\mu/k)^2$	$\frac{\mu^2T^3}{k^5}$
$n=1$	$(T/k)^2$	μ/k	$\frac{\mu T^2}{k^3}$
$n=2$	T/k	1	$\frac{T}{k}$
$n=3$	1	k/μ	$\frac{k}{\mu}$
$n\geq 4$	$(T/k)^{n-3}$	$(\mu/k)^{4-n}$	$\frac{\mu^4}{T^3k}(\frac{T}{\mu})^n$

Once again we find that the leading term comes from the $n = 3$ term, and there is no other term to cancel it. However, this time the J_n's in the prefactor can vanish; this happens when

$$J_2(x) = 0 \ , \tag{B.6}$$

where $x = \mu/T$. This equation has solutions with $\mu \sim T$. In order to see the correction from implementing a non-zero $\tilde{R}$, we need to look at sub-leading terms. The boundary condition at $z = 1/T$, up to 2nd order of $\tilde{R}T$, is

$$J_2(x) - \tilde{R}^2 T^2 \sum_{n=2}^{\infty} \frac{n!(n+2)2^n}{96x^{n-1}} J_{n+3}(x) = 0 \ . \tag{B.7}$$

Given this, we can calculate the correction to the original spectrum numerically. For the first excited mode, $5.13\,T - 0.40\,\tilde{R}^2T^3$; the second massive mode has $8.42\,T + 1.46\,\tilde{R}^2T^3$.

C. $\tilde{\mathbf{R}} \sim \mathbf{1/T}$

C.1 $\mu \ll T$

Now let us look at the case when the size of the compactified dimensions gets larger. Since for large x, $I_n(x)$ doesn't really depend on n, at $z = 1/k$, all the I_n's can be viewed as identical.

As $\mu\tilde{R} \ll 1$ still stands, (A.8) and (A.9) are still the coefficients for h_1 and h_2. Plug them into the boundary condition at $z = 1/k$, once again we get

$$A_1 = -\frac{1}{8}\mu\tilde{R}A_2 \tag{C.1}$$

So we can still use (B.4) to compute h, with the change that C_0 is now of order $\mu^2\tilde{R}/k$.

Inserting this into the boundary condition at $z = 1/T$ gives:

	J_n	I_{n-3}	C_n	$J*I*C$
$-n$	$(\mu/T)^n$	1	$(\mu/T)^{2-n}$	$\frac{\mu^2}{T^2}$
$n=0$	1	1	μ^2/Tk	$\frac{\mu^2}{Tk}$
$n=1$	μ/T	1	μ/T	$\frac{\mu^2}{T^2}$
$n=2$	$(\mu/T)^2$	1	1	$\frac{\mu^2}{T^2}$
$n=3$	$(\mu/T)^3$	1	T/μ	$\frac{\mu^2}{T^2}$
$n\geq 4$	$(\mu/T)^n$	1	$(\mu/T)^{4-n}$	$\frac{\mu^4}{T^4}$

The leading terms are $n = 1, 2, 3$ and all negative n's. Summing these contributions, the boundary condition becomes

$$\begin{aligned} 0 = &- \tfrac{3}{8}yI_2(y) + \left(\tfrac{1}{16}y^2 + \tfrac{11}{12}\right) I_1(y) - \left(\tfrac{1}{8}y + \tfrac{1}{y}\right) I_0(y) \\ &- \textstyle\sum_{n=1}^{\infty} \frac{(n+2)!2^n}{192n} y^{3-n} I_{n+4}(y) \ , \end{aligned} \tag{C.2}$$

where $y = \tilde{R}T$. The right-hand side is non-zero for all $\tilde{R}$, so there is no solution with $\mu \ll T$.

C.2 $\mu \sim T$

Now we have to face a situation where $\mu\tilde{R}$ is of order one. Therefore (A.8) and (A.9) are no longer a good approximation for the coefficients of h_1 and h_2 and we don't know a simple way to write them out. The only hope now is that the first few terms in the sum dominates and we can approximate in that way.

