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The Interval Approach to Braneworld Gravity

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Abstract: Gravity in five-dimensional braneworld backgrounds may exhibit extra scalar degrees of freedom with problematic features, including kinetic ghosts and strong coupling behavior. Analysis of such effects is hampered by the standard heuristic approaches to braneworld gravity, which use the equations of motion as the starting point, supplemented by orbifold projections and junction conditions. Here we develop the interval approach to braneworld gravity, which begins with an action principle. This shows how to implement general covariance, despite allowing metric fluctuations that do not vanish on the boundaries. We reproduce simple $\mathbf{Z_2}$ orbifolds of gravity, even though in this approach we never perform a $\mathbf{Z_2}$ projection. We introduce a family of "straight gauges", which are bulk coordinate systems in which both branes appear as straight slices in a single coordinate patch. Straight gauges are extremely useful for analyzing metric fluctations in braneworld models. By explicit gauge fixing, we show that a general AdS_5/AdS_4 setup with two branes has at most a radion, but no physical "brane-bending" modes.

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Contents

1. Introduction **2**

2. Orbifolds in field theory **3**
- 2.1 Scalars on a 5d orbifold 3
- 2.2 Abelian gauge theory on a warped orbifold 5
  - 2.2.1 massless modes 7
  - 2.2.2 massive modes 8

3. Gravity on a flat orbifold **9**
- 3.1 Orbifold picture 9
- 3.2 Interval picture 11
  - 3.2.1 $\mathbf{p^2 \neq 0}$ 13
  - 3.2.2 $\mathbf{p^2 = 0}$ 14

4. The Randall-Sundrum model in the interval picture **16**
- 4.1 $\mathbf{p^2 \neq 0}$ 17
- 4.2 $\mathbf{p^2 = 0}$ 19

5. Gravity in a general warped background **21**
- 5.1 Derivation of the equations of motion 22
- 5.2 Straight gauges 25
- 5.3 Background solutions 28
- 5.4 Gauge fixing 30
  - 5.4.1 “massive” case 33
  - 5.4.2 “massless” case 36

6. Conclusion **39**

1. Introduction

Models of braneworld gravity are so varied and popular that they are commonly denoted in a cryptic shorthand: ADD [1], RSI [2], RSII [3], LR [4], AED [5], UED [6], DGP [7], KR [8], etc. Most models are based upon an orbifold as the background geometry, usually $\mathbf{S^1/Z_2}$. The analysis of such models begins with making explicit orbifold projections on the equations of motion, and integrating the equations of motion through the orbifold fixed points to obtain junction conditions [9]-[14]. This is a convenient shortcut which produces correct results in simple analyses of simple systems.

However there are problems with this standard approach. The first is that general relativity (GR) is defined on manifolds, not orbifolds. An orbifold is a well-defined but singular limit of a smooth manifold. It is possible to treat a $\mathbf{Z_2}$ orbifold as a limit of a manifold with boundaries, however a generally covariant action principle for manifolds with boundaries is usually only discussed in the case where metric fluctuations are restricted to vanish on the boundaries [15]. In braneworld models we are specifically interested in the case where the metric fluctuations *do not* vanish on the boundaries. This problem is usually finessed by applying junction conditions [16], which does not address the status of general covariance in such a system, or resolve whether one can define an unambiguous action principle.

Another complication is that braneworld models often contain extra scalar degrees of freedom, coming from fluctuations of the higher dimensional metric. In many setups [17]-[24] it appears that these extra scalars are kinetic ghosts (*i.e.* they have kinetic terms with the wrong sign) or have a kinetic term with vanishing coefficient, leading to strong coupling behavior. To exhibit either kind of pathology explicitly requires computing the full gauge-fixed effective action of (at least) the linearized theory. This certainly requires a well-defined action principle as a starting point, and it requires an unambiguous understanding of the full general coordinate invariance of the model.

In this paper we provide a general set of definitions and methodologies for analyzing models of braneworld gravity. We begin with a number of familiar examples. In §2.1 we introduce basic concepts and notation of the interval picture using a simple 5d scalar field theory. In §2.2 we discuss 5d abelian gauge theory in a fixed braneworld background. In §3.1 and §3.2, we treat 5d gravity in a flat $\mathbf{S^1/Z_2}$ background. We contrast the usual orbifold techniques with the interval picture, where we never invoke $\mathbf{Z_2}$ projections or junction conditions. To simplify the presentation we employ limits of the general results derived in §5. In §4 we do a similar analysis for the original Randall-Sundrum model, in the interval picture.

We then proceed to analyze a general AdS_5/AdS_4 setup with two branes, including brane kinetic terms for gravity. Here it is already not obvious from previous work how to count the physical scalars coming from the metric. Orbifold projections by their very nature are only implemented in a coordinate system where the branes are "straight", *i.e.*, located

at fixed slices of the 5d coordinate y. We call such coordinate systems "straight gauges", and define them precisely. For setups with more than one brane, none of the standard gauge choices of gravity (axial, harmonic, de Donder, Gaussian normal) are straight gauges in a single coordinate patch. A coordinate transformation of the metric that violates the straight gauge condition has the appearance of a scalar metric perturbation, called a "brane-bending" mode. In the orbifold approach one cannot distinguish between the following two possibilities:

- orbifold gravity does not respect the full general coordinate invariance of gravity on manifolds, and thus some brane-bending modes are physical;
- orbifold gravity does implement the full general coordinate invariance, and brane-bending modes are always pure gauge.

The interval picture shows that the second alternative is the correct one. We show that warped two-brane setups have at most a single 4d scalar mode (a radion) coming from the metric.

Other authors have already introduced some of the concepts employed in this paper [25]-[29].

2. Orbifolds in field theory

2.1 Scalars on a 5d orbifold

Most of the literature on warped extra dimensions is based upon the idea of field theory on the simple orbifold $\mathbf{S^1/Z_2}$. For a nongravitational theory there is a simple unambiguous implementation of the orbifolding. Consider for example a real 5d scalar field $\phi(x^\mu, y)$. Compactifying the y direction on a circle with radius L/π implies that ϕ should be decomposed into the appropriate Fourier modes:

$$\phi(x,y) = \frac{a_0(x)}{\sqrt{2}} + \sum_{n>0} \left[a_n(x) \cos\left(\frac{\pi n y}{L}\right) + b_n(x) \sin\left(\frac{\pi n y}{L}\right) \right] \ . \tag{2.1}$$

The $\mathbf{Z_2}$ orbifolding around the point $y = 0$ then amounts to projecting out all of the odd modes, *i.e.*, setting all the 4d fields $b_n(x)$ to zero. It is also possible to define a different orbifolded theory in which all of the even modes are projected out. Note that the modes which are even around $y = 0$ are also even around $y = L$. These are the two fixed points of the orbifold. By periodicity, the fixed point $y = L$ is identified with the point $y = -L$. The orbifold can be regarded as extending from $-L$ to L, with two fixed points but no boundaries.

This definition of a field theory orbifold is certainly not adequate for a theory which includes gravity. In particular the fixed points of an orbifold lead to ambiguities in the formulation of GR. This is especially true if one introduces delta function sources at the

fixed points of the orbifold. It is not obvious in this case that there is a well-defined action principle, and the status of general coordinate invariance is murky.

To examine these issues, we first need a definition of the field theory orbifold at the level of the action, rather than as a projection on the equations of motion. We use essentially the same definition as [26]. Consider again a real 5d scalar field. Including polynomial sources located at the fixed points, the $\mathbf{S^1/Z_2}$ field theory orbifold is defined by the following action (our metric signature is $-++++$):

$$S = -\lim_{\varepsilon\to 0}\int d^4x \left(\int_{\varepsilon}^{L-\varepsilon} dy + \int_{-L+\varepsilon}^{-\varepsilon} dy\right)\Big\{\frac{1}{2}\partial^M\phi\partial_M\phi + V(\phi) \\ +\Big(\delta(y-\varepsilon)+\delta(y+\varepsilon)\Big)V_0(\phi) + \Big(\delta(y-L+\varepsilon)+\delta(y+L-\varepsilon)\Big)V_L(\phi)\Big\}\,. \quad (2.2)$$

The action comes from integrating over two intervals: $[-L+\varepsilon,-\varepsilon]$ and $[\varepsilon, L-\varepsilon]$. It is understood that we are imposing periodicity under $y\to y+2L$, as before. Thus the $\mathbf{S^1/Z_2}$ orbifold, which has two fixed points and no boundary, is here represented as a limit of a theory with two intervals and four boundary points. In this simple example there is a bulk potential $V(\phi)$ and two "brane" sources $V_0(\phi)$ and $V_L(\phi)$. These brane sources have support only at the four boundary points; they are introduced symmetrically to reproduce the usual delta function brane sources in the limit, *e.g.*:

$$\lim_{\varepsilon\to 0}\left(\int_{\varepsilon}^{L-\varepsilon} dy + \int_{-L+\varepsilon}^{-\varepsilon} dy\right)\Big(\delta(y-\varepsilon)+\delta(y+\varepsilon)\Big)V_0(\phi) \\ = \lim_{\varepsilon\to 0}\left(\int_{-L+\varepsilon}^{L-\varepsilon} dy\right)\frac{1}{2}\Big(\delta(y-\varepsilon)+\delta(y+\varepsilon)\Big)V_0(\phi) = \int_{-L}^{L} dy\,\delta(y)V_0(\phi)\,. \quad (2.3)$$

It is important to note that we are assuming that the brane sources are continuous, *e.g.*

$$\lim_{\varepsilon\to 0} V_0(\phi)|_{\varepsilon} = \lim_{\varepsilon\to 0} V_0(\phi)|_{-\varepsilon} \equiv V_0(\phi)|_0\,. \quad (2.4)$$

In order to have a well-defined action principle, a field theory with boundaries requires the imposition of appropriate boundary conditions. To see how this works for our simple example, consider the full variation of the action, keeping surface terms:

$$\delta S = -\lim_{\varepsilon\to 0}\int d^4x\Big\{\left(\int_{\varepsilon}^{L-\varepsilon} dy + \int_{-L+\varepsilon}^{-\varepsilon} dy\right)\left(-\partial^M\partial_M\phi + \frac{\delta V(\phi)}{\delta\phi}\right)\delta\phi \\ +\left[\left(\phi' + \frac{1}{2}\frac{\delta V_L}{\delta\phi}\right)\delta\phi\right]_{L-\varepsilon} + \left[\left(-\phi' + \frac{1}{2}\frac{\delta V_0}{\delta\phi}\right)\delta\phi\right]_{\varepsilon} \\ +\left[\left(\phi' + \frac{1}{2}\frac{\delta V_0}{\delta\phi}\right)\delta\phi\right]_{-\varepsilon} + \left[\left(-\phi' + \frac{1}{2}\frac{\delta V_L}{\delta\phi}\right)\delta\phi\right]_{-L+\varepsilon}\Big\}\,, \quad (2.5)$$

where prime denotes a derivative with respect to y.

The bulk equation of motion is

$$-\partial^M\partial_M\phi + \frac{\delta V(\phi)}{\delta\phi} = 0\,. \quad (2.6)$$

To make the action stationary, this must be supplemented by boundary conditions at the four boundary points. One option is to impose Dirichlet boundary conditions, *i.e.*, to require that $\delta\phi(x^\mu, y)$ vanishes at the boundaries. This is not usually what one wants for brane models, although it is the assumption used for general relativity with boundaries.

The other option is to supplement the bulk equations of motion by four "brane-boundary" equations:

$$\begin{aligned} \left[\phi' + \tfrac{1}{2}\tfrac{\delta V_L}{\delta\phi}\right]_{L-\varepsilon} &= 0\,; \\ \left[-\phi' + \tfrac{1}{2}\tfrac{\delta V_0}{\delta\phi}\right]_{\varepsilon} &= 0\,; \\ \left[\phi' + \tfrac{1}{2}\tfrac{\delta V_0}{\delta\phi}\right]_{-\varepsilon} &= 0\,; \\ \left[-\phi' + \tfrac{1}{2}\tfrac{\delta V_L}{\delta\phi}\right]_{-L+\varepsilon} &= 0\,. \end{aligned} \tag{2.7}$$

In the limit $\varepsilon \to 0$ this is equivalent to:

$$\begin{aligned} \phi'|_{0^+} &= -\phi'|_{0^-}\,; \\ \phi'|_{L^-} &= -\phi'|_{-L^+}\,; \\ -2\phi'|_{0^+} + \tfrac{\delta V_0}{\delta\phi}|_0 &= 0\,; \\ 2\phi'|_{L^-} + \tfrac{\delta V_L}{\delta\phi}|_L &= 0\,. \end{aligned} \tag{2.8}$$

The brane-boundary conditions (2.7) are invariant under interchanging the two intervals combined with $y \to -y$. From this it is clear that we can always restrict our attention to solving for ϕ in the interval $0 < y < L$, imposing the second two boundary equations of (2.8). The solution in the interval $-L < y < 0$ then follows by applying the first two relations of (2.8). In this paper we will always be content to display our solutions on $0 < y < L$.

Note that if we remove the brane sources, we get simple Neumann boundary conditions at the boundaries. Then in the limit $\varepsilon \to 0$, the brane-boundary equations are precisely equivalent to the usual orbifold projection.

Following earlier work [27, 29], we will use the name "interval picture" to refer to this approach to defining field theory orbifolds at the level of the action.

2.2 Abelian gauge theory on a warped orbifold

A more ambitious example is to consider a 5d abelian gauge theory, with brane kinetic terms, in a warped orbifold background with two branes. This setup was analyzed in the conventional orbifold picture in [30]. The interval picture action is given by:

$$\begin{aligned} S = -\lim_{\varepsilon\to 0}\int d^4x \left(\int_{\varepsilon}^{L-\varepsilon} dy + \int_{-L+\varepsilon}^{-\varepsilon} dy\right)\sqrt{-G}\,\frac{1}{8g_5^2}\Big\{ & G^{MP}G^{NQ}F_{MN}F_{PQ} \\ & +2r_U\{\delta(y-\varepsilon)+\delta(y+\varepsilon)\}G^{\mu\rho}G^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \\ & +2r_I\{\delta(y-L+\varepsilon)+\delta(y+L-\varepsilon)\}G^{\mu\rho}G^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}\Big\}\,. \end{aligned} \tag{2.9}$$

Here we have introduced a fixed background metric which is warped: $G_{\mu\nu} = a^2(y)\eta_{\mu\nu}$, $G_{44} = 1$, $G_{\mu 4} = 0$. The function $a(y)$ is the warp factor. For *e.g.* an AdS_5 background as in Randall-Sundrum (RS), we would have:

$$a(y) = \begin{cases} \mathrm{e}^{ky} & -L < y < 0\,, \\ \mathrm{e}^{-ky} & 0 < y < L\,, \end{cases} \tag{2.10}$$

where k is the inverse AdS_5 radius of curvature.

To be clear, let's pull out all the warp factors explicitly, and for the rest of this example we raise and lower indices with $\eta_{\mu\nu}$. Then:

$$\begin{aligned} F^{MN}F_{MN} = & \frac{2}{a^4}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right) \\ & + \frac{2}{a^2}\left(\partial_\mu A_4 \partial^\mu A_4 - 2\partial^\mu A_4 A'_\mu + A^{\mu\prime} A'_\mu\right)\ , \end{aligned} \tag{2.11}$$

where prime denotes a derivative with respect to y. The bulk equations of motion (EOM) are:

$$\partial^2 A_\mu - \partial_\mu \partial^\nu A_\nu - \left(a^2 \partial_\mu A_4\right)' + \left(a^2 A'_\mu\right)' = 0\ ; \tag{2.12}$$

$$a^2 \partial^2 A_4 - a^2 \partial^\mu A'_\mu = 0\,. \tag{2.13}$$

The two brane-boundary equations are:

$$\begin{aligned} -a^2 \partial_\mu A_4|_{0^+} + a^2 A'_\mu|_{0^+} + r_U \left[\partial^2 A_\mu - \partial_\mu \partial^\nu A_\nu\right]_{y=0} &= 0\ ; \\ a^2 \partial_\mu A_4|_{L^-} - a^2 A'_\mu|_{L^-} + r_I \left[\partial^2 A_\mu - \partial_\mu \partial^\nu A_\nu\right]_{y=L} &= 0\,. \end{aligned} \tag{2.14}$$

The 5d abelian gauge transformations are generated by $\Lambda(x, y)$:

$$\begin{aligned} A_\mu &\to A_\mu + \partial_\mu \Lambda\ ; \\ A_4 &\to A_4 + \Lambda'\,. \end{aligned} \tag{2.15}$$

We want to determine the physical degrees of freedom of this theory in the interval picture. To begin, we do a partial gauge-fixing by choosing

$$\Lambda^{(\mathrm{I})}(x, y) = -\int^y A_4 dy + \int^y F(y)\psi(x) dy\,. \tag{2.16}$$

With this partial gauge-fixing we have

$$A_4(x, y) = F(y)\psi(x)\,. \tag{2.17}$$

The function $F(y)$ is fixed but arbitrary; different choices of $F(y)$ correspond to different gauges. The 4d field $\psi(x)$, on the other hand, appears at this point to be a 4d scalar degree of freedom.

The bulk equation (2.13) becomes:

$$F\partial^2 \psi = \partial^\mu A'_\mu\,. \tag{2.18}$$

The 5d field $A_M(x, y)$ has a different 4d tensor decomposition depending upon whether or not it is a zero mode of the operator ∂^2, *i.e.*, whether it is a massless mode in the 4d sense. Thus we need to solve separately for the massless and massive modes.

2.2.1 massless modes

When $A_M(x,y)$ is a zero mode of ∂^2, we can write:

$$A_\mu(x,y) = A^T_\mu(x,y) + \partial_\mu\phi(x,y) + A^L_\mu(x,y) \ , \tag{2.19}$$

where $\phi(x,y)$ is pure gauge, $A^L_\mu(x,y)$ is the 4d longitudinal mode, and $A^T_\mu(x,y)$ are the two remaining transverse modes which are not pure gauge. In addition, we are only looking at the part of $\psi(x)$ which satisfies $\partial^\mu\partial_\mu\psi = 0$.

The bulk equation (2.18) reduces to:

$$\partial^\mu A^{L\prime}_\mu(x,y) = 0 \ \Rightarrow \partial^\mu A^L_\mu(x,y) = \rho(x) \ , \tag{2.20}$$

where $\rho(x)$ is an arbitrary function. Defining

$$\chi^{(0)\prime}_\mu(x,y) = A'_\mu - F(y)\partial_\mu\psi(x) \ , \tag{2.21}$$

the remaining bulk equation (2.12) gives:

$$(a^2\chi^{(0)\prime}_\mu)' = \partial_\mu\rho \ , \tag{2.22}$$

while the brane-boundary equations become:

$$\begin{aligned} \left[a^2\chi^{(0)\prime}_\mu - r_U\partial_\mu\rho\right]_0 &= 0 \ ; \\ \left[a^2\chi^{(0)\prime}_\mu + r_I\partial_\mu\rho\right]_L &= 0 \ , \end{aligned} \tag{2.23}$$

where we are employing a shorthand notation $(0,L)$ to distinguish the two independent brane-boundary conditions.

Provided that $r_U + r_I + L \neq 0$, the only simultaneous solution of (2.22-2.23) is

$$\chi^{(0)\prime}_\mu = 0 \ ; \quad A^L_\mu = 0 \ . \tag{2.24}$$

This in turn implies:

$$\begin{aligned} A^T_\mu(x,y) &= A^T_\mu(x) \ ; \\ \phi(x,y) &= \mathcal{F}(y)\psi(x) + \phi(x) \ , \end{aligned} \tag{2.25}$$

where $\mathcal{F}(y)$ is defined by

$$\mathcal{F}'(y) = F(y) \ , \tag{2.26}$$

with the integration constant set to zero.

To count massless degrees of freedom, we perform the gauge transformation defined by

$$\Lambda^{(\mathrm{II})}(x,y) = -\mathcal{F}(y)\psi(x) - \phi(x) \ . \tag{2.27}$$

This takes us to an axial gauge, which is also the unitary gauge for this model:

$$\begin{aligned} A_\mu(x,y) &= A^T_\mu(x) \ ; \\ A_4(x,y) &= 0 \ . \end{aligned} \tag{2.28}$$

There are no extra massless scalar modes, as expected.

2.2.2 massive modes

In this case we decompose

$$A_\mu = A_\mu^T + \partial_\mu \phi \,, \tag{2.29}$$

where A_μ^T is transverse. Then (2.18) becomes:

$$F\partial^2\psi = \partial^2\phi' \,. \tag{2.30}$$

However we already gauge-fixed $\psi(x)$ (massless and massive modes) to zero by the transformation (2.27). Since also we are looking only at massive modes of ϕ we can remove the ∂^2 and conclude:

$$\phi(x,y) = \phi(x) \,. \tag{2.31}$$

Note $\phi(x)$ just represents the residual 4d gauge freedom that preserves the axial gauge. Thus we can gauge-fix it to zero.

So far we have:

$$\begin{aligned} A_\mu(x,y) &= A_\mu^T(x,y) \,; \\ A_4(x,y) &= 0 \,. \end{aligned} \tag{2.32}$$

The massive KK modes have three physical polarizations (and no residual gauge freedom), as appropriate for a massive vector.

Plug this into the bulk equation (2.12):

$$\partial^2 A_\mu^T + (a^2 A_\mu^{T\prime})' = 0 \,. \tag{2.33}$$

The brane-boundary equations become:

$$\left[a^2 A_\mu^{T\prime} + r_U \partial^2 A_\mu^T\right]_0 = 0 \,; \tag{2.34}$$

$$\left[-a^2 A_\mu^{T\prime} + r_I \partial^2 A_\mu^T\right]_L = 0 \,. \tag{2.35}$$

We introduce a Kaluza-Klein (KK) decomposition for the massive transverse modes $A_\mu^T(x,y)$:

$$A_\mu^T(x,y) = \sum_{n=1}^{\infty} A_\mu^{(n)}(x)\chi^{(n)}(y) \,. \tag{2.36}$$

We can take the $A_\mu^{(n)}(x)$ to be on-shell in the 4d sense, so $\partial^2 \to -p^2 \to m_n^2$. The bulk equation of motion becomes:

$$(a^2 \chi^{(n)\prime}(y))' = -m_n^2 \chi^{(n)}(y) \,. \tag{2.37}$$

We can turn the above into Bessel's equation by making the substitutions:

$$\chi^{(n)} = \frac{1}{a(y)} f^{(n)} \,; \quad z_n = \frac{m_n}{ka(y)} \,, \tag{2.38}$$

where now we are going to restrict to the RS case, so $a'/a = -k$, $a''/a = k^2$. This produces

$$\left(z_n^2 \frac{d^2}{dz_n^2} + z_n \frac{d}{dz_n} + (z_n^2 - 1) \right) f^{(n)} = 0 \,. \tag{2.39}$$

The solutions are:

$$\chi^{(n)} = \frac{1}{a} N_n \left(J_1(z_n) + b Y_1(z_n) \right) \,, \tag{2.40}$$

where b is a constant and the N_n are normalization constants.

The brane-boundary equations now become:

$$\chi^{(n)\prime}|_0 = -\frac{r_U m_n^2}{a^2} \chi^{(n)}\Big|_0 \,; \quad \chi^{(n)\prime}|_L = \frac{r_I m_n^2}{a^2} \chi^{(n)}\Big|_L \,. \tag{2.41}$$

Using

$$\chi^{(n)\prime} = \frac{m_n}{a} N_n \left(J_0(z_n) + b Y_0(z_n) \right) \,, \tag{2.42}$$

we determine the constant b and the eigenvalues m_n:

$$b = -\left[\frac{J_0(z_n) + \frac{r_U m_n}{a} J_1(z_n)}{Y_0(z_n) + \frac{r_U m_n}{a} Y_1(z_n)} \right]_0 = -\left[\frac{J_0(z_n) - \frac{r_U m_n}{a} J_1(z_n)}{Y_0(z_n) - \frac{r_U m_n}{a} Y_1(z_n)} \right]_L \,. \tag{2.43}$$

These results are identical to those of [30], computed in the orbifold picture.