Let us look at $z = 1/k$ first. I_n's are identical as we stated before. We still have $\mu k \ll 1$ so (B.1) and (B.2) are still true. Therefore the leading term for the boundary condition comes from the $n = 0$ term. This gives us

$$A_1 = -\frac{1}{8}\mu\tilde{R}A_2 \ . \tag{C.3}$$

The complete set of C_n's for h can not be computed in a simple way. The first few terms are:

$$\begin{aligned}
C_{-2} &= -\frac{16}{\mu\tilde{R}} \\
C_{-1} &= 1 \\
C_0 &= 0 \\
C_1 &= 1 \\
C_2 &= -\frac{8}{\mu\tilde{R}} \\
C_3 &= 1 + \frac{64}{(\mu\tilde{R})^2} \\
C_4 &= -\frac{8}{\mu\tilde{R}} \\
C_5 &= 1 - \frac{64}{(\mu\tilde{R})^2} \ .
\end{aligned} \tag{C.4}$$

At $z = 1/T$ things are even more complicated. Now that both μz and $\tilde{R}/z$ are of order one, we have no approximation to use. And for large n, $C_n \sim (n!)^2$. However, J_n and I_n decrease much faster than $1/n!$. So for large n, the series converges reasonably quickly. Therefore, we can use a finite sum to get an approximation for h and compute the spectrum numerically. The results for the first two massive modes are shown in Figure 1.

D. $1/k \ll \tilde{R} \ll 1/T$

Having understood the spectrum of light modes for two specific ranges of $\tilde{R}$, the next step is to understand what happens in between. For $1/k \ll \tilde{R} \ll 1/T$, $\tilde{R}T$ is still much smaller than 1 so every approximation we made at $z = 1/T$ is still true. The only difference is that at $z = 1/k$, $\tilde{R}/z$ is no longer of order one. Instead, $\tilde{R}k \gg 1$. However, the only feature we used for $\tilde{R}k$ in section 2 is that $I_{n+3}(\tilde{R}k)$ are of the same order. This statement is still true when $\tilde{R}k \gg 1$. So what we got in the $\tilde{R} \sim 1/k$ case applies for this situation as well.

E. $\tilde{\mathbf{R}} \gg \mathbf{1/T}$

Now let us look at the situation when $\tilde{R}$ becomes large. Here for all range of z from $1/k$ to $1/T$, $\tilde{R}/z \gg 1$. Therefore at leading order all the I's could be regarded as identical

$$I_n\Big(\frac{\tilde{R}}{z}\Big) \approx \frac{e^{\frac{\tilde{R}}{z}}}{\sqrt{2\pi\frac{\tilde{R}}{z}}} \ . \tag{E.1}$$

The boundary condition then just yields

$$\sum C_n J_n(\mu z) = 0 \tag{E.2}$$

For $\mu \ll 1/\tilde{R}$, it is very similar to what we did previously. The C_n's are calculated in (A.8) and (A.9). We use the boundary condition at $z = 1/k$ to get the relation of A_1 and A_2 and use the one at $z = 1/T$ to check if we get a solution of μ in the region we assumed. The same is true for the case $\mu \sim \frac{1}{\tilde{R}}$, except that we need to use (C.4) this time. However, we don't find any solutions in either case.

For $\mu\tilde{R} \gg 1$ things are a bit different. The C_n's now can be separated into 2 part, even and odd. To the leading order all C_n's for even n are same, and all C_n's for odd n are the same. The boundary condition at $z = 1/k$ gives us $C_0 = 0$, so we only need to consider the odd ones.

On the other hand we notice that $J_{-n} = (-1)^n J_n$. So for odd n, J_n and J_{-n} cancel out. Then we have no leading term left and have to consider sub-leading terms. We use

$$I_n\Big(\frac{\tilde{R}}{z}\Big) \approx \frac{e^{\frac{\tilde{R}}{z}}}{\sqrt{\frac{2\pi\tilde{R}}{z}}}\Big(1 + \frac{1-4n^2}{\frac{8\tilde{R}}{z}}\Big) \ , \tag{E.3}$$

and for the C_n's given by boundary condition at $z = 1/k$:

$$\begin{aligned}
C_0 &= 0 \\
C_{2n+1} &= 1 - \frac{8}{9(\mu\tilde{R})^2}\, n(n+1)(16n^4 - 40n^3 - 25n^2 + 67n + 18) \\
C_{2n} &= \frac{4}{3\mu\tilde{R}}\, n(4n^2 - 9n - 1) \\
C_{-(2n+1)} &= 1 + \frac{8}{9(\mu\tilde{R})^2}\, n(n+1)(16n^4 + 104n^3 + 191n^2 + 67n - 18) \\
C_{-2n} &= -\frac{4}{3\mu\tilde{R}}\, n(4n^2 + 9n - 1) \ .
\end{aligned} \tag{E.4}$$

Now we plug these into the boundary condition and look for a solution for μ. However for $1/\tilde{R} \ll \mu \ll T$ there is none, and for $\mu \sim T$ we get

$$\sum_{n=1}^{\infty}(2n+1)J_{2n+1}(\mu T) = 0 \ , \tag{E.5}$$

which also has no solution.

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