3. Gravity on a flat orbifold

3.1 Orbifold picture

Consider 5d gravity on an $\mathbf{S^1/Z_2}$ orbifold in the simplest case where there are no brane sources and there is no bulk cosmological constant. This would be, *e.g.*, the gravity background for the simplest model of Universal Extra Dimensions [6]. Let's find the physical degrees of freedom coming from the 5d metric, using the conventional orbifold language. The background metric is flat:

$$G^{\mathbf{0}}_{\mu\nu} = \eta_{\mu\nu} \,; \; G^{\mathbf{0}}_{\mu 4} = 0 \,; \; G^{\mathbf{0}}_{44} = 1 \,. \tag{3.1}$$

Including linearized metric fluctuations, we write:

$$G_{MN} = G^{\mathbf{0}}_{MN} + h_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} h_{\mu\nu} & h_{\mu 4} \\ h_{4\nu} & h_{44} \end{pmatrix} \,. \tag{3.2}$$

Plugging this into the standard source-free 5d Einstein equation gives the following bulk equations of motion:

$$\begin{aligned} 0 &= \partial_P \partial_\mu h^P_\nu + \partial_P \partial_\nu h^P_\mu - \partial^P \partial_P h_{\mu\nu} - \partial_\mu \partial_\nu h^M_M \\ &\quad - \eta_{\mu\nu} \left(\partial^M \partial^N h_{MN} - \partial^P \partial_P h^M_M \right) \,; \end{aligned} \tag{3.3}$$

$$0 = \partial_P \partial_\mu h^P_4 + \partial_P h^{P\prime}_\mu - \partial^P \partial_P h_{\mu 4} - \partial_\mu h^{M\prime}_M \,; \tag{3.4}$$

$$0 = 2\partial_P h^{P\prime}_4 - \partial^P \partial_P h_{44} - h^{M\prime\prime}_M - \partial^M \partial^N h_{MN} + \partial^P \partial_P h^M_M \,, \tag{3.5}$$

where as always a prime indicates derivative with respect to y.

Because of the periodicity in y, all of the metric fluctuations are expanded in a tower of KK modes, which are just sines and cosines. In the orbifold picture, we impose the $\mathbf{Z_2}$ symmetry explicitly, by projecting out the sine modes for $h_{\mu\nu}$ and h_{44}, as well as the cosine modes for $h_{\mu 4}$:

$$\begin{aligned} h_{\mu\nu}(x,y) &= \frac{h^{(0)}_{\mu\nu}(x)}{\sqrt{2}} + \sum_{n>0}^{\infty} h^{(n)}_{\mu\nu}(x) \cos\left(\frac{n\pi y}{L}\right) \ ; \\ h_{\mu 4}(x,y) &= \sum_{n>0}^{\infty} h^{(n)}_{\mu 4}(x) \sin\left(\frac{n\pi y}{L}\right) \ ; \\ h_{44}(x,y) &= \frac{h^{(0)}_{44}(x)}{\sqrt{2}} + \sum_{n>0}^{\infty} h^{(n)}_{44}(x) \cos\left(\frac{n\pi y}{L}\right) \ . \end{aligned} \tag{3.6}$$

Note there are ten zero modes from $h_{\mu\nu}$ and one more from h_{44}.

Linearized general coordinate transformations (GCTs) $x^M \to x^M + \xi^M$ give:

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \ , \tag{3.7}$$

$$h_{\mu 4} \to h_{\mu 4} - \xi_\mu{}' - \partial_\mu \xi^4 \ , \tag{3.8}$$

$$h_{44} \to h_{44} - 2\xi^{4\prime} \ , \tag{3.9}$$

where the gauge parameters $\xi^M(x,y)$ are also expanded in KK modes and subjected to $\mathbf{Z_2}$ projections:

$$\begin{aligned} \xi^\mu(x,y) &= \frac{\xi^{\mu(0)}(x)}{\sqrt{2}} + \sum_{n>0}^{\infty} \xi^{\mu(n)}(x) \cos\left(\frac{n\pi y}{L}\right) \ ; \\ \xi^4(x,y) &= \sum_{n>0}^{\infty} \xi^{4(n)}(x) \sin\left(\frac{n\pi y}{L}\right) \ . \end{aligned} \tag{3.10}$$

We can go to a convenient coordinate system by choosing

$$\frac{2n\pi}{L}\xi^{4(n)} = h^{(n)}_{44} \ , \tag{3.11}$$

$$\frac{n\pi}{L}\xi^{(n)}_\mu = -h^{(n)}_{\mu 4} + \frac{L}{2n\pi}\partial_\mu h^{(n)}_{44} \ , \tag{3.12}$$

so that all $h^{(n)}_{\mu 4}$ and $h^{(n)}_{44}$ with $n > 0$ are gauged away, leaving only

$$h_{\mu\nu}(x,y) = h^{(0)}_{\mu\nu}(x) + \sum_{n>0}^{\infty} h^{(n)}_{\mu\nu}(x) \cos\left(\frac{n\pi y}{L}\right) \ ; \tag{3.13}$$

$$h_{44}(x,y) = h^{(0)}_{44}(x) \, . \tag{3.14}$$

The residual gauge freedom is generated by $\xi^M(x,y)$ that satisfy

$$\xi^{4\prime} = 0 \, , \ \ \xi'_\mu = -\partial_\mu \xi^4 \, . \tag{3.15}$$

In addition, we began with a coordinate system in which the branes are straight, *i.e.*, they are located at fixed slices of y. In the orbifold picture it is not obvious whether we are allowed to deviate from this "straight gauge". In the literature it is usually assumed (implicitly) that one should not deviate from straight gauges. This assumption implies that ξ^4 should vanish at the brane locations, reducing (3.15) to:

$$\xi^4 = 0\,, \;\; \xi'_\mu = 0\,. \tag{3.16}$$

Thus the remaining gauge freedom is just the 4d GCTs generated by $\xi^{\mu(0)}(x)$. The equations of motion for the gauge-fixed degrees of freedom can be decoupled by a standard analysis [31]. From (3.13) we read off the physical degrees of freedom (DOF):

- a massless graviton $h^{(0)}_{\mu\nu}(x)$ with two on-shell degrees of freedom,
- a massless radion $h^{(0)}_{44}(x)$,
- a Kaluza-Klein tower of massive gravitons $h^{(n)}_{\mu\nu}(x)$ with 5 DOF each.

3.2 Interval picture

In the interval picture we begin with an action:

$$S = \int d^4x \left(\int_{0^+}^{L^-} dy + \int_{-L^+}^{0^-} dy \right) \sqrt{-G}\, 2M^3 R \; + 4M^3 \oint_{\partial\mathcal{M}} K\,, \tag{3.17}$$

where R is the Ricci scalar, K is extrinsic curvature, and M is the 5d Planck mass. We are using coordinates in which the branes (*i.e.* the boundaries) are straight, that is, they are located at fixed slices of y.

The second term in the action is the usual Gibbons-Hawking modification of GR for the case of manifolds with boundaries [15]. The addition of this term ensures that the bulk EOM is the usual Einstein equation, for metric variations which vanish on the boundaries. Since we need an action principle for metric variations which *do not* necessarily vanish at the boundaries, we must supplement the bulk Einstein equation by appropriate brane-boundary equations.

We want to compare the orbifold results of the previous section with what we obtain using the interval picture. Keep in mind that in the interval picture we *do not* impose any $\mathbf{Z_2}$ projections, on either the metric fluctuations or on the generators of general coordinate transformations. We will only quote results since a general derivation is given in §5. There we will also generalize to the case of non-straight gauges.

As in our previous example of the 5d gauge field, we begin by performing a partial gauge fixing. The GCT with $\xi^{(\mathrm{I})\mu} = 0$ and

$$\xi^{(\mathrm{I})4} = \frac{1}{2}\int^y h_{44}\, dy - \frac{1}{2}\int^y F(y)\psi(x)\, dy\,, \tag{3.18}$$

with $F(y)$ a fixed but arbitrary function of y, transforms an arbitrary h_{44} into

$$h_{44} = F(y)\psi(x)\,. \tag{3.19}$$

Since we want to be in a straight gauge, we must require that $\xi^{(\mathrm{I})4}$ vanishes at the locations of the branes. On the interval $0 < y < L$, this fixes the y-independent part of (3.18) to be

$$\xi^{(\mathrm{I})4} = \frac{1}{2}\int_0^y h_{44}\,dy - \frac{1}{2}\int_0^y F(y)\psi(x)\,dy\ , \tag{3.20}$$

and fixes a relation between the radion field $\psi(x)$, $F(y)$, and the original metric fluctuation $h_{44}(x,y)$:

$$\psi(x) = \frac{\int_0^L h_{44}\,dy}{\int_0^L F(y)\,dy}\ . \tag{3.21}$$

From (3.21) we see that $F(y)$, though arbitrary, must be nonzero. More precisely, the straight gauge condition requires:

$$\int_0^L F(y)dy \neq 0\ . \tag{3.22}$$

Thus the analog of axial gauge is not a straight gauge.

Next we can perform an additional partial gauge-fixing to eliminate $h_{\mu 4}$. Choose $\xi^{(\mathrm{II})4} = 0$ and

$$\xi^{(\mathrm{II})\mu} = \int^y h^{\mu 4}dy\,, \tag{3.23}$$

which leaves h_{44} unaffected and gives

$$h_{\mu 4} = 0\,. \tag{3.24}$$

The remaining gauge freedom is just the 4d general coordinate transformation generated by

$$\xi^4 = 0\ , \quad \xi^\mu = \xi^\mu(x)\ . \tag{3.25}$$

Note that the coordinate transformation generated by

$$\xi^4 \equiv \epsilon(x)\ , \quad \xi^\mu = -y\partial_\mu\epsilon(x)\ , \tag{3.26}$$

respects the gauge conditions (3.19) and (3.24) but does not keep us in a straight gauge. Treated as a scalar metric perturbation, $\epsilon(x)$ is the putative brane-bending mode.

To identify the physical DOF, we examine the bulk equations of motion obtained from (3.3-3.5):

$$\begin{aligned}
0 &= \partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu\tilde{h} \\
&\quad -\eta_{\mu\nu}(\partial_\rho\partial_\sigma h^{\rho\sigma} - \partial^2\tilde{h}) - h''_{\mu\nu} + \eta_{\mu\nu}\tilde{h}'' - F\partial_\mu\partial_\nu\psi + \eta_{\mu\nu}F\partial^2\psi\ ; \\
0 &= \partial_\nu h^{\nu\prime}_\mu - \partial_\mu\tilde{h}'\ ; \\
0 &= -\partial_\mu\partial_\nu h^{\mu\nu} + \partial^2\tilde{h}\ ; \\
0 &= \tilde{h}'' + F\partial^2\psi\,,
\end{aligned} \tag{3.27}$$

where $\tilde{h} = \eta^{\mu\nu} h_{\mu\nu}$, and the fourth equation is an auxiliary relation obtained from twice the third equation subtracted from the trace of the first. From the general formula (5.88) that we will derive in §5, the brane-boundary equations are

$$0 = \left[h'_{\mu\nu} - \eta_{\mu\nu} \tilde{h}' \right]_{y=0,L} \ , \tag{3.28}$$

As in the gauge field example of the previous section, we will need to solve these equations separately for the cases where the metric perturbations are massless or massive in the 4d sense. To be completely explicit, we will Fourier transform to a 4d momentum space representation of the metric perturbations $\bar{h}_{\mu\nu}(p, y)$. The bulk and brane-boundary equations become:

$$\begin{aligned} 0 &= -p_\mu p_\rho \bar{h}^\rho_\nu - p_\nu p_\rho \bar{h}^\rho_\mu + p^2 \bar{h}_{\mu\nu} + p_\mu p_\nu \bar{h} \\ &\quad + \eta_{\mu\nu}(p_\rho p_\sigma \bar{h}^{\rho\sigma} - p^2 \bar{h}) - \bar{h}''_{\mu\nu} + \eta_{\mu\nu} \bar{h}'' + F p_\mu p_\nu \bar{\psi} - \eta_{\mu\nu} F p^2 \bar{\psi} \ ; \end{aligned} \tag{3.29}$$

$$0 = p_\nu \bar{h}^{\nu\prime}_\mu - p_\mu \bar{h}' \ ; \tag{3.30}$$

$$0 = p_\mu p_\nu \bar{h}^{\mu\nu} - p^2 \bar{h} \ ; \tag{3.31}$$

$$0 = \bar{h}'' - F p^2 \bar{\psi} \, , \tag{3.32}$$

$$0 = \left[\bar{h}'_{\mu\nu} - \eta_{\mu\nu} \bar{h}' \right]_{y=0,L} \ . \tag{3.33}$$

3.2.1 $\mathbf{p^2 \neq 0}$

As discussed in the Appendix, for $p^2 \neq 0$ the tensor $\bar{h}_{\mu\nu}(p, y)$ can be decomposed as

$$\bar{h}_{\mu\nu}(p, y) = \bar{b}_{\mu\nu}(p, y) + i p_\mu \bar{V}_\nu(p, y) + i p_\nu \bar{V}_\mu(p, y) - p_\mu p_\nu \bar{\phi}_1(p, y) + \eta_{\mu\nu} \bar{\phi}_2(p, y) \, , \tag{3.34}$$

where $\bar{b}_{\mu\nu}(p, y)$ is traceless transverse, $\bar{V}_\mu(p, y)$ is transverse and pure gauge, $\bar{\phi}_1(p, y)$ is a pure gauge scalar, and $\bar{\phi}_2(p, y)$ is another scalar. The bulk equation (3.31) immediately gives the constraint

$$\bar{\phi}_2 = 0 \ . \tag{3.35}$$

Then (3.30) gives

$$i p^2 \bar{V}'_\mu = 0 \ . \tag{3.36}$$

Since $\bar{V}_\mu$ is both pure gauge and y-independent, we can gauge it away using the transverse modes of our residual gauge freedom (3.25).

Integrating (3.32) twice in y gives

$$\bar{\phi}_1 = \bar{f}_2(p) + y \bar{f}_1(p) - \mathfrak{F} \bar{\psi} \ , \tag{3.37}$$

where $\bar{f}_1(p)$, $\bar{f}_2(p)$ are integration "constants", and $\mathfrak{F}(y)$ is defined by $\mathfrak{F}''(y) = F(y)$, with no integration constants. We can remove $\bar{f}_2(p)$ using the longitudinal mode of the residual gauge freedom (3.25).

The trace of the brane-boundary equation (3.33) gives

$$\left[3p^2\bar{\phi}_1'\right]_{y=0,L} = \left[3p^2(\bar{f}_1 - \mathcal{F}\bar{\psi})\right]_{y=0,L} = 0\,, \tag{3.38}$$

with $\mathcal{F}'(y) = F(y)$. Due to (3.22), $\mathcal{F}(0) \neq \mathcal{F}(L)$, and thus the only solution of (3.38) is

$$\bar{f}_1(p) = 0\,, \quad \bar{\psi}(p) = 0\,. \tag{3.39}$$

Now only $\bar{b}_{\mu\nu}$ is left, and (3.29) and (3.33) determine its mass spectrum. By going on-shell, *i.e.*, $p^2 = -m^2$:

$$\begin{aligned} & m^2\bar{b}_{\mu\nu} + \bar{b}_{\mu\nu}'' = 0\,, \quad \left[\,\bar{b}_{\mu\nu}'\,\right]_{y=0,L} = 0 \\ & \Rightarrow \quad \bar{b}_{\mu\nu}(p,y) = \bar{B}_{\mu\nu}(p)\cos\frac{n\pi}{L}y\,, \quad n = 1,2,\cdots\,, \end{aligned} \tag{3.40}$$

which agrees with the results obtained for the massive sector in the orbifold approach.

3.2.2 $\mathbf{p^2 = 0}$

The massless modes of $\bar{h}_{\mu\nu}$ have the more complicated tensor decomposition given by (9):

$$\begin{aligned} \bar{h}_{\mu\nu}(p,y) = {} & \bar{\beta}_{\mu\nu}(p,y) + ip_\mu\bar{v}_\nu(p,y) + ip_\nu\bar{v}_\mu(p,y) - p_\mu p_\nu\bar{\varphi}_1(p,y) \\ & + ip_\mu\bar{n}_\nu(p,y) + ip_\nu\bar{n}_\mu(p,y) + \bar{c}_{\mu\nu}(p,y) + \eta_{\mu\nu}\bar{\varphi}_2(p,y)\,. \end{aligned} \tag{3.41}$$

Here $\bar{v}_\mu(p,y)$ is transverse and pure gauge (2 DOF), $\bar{\varphi}_1(p,y)$ is also pure gauge (1 DOF), and $\bar{n}_\mu(p,y)$ is pure gauge but not transverse (1 DOF). Also, $\bar{c}_{\mu\nu}(p,y)$ is traceless but not transverse (3 DOF), $\bar{\varphi}_2(p,y)$ is a scalar (1 DOF), and $\bar{\beta}_{\mu\nu}(p,y)$ are the remaining traceless transverse components (2 DOF).

The bulk equation (3.31) gives the constraint:

$$p_\mu p_\nu \bar{c}^{\mu\nu} = 0\,, \tag{3.42}$$

which, because $p_\nu\bar{c}^{\mu\nu} \neq 0$, implies

$$\bar{c}^{\mu\nu} = 0\,. \tag{3.43}$$

Then, (3.30) becomes

$$p_\mu(ip_\nu\bar{n}^{\nu\prime} + 3\bar{\varphi}_2') = 0 \quad \Rightarrow \quad ip_\nu\bar{n}^{\nu\prime} + 3\bar{\varphi}_2' = 0\,, \tag{3.44}$$

while (3.32) gives

$$ip_\nu\bar{n}^{\nu\prime\prime} + 2\bar{\varphi}_2'' = 0 \quad \Rightarrow \quad ip_\nu\bar{n}^{\nu\prime} + 2\bar{\varphi}_2' = \bar{f}_1(p)\,. \tag{3.45}$$

Solving for $p_\nu\bar{n}^{\nu\prime}$ and $\bar{\varphi}_2'$, we get

$$ip_\nu\bar{n}^{\nu\prime} = 3\bar{f}_1\,, \quad \bar{\varphi}_2' = -\bar{f}_1\,. \tag{3.46}$$

Taking the trace of the brane-boundary equation (3.33):

$$0 = \left[\bar{h}'\right]_{y=0,L} = \left[2ip_\nu \bar{n}^{\nu\prime} + 4\bar{\varphi}_2'\right]_{y=0,L} = 2\bar{f}_1 \ . \tag{3.47}$$

Then

$$\bar{\varphi}_2 = \bar{f}_2(p) \, , \tag{3.48}$$

and due to $p_\mu \bar{n}^\mu \neq 0$,

$$\bar{n}^{\mu\prime} = 0 \ . \tag{3.49}$$

Since $\bar{n}^\mu$ is pure gauge and y-independent, it can be eliminated by the longitudinal part of the residual gauge freedom.

Finally, (3.29) gives

$$\bar{\beta}''_{\mu\nu} + ip_\mu \bar{v}''_\nu + ip_\nu \bar{v}''_\mu - p_\mu p_\nu (\bar{\varphi}''_1 + 2\bar{f}_2 + F\bar{\psi}) = 0 \ . \tag{3.50}$$

Now we contract (3.50) with $\bar{n}^\mu \bar{n}^\nu$. In the Appendix, we show that $\bar{n}^\mu \bar{\beta}_{\mu\nu} = 0$, $\bar{n}^\mu \bar{v}_\mu = 0$, and $p_\mu \bar{n}^\mu \neq 0$; these are 4d tensor relations which are unchanged if we replace $\bar{\beta}_{\mu\nu}$ by $\bar{\beta}''_{\mu\nu}$ or $\bar{v}_\mu$ by $\bar{v}''_\mu$. Thus we get

$$\bar{\varphi}''_1 + 2\bar{f}_2 + F\bar{\psi} = 0 \ , \tag{3.51}$$

and

$$\bar{\beta}''_{\mu\nu} + ip_\mu \bar{v}''_\nu + ip_\nu \bar{v}''_\mu = 0 \ . \tag{3.52}$$

For convenience, let's define $\bar{t}_{\mu\nu} = \bar{\beta}_{\mu\nu} + ip_\mu \bar{v}_\nu + ip_\nu \bar{v}_\mu$. Then (3.33) gives

$$0 = \left[\bar{t}_{\mu\nu}{}' - p_\mu p_\nu \bar{\varphi}_1'\right]_{y=0,L} \ , \tag{3.53}$$

thus

$$0 = \left[\bar{t}_{\mu\nu}{}'\right]_{y=0,L} \ , \tag{3.54}$$

$$0 = \left[\bar{\varphi}_1'\right]_{y=0,L} \ . \tag{3.55}$$

From (3.52) and (3.54), we see that $\bar{t}_{\mu\nu}$ is y-independent. Then since $\bar{v}_\nu$ is pure gauge, we can gauge it away using two transverse components of the residual gauge freedom.

Next, from (3.51) we get

$$\bar{\varphi}_1(p,y) = \bar{f}_4(p) + y\bar{f}_3(p) - y^2 \bar{f}_2(p) - \mathfrak{F}(y)\bar{\psi} \ . \tag{3.56}$$

Since $\bar{\varphi}_1$ is pure gauge, we can eliminate $\bar{f}_4$ by the remaining transverse component of the residual gauge freedom. Using (3.55), $\bar{\varphi}_1$ and $\bar{f}_2$ can be written in terms of $\bar{\psi}$:

$$\bar{f}_2 = -\frac{\mathcal{F}(L) - \mathcal{F}(0)}{2L}\bar{\psi} \, , \tag{3.57}$$

$$\bar{\varphi}_1 = \left\{\frac{\mathcal{F}(L) - \mathcal{F}(0)}{2L} y^2 - (\mathfrak{F}(y) - \mathcal{F}(0)y)\right\}\bar{\psi} \ . \tag{3.58}$$

Recall that $F(y)$ is an arbitrary function satisfying (3.22). For example we can choose $F(y) = 1$, in which case the above reduces to $f_2(x) = -\psi(x)/2$, $\varphi_1(x) = 0$.

In short, the physical degrees of freedom of the massless sector consist of a massless graviton $\beta_{\mu\nu}(x)$ with two on-shell degrees of freedom, together with the massless radion $\psi(x)$. This agrees with the results of the orbifold approach.

4. The Randall-Sundrum model in the interval picture

Let's repeat the same exercise for the case of the RSI background [2]. We want to reproduce the well-known results from the orbifold approach using the interval picture analysis. Here we have a nonzero bulk cosmological constant and brane tensions, which are tuned to give a warped background solution with flat 4d slices. The interval picture action is

$$\begin{aligned}\frac{S}{2M^3} &= \int d^4x \left(\int_{0^+}^{L^-} dy + \int_{-L^+}^{0^-} dy\right) \sqrt{-G}\Big(R + 12k^2\Big) \\ &\quad + \sum_i \int_{y=y_i} d^4x \sqrt{-g}\; 12k\theta_i + 2\oint_{\partial\mathcal{M}} K\,, \end{aligned} \tag{4.1}$$

where $-\theta_1 = \theta_2 = 1$, $y_1 = 0$, $y_2 = L$, and we have already inserted the tuned values for the two brane tensions. The background solution is

$$G^{\mathbf{0}}_{MN} = \begin{pmatrix} g^{\mathbf{0}}_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{4.2}$$

where $g^{\mathbf{0}}_{\mu\nu} = a^2(y)\,\eta_{\mu\nu}$ with

$$a(y) = \begin{cases} \mathrm{e}^{ky} & -L < y < 0\,, \\ \mathrm{e}^{-ky} & 0 < y < L\,. \end{cases} \tag{4.3}$$

As in all our examples we will restrict our attention to the interval $0 < y < L$. Indices are raised and lowered with the warped background metric $G^{\mathbf{0}}_{MN}$. Following the procedure of the previous section, we gauge fix to

$$h_{\mu 4} = 0\,,\;\; h_{44} = F(y)\psi(x)\,, \tag{4.4}$$

with the residual gauge freedom generated by

$$\xi^4 = 0\,,\; \xi^\mu = \xi^\mu(x)\,. \tag{4.5}$$

Bulk and brane-boundary equations of motion are obtained from (5.84-5.88):

$$
\begin{aligned}
0 &= \partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu\tilde{h} - g^{\mathbf{0}}_{\mu\nu}(\partial_\rho\partial_\sigma h^{\rho\sigma} - \partial^2\tilde{h}) \\
&\quad -h''_{\mu\nu} + g^{\mathbf{0}}_{\mu\nu}\tilde{h}'' - 4kg^{\mathbf{0}}_{\mu\nu}\tilde{h}' + 4k^2 h_{\mu\nu} \\
&\quad -F\partial_\mu\partial_\nu\psi + g^{\mathbf{0}}_{\mu\nu}F\partial^2\psi + 3kg^{\mathbf{0}}_{\mu\nu}F'\psi - 12k^2 g^{\mathbf{0}}_{\mu\nu}F\psi\,, \\
0 &= \partial_\nu h^{\nu\prime}_\mu - \partial_\mu\tilde{h}' - 3kF\partial_\mu\psi\,, \\
0 &= -\partial_\mu\partial_\nu h^{\mu\nu} + \partial^2\tilde{h} - 3k\tilde{h}' - 12k^2F\psi\,, \\
0 &= \frac{(a^2\tilde{h}')'}{a^2} + F\partial^2\psi + 4kF'\psi - 8k^2F\psi\,, \\
0 &= \Big[h_{\mu\nu}{}' - g^{\mathbf{0}}_{\mu\nu}\tilde{h}' + 2kh_{\mu\nu} - 3kg^{\mathbf{0}}_{\mu\nu}F\psi\Big]_{y=y_i}\,,
\end{aligned}
\tag{4.6}
$$

with $\partial^2 = g^{\mathbf{0}\mu\nu}\partial_\mu\partial_\nu$, $\tilde{h} = g^{\mathbf{0}\mu\nu}h_{\mu\nu}$.

Although the spacetime is warped, we can still perform a 4d Fourier analysis on the flat 4d slices. Using p^2 to denote $a^2 g^{\mathbf{0}\mu\nu}p_\mu p_\nu$ we write:

$$
\begin{aligned}
0 &= -p_\rho p_\mu\bar{h}^\rho_\nu - p_\rho p_\nu\bar{h}^\rho_\mu + a^{-2}p^2\bar{h}_{\mu\nu} + p_\mu p_\nu\bar{h} + g^{\mathbf{0}}_{\mu\nu}(p_\rho p_\sigma\bar{h}^{\rho\sigma} - a^{-2}p^2\bar{h}) \\
&\quad -\bar{h}''_{\mu\nu} + g^{\mathbf{0}}_{\mu\nu}\bar{h}'' - 4kg^{\mathbf{0}}_{\mu\nu}\bar{h}' + 4k^2\bar{h}_{\mu\nu} \\
&\quad +Fp_\mu p_\nu\bar{\psi} - g^{\mathbf{0}}_{\mu\nu}Fa^{-2}p^2\bar{\psi} + 3kg^{\mathbf{0}}_{\mu\nu}F'\bar{\psi} - 12k^2g^{\mathbf{0}}_{\mu\nu}F\bar{\psi}\,,
\end{aligned}
\tag{4.7}
$$

$$
0 = p_\nu\bar{h}^{\nu\prime}_\mu - p_\mu\bar{h}' - 3kFp_\mu\bar{\psi}\,, \tag{4.8}
$$

$$
0 = p_\mu p_\nu\bar{h}^{\mu\nu} - a^{-2}p^2\bar{h} - 3k\bar{h}' - 12k^2F\bar{\psi}\,, \tag{4.9}
$$

$$
0 = \frac{(a^2\bar{h}')'}{a^2} - Fa^{-2}p^2\bar{\psi} + 4kF'\bar{\psi} - 8k^2F\bar{\psi}\,, \tag{4.10}
$$

$$
0 = \Big[\bar{h}_{\mu\nu}{}' - g^{\mathbf{0}}_{\mu\nu}\bar{h}' + 2k\bar{h}_{\mu\nu} - 3kg^{\mathbf{0}}_{\mu\nu}F\bar{\psi}\Big]_{y=y_i}\,. \tag{4.11}
$$

4.1 $\mathbf{p^2 \neq 0}$

We use the tensor decomposition from the Appendix:

$$
\bar{h}_{\mu\nu} = \bar{b}_{\mu\nu} + ip_\mu\bar{V}_\nu + ip_\nu\bar{V}_\mu - a^2p_\mu p_\nu\bar{\phi}_1 + g^{\mathbf{0}}_{\mu\nu}\bar{\phi}_2\,, \tag{4.12}
$$

where we put a^2 in front of $\bar{\phi}_1$ for convenience. Integrating (4.10), we get

$$
\bar{h}' = -p^2\bar{\phi}_1'(p,y) + 4\bar{\phi}_2'(p,y) = e^{2ky}\bar{f}_1(p) + e^{2ky}\mathcal{F}(y)p^2\bar{\psi}(p) - 4kF(y)\bar{\psi}(p)\,, \tag{4.13}
$$

with $\mathcal{F}'(y) = F(y)$. Then (4.9) and (4.8) are written as

$$
-p^2\bar{\phi}_2(p,y) = k\Big(\bar{f}_1(p) + \mathcal{F}p^2\bar{\psi}(p)\Big)\,, \tag{4.14}
$$

$$
-\Big(e^{2ky}p^2\bar{V}_\mu(p,y)\Big)' - 3ip_\mu\bar{\phi}_2'(p,y) = 3ikFp_\mu\bar{\psi}(p)\,. \tag{4.15}
$$

We can solve (4.13-4.15) for $\bar{\phi}_1$, $\bar{\phi}_2$ and $\bar{V}_\mu$, to get

$$\bar{\phi}_1 = \bar{f}_2(p) - \frac{1}{p^2}\Big(\frac{4k}{p^2} + \frac{e^{2ky}}{2k}\Big)\bar{f}_1 - \mathfrak{F}\bar{\psi} \,, \tag{4.16}$$

$$\bar{\phi}_2 = -k\Big(\frac{1}{p^2}\bar{f}_1 + \mathcal{F}\bar{\psi}\Big) \,, \tag{4.17}$$

$$0 = p^2\Big(e^{2ky}\bar{V}_\mu\Big)' \,, \tag{4.18}$$

where we have defined $\mathfrak{F}'(y) = e^{2ky}\mathcal{F}(y)$.

Equation (4.18) fixes the y-dependence of $\bar{V}_\mu$ to be e^{-2ky}, which allows us to eliminate $\bar{V}_\mu$ by the transverse part of the residual gauge freedom. Similarly, $\bar{f}_2(p)$ is removed by the longitudinal part of the residual gauge freedom. Then, $\bar{h}_{\mu\nu}$ becomes

$$\bar{h}_{\mu\nu} = \bar{b}_{\mu\nu} + e^{-2ky}p_\mu p_\nu\Big\{\frac{1}{p^2}\Big(\frac{4k}{p^2} + \frac{e^{2ky}}{2k}\Big)\bar{f}_1 + \mathfrak{F}\bar{\psi}\Big\} - kg^{\mathbf{0}}_{\mu\nu}\Big(\frac{1}{p^2}\bar{f}_1 + \mathcal{F}\bar{\psi}\Big) \,. \tag{4.19}$$

Plugging this into the Fourier-transformed version of the bulk $\mu\nu$-EOM and the boundary EOM, we have

$$0 = e^{2ky}p^2\bar{b}_{\mu\nu} - \bar{b}''_{\mu\nu} + 4k^2\bar{b}_{\mu\nu} \,, \tag{4.20}$$

$$0 = \Big[\bar{b}'_{\mu\nu} + 2k\bar{b}_{\mu\nu} + (p_\mu p_\nu - \eta_{\mu\nu}p^2)\Big(\frac{1}{p^2}\bar{f}_1 + \mathcal{F}\bar{\psi}\Big)\Big]_{y=y_i} \,. \tag{4.21}$$

Contracting (4.21) with $g^{\mathbf{0}\mu\nu}$ gives

$$\frac{1}{p^2}\bar{f}_1 + \mathcal{F}(0)\,\bar{\psi} = 0 \,, \quad \frac{1}{p^2}\bar{f}_1 + \mathcal{F}(L)\,\bar{\psi} = 0 \,. \tag{4.22}$$

Since $\mathcal{F}(0) \neq \mathcal{F}(L)$, this implies $\bar{f}_1 = \bar{\psi} = 0$.

Going on-shell, we substitute $-m^2$ for p^2:

$$0 = \bar{b}''_{\mu\nu} + \Big(m^2e^{2ky} - 4k^2\Big)\bar{b}_{\mu\nu} \,, \tag{4.23}$$

$$0 = \Big[\bar{b}'_{\mu\nu} + 2k\bar{b}_{\mu\nu}\Big]_{y=y_i} \,. \tag{4.24}$$

The solution of (4.23) is

$$\bar{b}_{\mu\nu}(p,y) = \bar{A}_{\mu\nu}(p)\,J_2\Big(\frac{m}{k}e^{ky}\Big) + \bar{B}_{\mu\nu}(p)\,Y_2\Big(\frac{m}{k}e^{ky}\Big) \,, \tag{4.25}$$

where $J_n(Y_n)$ is the Bessel function of the first(second) kind. Equation (4.24) provides boundary conditions:

$$0 = m\Big\{\bar{A}_{\mu\nu}(p)\,J_1\Big(\frac{m}{k}\Big) + \bar{B}_{\mu\nu}(p)\,Y_1\Big(\frac{m}{k}\Big)\Big\} \,, \tag{4.26}$$

$$0 = me^{kL}\Big\{\bar{A}_{\mu\nu}(p)\,J_1\Big(\frac{m}{k}e^{kL}\Big) + \bar{B}_{\mu\nu}(p)\,Y_1\Big(\frac{m}{k}e^{kL}\Big)\Big\} \,. \tag{4.27}$$

These can have a non-trivial solution only when

$$J_1\Big(\frac{m}{k}\Big)Y_1\Big(\frac{m}{k}e^{kL}\Big) - Y_1\Big(\frac{m}{k}\Big)J_1\Big(\frac{m}{k}e^{kL}\Big) = 0 \,, \tag{4.28}$$

which determines the discrete spectrum of massive graviton modes. Then, (4.25) becomes

$$\bar{b}_{\mu\nu}(p,y) = \bar{B}_{\mu\nu}(p)\left\{Y_1\Big(\frac{m}{k}\Big)\,J_2\Big(\frac{m}{k}e^{ky}\Big) - J_1\Big(\frac{m}{k}\Big)Y_2\Big(\frac{m}{k}e^{ky}\Big)\right\}, \tag{4.29}$$

up to an overall normalization. Note $\bar{b}_{\mu\nu}(p,y)$ has five polarizations from the transverse-traceless $\bar{B}_{\mu\nu}$. Thus the physical content of the massive sector is a Kaluza-Klein tower of massive gravitons coming from $b_{\mu\nu}(x,y)$, in agreement with the Randall-Sundrum result.

4.2 $\mathbf{p^2 = 0}$

We use the same massless tensor decomposition as in the flat orbifold case:

$$\bar{h}_{\mu\nu} = \bar{t}_{\mu\nu} - a^2 p_\mu p_\nu \bar{\varphi}_1 + ip_\mu \bar{n}_\nu + ip_\nu \bar{n}_\mu + \bar{c}_{\mu\nu} + g^{\mathbf{0}}_{\mu\nu}\bar{\varphi}_2\,, \tag{4.30}$$

where $\bar{t}_{\mu\nu} = \bar{\beta}_{\mu\nu} + ip_\mu \bar{v}_\nu + ip_\nu \bar{v}_\mu$. Equation (4.10) gives

$$2ip_\mu \bar{n}^{\mu\prime} + 4\bar{\varphi}_2' = e^{2ky}\bar{f}_1(p) - 4kF(y)\bar{\psi}(p)\,, \tag{4.31}$$

and (4.9) and (4.8) become

$$p_\mu p_\nu \bar{c}^{\mu\nu} = 3ke^{2ky}\bar{f}_1\,, \tag{4.32}$$

$$p_\nu \bar{c}_\mu^{\nu\prime} = p_\mu(e^{2ky}\bar{f}_1 - kF\bar{\psi} - ip_\nu \bar{n}^{\nu\prime} - \bar{\varphi}_2')\,. \tag{4.33}$$

Now we contract (4.33) with $\bar{n}^\mu$. In the Appendix, we show that $\bar{n}^\mu \bar{c}_{\mu\nu} = 0$ and $p_\mu \bar{n}^\mu \neq 0$; these are 4d tensor relations which are unchanged if replace $\bar{n}^\mu$ by $\bar{n}^{\mu\prime}$ or $\bar{c}_{\mu\nu}$ by $\bar{c}_{\mu\nu}{}'$. Thus we get

$$ip_\nu \bar{n}^{\nu\prime} + \bar{\varphi}_2' = e^{2ky}\bar{f}_1 - kF\bar{\psi}\,. \tag{4.34}$$

Solving this and (4.31) for $p_\nu \bar{n}^{\nu\prime}$ and $\bar{\varphi}_2'$,

$$ip_\nu \bar{n}^{\nu\prime} = \frac{3}{2}e^{2ky}\bar{f}_1\,, \tag{4.35}$$

$$\bar{\varphi}_2' = -\frac{e^{2ky}}{2}\bar{f}_1 - kF\bar{\psi}\,. \tag{4.36}$$

Since the trace of (4.11) gives

$$\Big[-3\bar{h}' - 12kF\bar{\psi}\Big]_{y=y_i} = \Big[-3e^{2ky}\bar{f}_1\Big]_{y=y_i} = 0 \;\;\Rightarrow\;\; \bar{f}_1(p) = 0\,, \tag{4.37}$$

then (4.32) dictates

$$\bar{c}_{\mu\nu} = 0\,, \tag{4.38}$$

and from (4.35), we see $\bar{n}_\mu$ can be gauged away by the longitudinal part of the residual gauge freedom.

Equation (4.36) is integrated to give

$$\bar{\varphi}_2 = \bar{f}_2(p) - k\mathcal{F}\bar{\psi}\,. \tag{4.39}$$

Then, contracting (4.7),

$$0 = -\bar{t}_{\mu\nu}{}'' + 4k^2\bar{t}_{\mu\nu} + p_\mu p_\nu\Big\{e^{-2ky}\bar{\varphi}_1'' - 4ke^{-2ky}\bar{\varphi}_1' + 2\bar{f}_2 + \Big(F - 2k\mathcal{F}\Big)\bar{\psi}\Big\}\,, \quad (4.40)$$

with $\bar{n}^\mu\bar{n}^\nu$, we get

$$0 = -\bar{t}_{\mu\nu}{}'' + 4k^2\bar{t}_{\mu\nu}\,, \quad (4.41)$$

$$0 = e^{-2ky}\bar{\varphi}_1'' - 4ke^{-2ky}\bar{\varphi}_1' + 2\bar{f}_2 + \Big(F - 2k\mathcal{F}\Big)\bar{\psi}\,. \quad (4.42)$$

Similarly for (4.11);

$$0 = \Big[\,\bar{t}_{\mu\nu}' + 2k\bar{t}_{\mu\nu}\,\Big]_{y=y_i}\,, \quad (4.43)$$

$$0 = \Big[\,e^{-2ky}\bar{\varphi}_1'\,\Big]_{y=y_i}\,. \quad (4.44)$$

Using (4.41) and (4.43), we obtain

$$\bar{t}_{\mu\nu}(p,y) = \bar{B}_{\mu\nu}(p)e^{-2ky}\,. \quad (4.45)$$

This means that $\bar{v}_\mu$ has the correct y-dependence to be gauged away by two transverse components of the residual gauge freedom.

Finally, solving (4.42), we get

$$\bar{\varphi}_1 = \bar{f}_4(p) + e^{4ky}\bar{f}_3(p) + \frac{e^{2ky}}{2k^2}\bar{f}_2(p) - \mathfrak{F}\bar{\psi}(p)\,, \quad (4.46)$$

where we can gauge away $\bar{f}_4$ by the remaining transverse component of the residual gauge freedom. Using (4.44), $\bar{f}_2$ and $\bar{f}_3$ can be written in terms of $\bar{\psi}$:

$$\bar{f}_2 = k\frac{e^{2kL}\mathcal{F}(0) - \mathcal{F}(L)}{e^{2kL}-1}\bar{\psi}\,, \quad \bar{f}_3 = \frac{\mathcal{F}(L) - \mathcal{F}(0)}{4k(e^{2kL}-1)}\bar{\psi}\,, \quad (4.47)$$

and then

$$\bar{\varphi}_1 = \Big\{\frac{\mathcal{F}(L) - \mathcal{F}(0)}{4k(e^{2kL}-1)}e^{4ky} + \frac{e^{2kL}\mathcal{F}(0) - \mathcal{F}(L)}{2k(e^{2kL}-1)}e^{2ky} - \mathfrak{F}\Big\}\bar{\psi}\,. \quad (4.48)$$

Thus all the surviving scalars are linearly dependent on $\bar{\psi}$. Since $F(y)$ is an arbitrary function satisfying (3.22), we can simplify the above expressions. For example, choosing $F(y) = 1/a^2$, the above reduces to $f_2(x) = 0$, $\varphi_1(x) = 0$, and $h_{\mu\nu} = a^2(y)B_{\mu\nu}(x) - (1/2)\eta_{\mu\nu}\psi(x)$.

We see that the physical content of the massless sector consists of a massless graviton $B_{\mu\nu}(x)$ with two on-shell degrees of freedom, and a massless radion $\psi(x)$. This agrees with the standard results [2, 10].

5. Gravity in a general warped background

We are interested in warped background solutions which are generalizations of the original two brane setup of Randall and Sundrum [2]. We have a 5d spacetime, $\mathcal{M}$, which extends to infinity along the usual (1+3) dimensions (denoted by x^μ) and has an extra spatial dimension (denoted by y) compactified on a circle with circumference $2L$. There are two branes, which are nonintersecting codimension one hypersurfaces described by $\Phi_1(x,y)=0$ and $\Phi_2(x,y)=0$. The branes divide the 5d spacetime $\mathcal{M}$ into two pieces: $\mathcal{M}_1$, which extends from $\Phi_1(x,y)=0^+$ to $\Phi_2(x,y)=0^-$, and $\mathcal{M}_2$, which extends from $\Phi_1(x,y)=0^-$ to $\Phi_2(x,y)=0^+$. The branes have tension, which may be positive or negative, and the brane actions have kinetic terms for gravity, which in a complete model would be induced by radiative corrections involving brane matter [32]-[34].

The bulk part of the action will be written with a bulk metric

$$G_{MN}=\begin{pmatrix} g_{\mu\nu} & G_{\mu 4} \\ G_{4\nu} & G_{44} \end{pmatrix},$$

whereas brane parts are written in terms of the induced metric

$$g^{(i)}_{\alpha\beta}=\Big[\frac{\partial x^M}{\partial x^{(i)\alpha}}\frac{\partial x^N}{\partial x^{(i)\beta}}G_{MN}\Big]_{\Phi_i=0},$$

with $x^{(i)\alpha}$ a coordinate on the boundary hypersurface $\Phi_i=0$, *i.e.*, a "brane coordinate". Since the superscript $^{(i)}$ on any entity always implies that it is evaluated on the $\Phi_i=0$ hypersurface, we will omit $[\ ]_{\Phi_i=0}$ hereafter unless there is room for confusion. The inverse of the bulk and induced metrics satisfy the relation [35]:

$$[\,G^{MN}\,]_{\Phi_i=0}=N^{(i)M}N^{(i)N}+g^{(i)\alpha\beta}\frac{\partial x^M}{\partial x^{(i)\alpha}}\frac{\partial x^N}{\partial x^{(i)\beta}}\,, \tag{5.1}$$

where $N^{(i)M}$ is the unit vector outward-normal to $\Phi_i=0$, which can be written as

$$N^{(i)}_M=\frac{\theta_i\partial_M\Phi_i}{\sqrt{G^{PQ}\partial_P\Phi_i\partial_Q\Phi_i}}\,. \tag{5.2}$$

Our convention for "outward" is that θ_1 is chosen to be -1, while θ_2 is $+1$.

The brane coordinate system together with $N^{(i)}_M$ naturally induces a bulk coordinate system *on the brane*, which we will call the "boundary normal coordinates" (BNCs) denoted by $x^{(i)\bar{M}}$: on the $\Phi_i=0$ hypersurface, we have $x^{(i)\alpha}$-coordinates and $N^{(i)}_M$ defining the directions orthogonal to them. Then at every point on the $\Phi_i=0$ hypersurface, we choose the $x^{(i)\bar{\alpha}}$ to be in the directions of the $x^{(i)\alpha}$'s, and $\bar{y}^{(i)}$ to be in the direction of $N^{(i)}_M$. One of the useful features of this BNC is that since the $\bar{y}^{(i)}$-coordinate is orthogonal the to $x^{(i)\bar{\alpha}}$-ones,

$$G^{(i)}_{\bar{\alpha}\bar{4}}=0\,, \tag{5.3}$$

which, in turn, implies that $\bar{\mu}$-indices are raised and lowered by $g^{(i)}_{\bar{\alpha}\bar{\beta}}$ only, and the $\bar{4}$-index by $G^{(i)}_{\bar{4}\bar{4}}$ only. Note that the BNC is not necessarily the same as Gaussian normal coordinates on the brane, since we don't require $G^{(i)}_{\bar{4}\bar{4}} = 1$. Also

$$g^{(i)}_{\alpha\beta} = \frac{\partial x^{(i)\bar{M}}}{\partial x^{(i)\alpha}} \frac{\partial x^{(i)\bar{N}}}{\partial x^{(i)\beta}} G^{(i)}_{\bar{M}\bar{N}} = \delta^{\bar{M}}_{\bar{\alpha}} \delta^{\bar{N}}_{\bar{\beta}} G^{(i)}_{\bar{M}\bar{N}} = g^{(i)}_{\bar{\alpha}\bar{\beta}} \,, \tag{5.4}$$

i.e., the $\bar{\alpha}\bar{\beta}$-components of the bulk metric are the same as the induced metric. By construction we have

$$N^{(i)}_{\bar{M}} = (0,0,0,0,\theta_i\sqrt{G^{(i)}_{\bar{4}\bar{4}}}) \,, \quad N^{(i)\bar{M}} = (0,0,0,0,\theta_i\sqrt{G^{(i)\bar{4}\bar{4}}}) \,. \tag{5.5}$$

To summarize, we have three types of coordinate system in this section: Roman indices denote bulk coordinates, barred Roman indices with superscript $^{(i)}$ denote BNCs on the i-th brane, and Greek indices with superscript $^{(i)}$ denote brane coordinates on the i-th brane. One exception to these rules is $N^{(i)}_M$: even though $N^{(i)}_\alpha$ has a Greek index and a superscript $^{(i)}$, it denotes part of a bulk vector.

5.1 Derivation of the equations of motion

The most general interval picture action for 5d braneworld gravity, up to second order in derivatives, is

$$\begin{aligned} S = & \left(\int_{\mathcal{M}_1} d^5x + \int_{\mathcal{M}_2} d^5x\right)\sqrt{-G}\Big(2M^3R - \Lambda\Big) \\ & +2M^3\sum_i \int_{\Phi_i=0} d^4x^{(i)}\sqrt{-g^{(i)}}(\lambda_i\tilde{\mathcal{R}}^{(i)} - U_i) + 4M^3\oint_{\partial\mathcal{M}_1+\partial\mathcal{M}_2} K \,. \end{aligned} \tag{5.6}$$

R is a Ricci scalar constructed from G_{MN}, while $\tilde{\mathcal{R}}^{(i)}$ is a 4d Ricci scalar made of only $g^{(i)}_{\mu\nu}$, with $\tilde{}$ indicating that it is a 4d quantity. The brane tensions V_i have been rescaled: $U_i = V_i/2M^3$, as have the coefficients of the brane kinetic terms: $\lambda_i = M_i^2/M^3$.

$K_{\alpha\beta}$ is the extrinsic curvature, defined on the boundary hypersurface [35]:

$$K^{(i)}_{\alpha\beta} = \nabla_M N^{(i)}_N \frac{\partial x^M}{\partial x^{(i)\alpha}} \frac{\partial x^N}{\partial x^{(i)\beta}} \,. \tag{5.7}$$

So

$$K^{(i)} = g^{(i)\alpha\beta}K^{(i)}_{\alpha\beta} = (G^{MN} - N^{(i)M}N^{(i)N})\nabla_M N^{(i)}_N = G^{MN}\nabla_M N^{(i)}_N \,, \tag{5.8}$$

where the last equality is because $N^{(i)M}N^{(i)}_M = 1$, implying $N^{(i)N}\nabla_M N^{(i)}_N = 0$.

The Gibbons-Hawking (GH) extrinsic curvature term in (5.6) is essential for a gravity analysis in spaces with nontrivial boundary. It ensures that, in the absence of boundary/brane sources, the EOM reduce to the usual Einstein equations for variations of the metric which vanish on the boundary. However in brane setups such as we are considering,

the variations of the metric *do not* vanish on the boundary. As a result, the GH term will make a nontrivial contribution to the boundary part of the EOM.

Let's find the equations of motion for (5.6). Replacing G_{MN} by $G_{MN}+\delta G_{MN}$ and expanding up to first order in δG_{MN}, the first term of (5.6) gives

$$\sqrt{-G}\Big(R-\frac{\Lambda}{2M^3}\Big)\to\sqrt{-G}\Big\{\delta R+\delta G^{MN}R_{MN}+\Big(R-\frac{\Lambda}{2M^3}\Big)\frac{\delta G}{2}\Big\}\,, \tag{5.9}$$

where

$$\delta G = G^{MN}\delta G_{MN} = -G_{MN}\delta G^{MN}\ , \tag{5.10}$$

$$\delta R = -\nabla_M(\nabla_N\delta G^{MN}-G_{PQ}\nabla^M\delta G^{PQ})\ . \tag{5.11}$$

The last two terms of (5.9) give the bulk part of the variation, which contains the Einstein tensor:

$$\frac{\delta S}{2M^3}\Big|_{\rm bulk}=\int d^5x\sqrt{-G}\Big\{R_{MN}-\frac{G_{MN}}{2}\Big(R-\frac{\Lambda}{2M^3}\Big)\Big\}\delta G^{MN}\,. \tag{5.12}$$

Next, from the brane part of (5.6) we get

$$\sqrt{-g^{(i)}}\Big\{\lambda_i\delta\tilde{\mathcal{R}}^{(i)}+\lambda_i\delta g^{(i)\alpha\beta}\tilde{\mathcal{R}}^{(i)}_{\alpha\beta}-\frac{g^{(i)}_{\alpha\beta}}{2}\Big(\lambda_i\tilde{\mathcal{R}}^{(i)}-U_i\Big)\delta g^{(i)\alpha\beta}\Big\}\,, \tag{5.13}$$

where

$$\delta\tilde{\mathcal{R}}^{(i)}=-\tilde{\nabla}^{(i)}_\alpha(\tilde{\nabla}^{(i)}_\beta\delta g^{(i)\alpha\beta}-g^{(i)}_{\gamma\delta}\tilde{\nabla}^{(i)\alpha}\delta g^{(i)\gamma\delta})\,, \tag{5.14}$$

with $\tilde{\nabla}^{(i)}$ a covariant derivative with respect to $g^{(i)}_{\alpha\beta}$. Since our bent branes extend to infinity along $x^{(i)\mu}$-directions, we can drop 4d total derivatives, and the brane part of the variation is

$$\frac{\delta S}{2M^3}\Big|_{\rm brane}=\sum_i\int_{\Phi_i=0}d^4x^{(i)}\sqrt{-g^{(i)}}\,\Big(\lambda_i\tilde{\mathcal{R}}^{(i)}_{\alpha\beta}-\frac{g^{(i)}_{\alpha\beta}}{2}(\lambda_i\tilde{\mathcal{R}}^{(i)}-U_i)\Big)\delta g^{(i)\alpha\beta}\ . \tag{5.15}$$

The δR term in (5.9) and the last term of (5.6) produce the boundary part of the variation: applying the Gauss theorem in the curved spacetime, the δR term gives

$$\begin{aligned}\frac{\delta S}{2M^3}\Big|_{\delta R}&=\int d^5x\sqrt{-G}\,\delta R\\ &=-2\sum_i\int^{(\rm bdy)}_{\Phi_i=0}d^4x^{(i)}\sqrt{-g^{(i)}}\,N^{(i)}_M\Big(\nabla_N\delta G^{MN}-G_{PQ}\nabla^M\delta G^{PQ}\Big)\,,\end{aligned} \tag{5.16}$$

where the factor of 2 is because we have used the symmetry discussed in §2.1 to write four boundary contributions in terms of two, and

$$\int^{(\rm bdy)}_{\Phi_1=0}=\int_{\Phi_1=0^+}\,,\qquad \int^{(\rm bdy)}_{\Phi_2=0}=\int_{\Phi_2=0^-}\,.$$

The K-term is a bit more complicated; we get

$$\frac{\delta S}{2M^3}\Big|_K = 4\sum_i \int_{\Phi_i=0}^{(\mathrm{bdy})} d^4x^{(i)}\, \delta\Big(\sqrt{-g^{(i)}}\, G^{MN}\nabla_M N_N^{(i)}\Big)$$
$$= 2\sum_i \int_{\Phi_i=0}^{(\mathrm{bdy})} d^4x^{(i)} \sqrt{-g^{(i)}}\Big\{2N_M^{(i)}\nabla_N\delta G^{MN} - (G_{MN} + N_M^{(i)}N_N^{(i)})N^{(i)P}\nabla_P\delta G^{MN}$$
$$+\Big(2\nabla_M N_N^{(i)} - \nabla_P(N^{(i)P}N_M^{(i)}N_N^{(i)}) - (G_{MN} - N_M^{(i)}N_N^{(i)})\nabla_P N^{(i)P}\Big)\delta G^{MN}\Big\}. \quad (5.17)$$

Note that we are varying $g_{\alpha\beta}^{(i)}$ and $N_M^{(i)}$ as well because δG^{MN} does not vanish on the boundary. Then, combining (5.16) and (5.17) gives

$$\frac{\delta S}{2M^3}\Big|_{\mathrm{bdy}} = 2\sum_i \int_{\Phi_i=0}^{(\mathrm{bdy})} d^4x^{(i)} \sqrt{-g^{(i)}}\Big\{N_M^{(i)}\Big(\nabla_N\delta G^{MN} - N_N^{(i)}N^{(i)P}\nabla_P\delta G^{MN}\Big)$$
$$+\Big(2\nabla_M N_N^{(i)} - 2N^{(i)P}N_M^{(i)}\nabla_P N_N^{(i)} - G_{MN}\nabla_P N^{(i)P}\Big)\delta G^{MN}\Big\}. \quad (5.18)$$

By introducing the projection operator, $P^{(i)}$, onto the i-th hypersurface, defined by

$$P_{MN}^{(i)} \equiv G_{MN} - N_M^{(i)}N_N^{(i)}\,, \quad (5.19)$$

(5.18) can be further simplified into

$$\frac{\delta S}{2M^3}\Big|_{\mathrm{bdy}} = 2\sum_i \int_{\Phi_i=0}^{(\mathrm{bdy})} d^4x^{(i)} \sqrt{-g^{(i)}}\Big\{P^{(i)}{}_M^P\nabla_P(N_N^{(i)}\delta G^{MN})$$
$$+\Big(P^{(i)}{}_M^P\nabla_P N_N^{(i)} - G_{MN}\nabla_P N^{(i)P}\Big)\delta G^{MN}\Big\}. \quad (5.20)$$

Using the identity

$$P_Q^{(i)P}\nabla_P P_M^{(i)Q} = -N_M^{(i)}\nabla_P N^{(i)P}\,, \quad (5.21)$$

we can rewrite (5.20) as follows:

$$\frac{\delta S}{2M^3}\Big|_{\mathrm{bdy}} = 2\sum_i \int_{\Phi_i=0}^{(\mathrm{bdy})} d^4x^{(i)} \sqrt{-g^{(i)}}\Big\{P^{(i)}{}_Q^P\nabla_P(P_M^{(i)Q}N_N^{(i)}\delta G^{MN})$$
$$+\Big(P^{(i)}{}_M^P\nabla_P N_N^{(i)} - P_{MN}^{(i)}\nabla_P N^{(i)P}\Big)\delta G^{MN}\Big\}. \quad (5.22)$$

At this point (5.22) does not seem to give us an EOM because of the first term of the integrand, which contains a derivative of δG^{MN}. However $P_M^{(i)P}\nabla_P$ is the tangential covariant derivative along the boundary hypersurface and $P_M^{(i)Q}N_N^{(i)}\delta G^{MN}$ is a vector tangential to the hypersurface. Thus the first term in (5.22) is a total tangential divergence, which is equivalent to a 4d total divergence, and can be dropped.

Now the complete variation of the action is

$$\begin{aligned}\frac{\delta S}{2M^3} &= \int d^5x\sqrt{-G}\left\{R_{MN} - \frac{G_{MN}}{2}\Big(R - \frac{\Lambda}{2M^3}\Big)\right\}\delta G^{MN} \\ &+ \sum_i \int_{\Phi_i=0} d^4x^{(i)}\sqrt{-g^{(i)}}\left(\lambda_i\tilde{\mathcal{R}}^{(i)}_{\alpha\beta} - \frac{g^{(i)}_{\alpha\beta}}{2}(\lambda_i\tilde{\mathcal{R}}^{(i)} - U_i)\right)\delta g^{(i)\alpha\beta} \\ &+ \sum_i \int^{(\mathrm{bdy})}_{\Phi_i=0} d^4x^{(i)}\sqrt{-g^{(i)}}\Big(2P^{(i)}{}^P_M\nabla_P N^{(i)}_N - 2P^{(i)}_{MN}\nabla_P N^{(i)P}\Big)\delta G^{MN}\ .\end{aligned} \tag{5.23}$$

From the arguments presented in §2.1, we can drop the distinction between brane and boundary contributions, obtaining

$$\begin{aligned}\frac{\delta S}{2M^3} &= \int d^5x\sqrt{-G}\left\{R_{MN} - \frac{G_{MN}}{2}\Big(R - \frac{\Lambda}{2M^3}\Big)\right\}\delta G^{MN} \\ &+ \sum_i \int_{\Phi_i=0} d^4x^{(i)}\sqrt{-g^{(i)}}\left\{\left(\lambda_i\tilde{\mathcal{R}}^{(i)}_{\alpha\beta} - \frac{g^{(i)}_{\alpha\beta}}{2}(\lambda_i\tilde{\mathcal{R}}^{(i)} - U_i)\right)e^{(i)\bar{\alpha}}{}_M\, e^{(i)\bar{\beta}}{}_N\right. \\ &\qquad \left. + 2P^{(i)}{}^P_M\nabla_P N^{(i)}_N - 2P^{(i)}_{MN}\nabla_P N^{(i)P}\right\}\delta G^{MN}\ ,\end{aligned} \tag{5.24}$$

where

$$e^{(i)\bar{M}}_M = \frac{\partial x^{(i)\bar{M}}}{\partial x^M} \tag{5.25}$$

transforms M-indices into $\bar{M}$-ones. Thus we have the bulk equations of motion:

$$R_{MN} - \frac{G_{MN}}{2}\Big(R - \frac{\Lambda}{2M^3}\Big) = 0\ , \tag{5.26}$$

supplemented by the brane-boundary equations:

$$\begin{aligned}&\Big[\Big(\lambda_i\tilde{\mathcal{R}}^{(i)}_{\alpha\beta} - \frac{g^{(i)}_{\alpha\beta}}{2}(\lambda_i\tilde{\mathcal{R}}^{(i)} - U_i)\Big)e^{(i)\bar{\alpha}}{}_M\, e^{(i)\bar{\beta}}{}_N \\ &\qquad + 2P^{(i)}{}^P_M\nabla_P N^{(i)}_N - 2P^{(i)}_{MN}\nabla_P N^{(i)P}\Big]_{\Phi_i=0} = 0\ .\end{aligned} \tag{5.27}$$

Equations (5.26-5.27) are completely general and can be applied to arbitrary boundaries. It is completely covariant under general coordinate transformations, including those that bend the branes.

5.2 Straight gauges

It is extremely convenient to work in straight gauges. These are defined as follows:

straight gauge: a choice of 5d bulk coordinate system such that:

- both of the branes are described by straight slices $y = y_i$;
- $G^{(i)}_{\mu 4} = [G_{\mu 4}]_{y=y_i} = 0$ for $i = 1, 2$.

From (5.2), straight slices at $y = y_i$ implies that

$$N_\mu^{(i)} = 0\,, \quad N_4^{(i)} = \frac{\theta_i}{\sqrt{G^{(i)44}}}\,. \tag{5.28}$$

Then using

$$e_{\bar\alpha}^{(i)M} N_M^{(i)} = e_{\bar\alpha}^{(i)\bar M} N_{\bar M}^{(i)} = N_{\bar\alpha}^{(i)} = 0\,, \tag{5.29}$$

and (5.28), we get

$$e_{\bar\alpha}^{(i)4} = 0\,. \tag{5.30}$$

Note however that $G_{44}^{(i)} = 1/G^{(i)44}$ is still arbitrary.

Thus we see that an equivalent (and perhaps more intuitive) definition of straight gauges is:

> **straight gauge:** a choice of 5d bulk coordinate system such that $N^{(i)\mu} = 0$ and $N_\mu^{(i)} = 0$.

Yet another equivalent definition is that a straight gauge is any choice of 5d bulk coordinates such that the bulk coordinates are BNC's at the locations of both branes.

A natural question is whether it is always possible to impose a straight gauge, starting from an arbitrary bulk coordinate system. We can prove this, without loss of generality, by starting from a bulk coordinate system where the first brane is at $y = 0$ with $N_\mu^{(1)} = 0$ and $[G_{\mu 4}]_{y=0} = 0$, while the second brane is bent:

$$\Phi_1 = y\,, \quad \Phi_2 = y - L - \rho(x)\,, \tag{5.31}$$

and $N_\mu^{(2)}$, $[G_{\mu 4}]_{\Phi_2=0}$ do not necessarily vanish.

To get to a straight gauge, we first perform a GCT defined by

$$\breve x^\mu = x^\mu\,, \quad \breve y = y - \frac{\rho(x)}{L + \rho(x)} y\,, \tag{5.32}$$

under which

$$\breve\Phi_1 = \breve y\,, \quad \breve\Phi_2 = \breve y - L\,, \tag{5.33}$$

but

$$\begin{aligned}
[\breve G_{\mu 4}]_{\breve y = y_i} &= \left[\frac{\partial x^M}{\partial \breve x^\mu}\frac{\partial x^N}{\partial \breve y} G_{MN}\right]_{\breve y = y_i} = \left[\frac{\partial y}{\partial \breve y}\left(G_{\mu 4} + \frac{\partial y}{\partial \breve x^\mu} G_{44}\right)\right]_{\breve y = y_i} \\
&= \frac{L + \rho(x)}{L}\left[G_{\mu 4} + \frac{\breve y}{L}\frac{\partial \rho}{\partial \breve x^\mu} G_{44}\right]_{\breve y = y_i}\,.
\end{aligned} \tag{5.34}$$

That is, the first condition in our definition of a straight gauge is satisfied but $[\breve G_{\mu 4}]_{\breve y = L}$ is still non-vanishing. Now we perform a second GCT such that

$$\hat y = \breve y\,, \quad \hat x^\mu = f^\mu(\breve x, \breve y)\,; \tag{5.35}$$

both branes are still described by $\hat{y} = y_i$ and

$$[\hat{G}_{\mu 4}]_{\breve{y}=y_i} = \left[\frac{\partial \breve{x}^\alpha}{\partial \hat{x}^\mu}\left(\frac{\partial \breve{x}^\beta}{\partial \hat{y}}\,\breve{g}_{\alpha\beta} + \breve{G}_{\alpha 4}\right)\right]_{\hat{y}=y_i} . \tag{5.36}$$

(5.36) does not necessarily vanish at $\hat{y} = 0, L$ for arbitrary $\breve{G}_{MN}$. But for any fixed $\breve{G}_{MN}$, the quantity inside the parentheses can be set to be zero by choosing, for example,

$$\hat{x}^\alpha = \breve{x}^\alpha + \int d\hat{y}\; \breve{g}^{\alpha\beta}\breve{G}_{\beta 4} . \tag{5.37}$$

Therefore it is always possible to find a bulk coordinate system satisfying straight gauge conditions.

The general brane-boundary equations (5.27) simplify quite a bit in a straight gauge. To see this, contract the tensor equations (5.27) with $e^{(i)M}_{\bar{M}} e^{(i)N}_{\bar{N}}$:

$$\begin{aligned}&\Big[\Big(\lambda_i \tilde{\mathcal{R}}^{(i)}_{\alpha\beta} - \frac{g^{(i)}_{\alpha\beta}}{2}(\lambda_i \tilde{\mathcal{R}}^{(i)} - U_i)\Big)\delta^\alpha_{\bar{M}}\delta^\beta_{\bar{N}} \\ &\quad + e^{(i)M}_{\bar{M}} e^{(i)N}_{\bar{N}}\left(P^{(i)}{}^P_M \nabla_P N^{(i)}_N + P^{(i)}{}^P_N \nabla_P N^{(i)}_M - 2P^{(i)}_{MN}\nabla_P N^{(i)P}\right)\Big]_{\Phi_i=0} = 0 .\end{aligned} \tag{5.38}$$

These brane-boundary equations break up into three tensor equations each. The $\bar{4}\bar{4}$ equation is:

$$\left[e^{(i)M}_{\bar{4}} e^{(i)N}_{\bar{4}}\left(P^{(i)}{}^P_M \nabla_P N^{(i)}_N + P^{(i)}{}^P_N \nabla_P N^{(i)}_M - 2P^{(i)}_{MN}\nabla_P N^{(i)P}\right)\right]_{\Phi_i=0} = 0 . \tag{5.39}$$

This is trivially satisfied, since $e^{(i)M}_{\bar{4}}$ is parallel to $N^{(i)M}$,[1] and $N^{(i)M}$ contracted with a projection operator $P^{(i)P}_M$ vanishes.

The $\bar{\mu}\bar{4}$ part is:

$$\left[e^{(i)M}_{\bar{\mu}} e^{(i)N}_{\bar{4}}\left(P^{(i)}{}^P_M \nabla_P N^{(i)}_N + P^{(i)}{}^P_N \nabla_P N^{(i)}_M - 2P^{(i)}_{MN}\nabla_P N^{(i)P}\right)\right]_{\Phi_i=0} = 0 . \tag{5.40}$$

The second and third terms vanish for the same reason as above, leaving only the first term, which is proportional to $N^{(i)N}\nabla_P N^{(i)}_N = 0$.

So only the $\bar{\mu}\bar{\nu}$ brane-boundary equation has any content. It can be simplified using (5.29):

$$\begin{aligned}&\Big[\Big(\lambda_i \tilde{\mathcal{R}}^{(i)}_{\bar{\alpha}\bar{\beta}} - \frac{g^{(i)}_{\bar{\alpha}\bar{\beta}}}{2}(\lambda_i \tilde{\mathcal{R}}^{(i)} - U_i)\Big) \\ &\quad + e^{(i)M}_{\bar{\alpha}} e^{(i)N}_{\bar{\beta}}\left(\nabla_M N^{(i)}_N + \nabla_N N^{(i)}_M\right) - 2g_{\bar{\alpha}\bar{\beta}}\nabla_P N^{(i)P}\Big]_{\Phi_i=0} = 0 .\end{aligned} \tag{5.41}$$

[1]For any 5-vector T_M tangential to $\Phi_i = 0$-hypersurface,

$$e^{(i)M}_{\bar{4}} T_M = e^{(i)\bar{M}}_{\bar{4}} T_{\bar{M}} = T_{\bar{4}} = 0 .$$

Now we impose a straight gauge. Then because of (5.30), we can always choose the BNCs such that

$$e_{\bar{\alpha}}^{(i)M} = \delta_{\bar{\alpha}}^{M} \ . \tag{5.42}$$

Furthermore, in the straight gauge the y-direction of the bulk coordinate system on the i-th brane is parallel to $N_M^{(i)}$ which is in the $\bar{y}^{(i)}$-direction, and thus we can take

$$e_{\bar{4}}^{(i)M} = \delta_{\bar{4}}^{M} \ . \tag{5.43}$$

That is, $e^{(i)} = 1$ and we need not distinguish between $x^{(i)\bar{M}}$-system and $[x^M]_{\Phi_i=0}$-one; one bulk coordinate patch can describe the whole spacetime including the boundary while keeping a straight gauge, which justifies dropping bars on indices in (5.41).

Due to (5.28) and the second condition in our definition of a straight gauge, we get

$$\nabla_\alpha N_\beta^{(i)} + \nabla_\beta N_\alpha^{(i)} = -2\Gamma^4_{\alpha\beta} N_4^{(i)} = \theta_i \sqrt{G^{44}} g'_{\alpha\beta} \ . \tag{5.44}$$

Similarly:

$$-2g_{\alpha\beta}\nabla_P N^{(i)P} = -2\theta_i\, g_{\alpha\beta}(\sqrt{G^{44}}' + \sqrt{G^{44}}\Gamma^P_{P4}) = -\theta_i\sqrt{G^{44}} g_{\alpha\beta} g^{\rho\sigma} g'_{\rho\sigma} \ . \tag{5.45}$$

Putting together (5.41-5.45) we get the full EOM in an arbitrary straight gauge:

$$\text{bulk}: \qquad R_{MN} - \frac{1}{2}G_{MN}\Big(R - \frac{\Lambda}{2M^3}\Big) = 0 \ , \tag{5.46}$$

$$\text{brane–boundary}: \ \Big[\lambda_i \tilde{\mathcal{R}}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\lambda_i\tilde{\mathcal{R}} - U_i) + \theta_i\sqrt{G^{44}}(g'_{\mu\nu} - g_{\mu\nu}g'_{\rho\sigma}g^{\rho\sigma})\Big]_{y=y_i} = 0 \ . \tag{5.47}$$

Recall that $-\theta_1 = \theta_2 = 1$, and that strictly speaking the terms multiplying θ_i are evaluated at $y = 0^+$, L^-, not at $y = 0$, L.

5.3 Background solutions

For a linearized analysis, we write

$$G_{MN} = G^{\mathbf{0}}_{MN} + h_{MN} = \begin{pmatrix} g^{\mathbf{0}}_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} h_{\mu\nu} & h_{\mu 4} \\ h_{4\nu} & h_{44} \end{pmatrix} . \tag{5.48}$$

We can solve (5.46-5.47) with a straight gauge ansatz for a general warped AdS_4 background metric:

$$g^{\mathbf{0}}_{\mu\nu} = \frac{a(y)^2}{(1 - \frac{H^2x^2}{4})^2}\eta_{\mu\nu} \ , \tag{5.49}$$

with $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and $x^2 = \eta_{\mu\nu}x^\mu x^\nu$. This corresponds to a warped geometry where each slice is AdS_4 (or 4d Minkowski space in the limit $H^2 \to 0$). [2]

[2] It is also possible to obtain factorizable backgrounds with dS_4 slices, but we will not consider these solutions here. See [36, 24].

It is easy to show that

$$\Gamma^{\mathbf{0}\mu}_{\nu\lambda} = \frac{H^2}{2(1-\frac{H^2x^2}{4})}(\delta^\mu_\lambda\eta_{\nu\sigma}x^\sigma + \delta^\mu_\nu\eta_{\lambda\sigma}x^\sigma - \eta_{\nu\lambda}x^\mu)\,, \tag{5.50}$$

$$\Gamma^{\mathbf{0}\mu}_{\nu 4} = \frac{a'}{a}\delta^\mu_\nu\,, \tag{5.51}$$

$$\Gamma^{\mathbf{0}4}_{\mu\nu} = -\frac{a'}{a}g^{\mathbf{0}}_{\mu\nu}\,, \tag{5.52}$$

while all the other components of $\Gamma^{\mathbf{0}M}_{NP}$ vanish.

Also we find

$$R^{\mathbf{0}}_{\mu\nu} = -\frac{3H^2+3a'^2+aa''}{a^2}g^{\mathbf{0}}_{\mu\nu}\,, \tag{5.53}$$

$$R^{\mathbf{0}}_{44} = -\frac{4a''}{a}\,, \tag{5.54}$$

$$R^{\mathbf{0}} = -\frac{4(3H^2+3a'^2+2aa'')}{a^2}\,, \tag{5.55}$$

$$\tilde{\mathcal{R}}^{\mathbf{0}}_{\mu\nu}\Big|_{y=y_i} = -\Big[\frac{3H^2}{a^2}g^{\mathbf{0}}_{\mu\nu}\Big]_{y=y_i}\,, \tag{5.56}$$

$$\tilde{\mathcal{R}}^{\mathbf{0}}\Big|_{y=y_i} = -\Big[\frac{12H^2}{a^2}\Big]_{y=y_i}\,, \tag{5.57}$$

and $R^{\mathbf{0}}_{\mu 4} = 0$. Then, with G_{MN} replaced by $G^{\mathbf{0}}_{MN}$, (5.46) gives

$$\Big(H^2 + a'^2 - 2k^2a^2 + aa''\Big)g^{\mathbf{0}}_{\mu\nu} = 0\,, \tag{5.58}$$

$$H^2 + a'^2 - k^2a^2 = 0\,, \tag{5.59}$$

where $k^2 = -\Lambda/24M^3$. We will restrict our consideration to models with a negative bulk cosmological constant. From (5.47) we get

$$\Big[\Big(\frac{U_i}{6} + \frac{\lambda_i H^2}{a^2} - \theta_i\frac{2a'}{a}\Big)g^{\mathbf{0}}_{\mu\nu}\Big]_{y=y_i} = 0\,. \tag{5.60}$$

The general solution of (5.59) with normalization $a(0) = 1$ has the form:

$$a(y) = \frac{\cosh k(y-y_0)}{\cosh ky_0}\,, \quad 0 < y < L\,, \tag{5.61}$$

where

$$\cosh ky_0 = \frac{k}{H}\,. \tag{5.62}$$

With this solution, (5.58) is automatically satisfied. (5.60) gives boundary conditions at $y = 0$ and L:

$$y = 0 : 2kT_0 = \frac{U_0}{6} + \frac{\lambda_0 H^2}{a(0)^2}\,, \tag{5.63}$$

$$y = L : 2kT_L = \frac{U_L}{6} + \frac{\lambda_L H^2}{a(L)^2}\,, \tag{5.64}$$

where $T_0 = \tanh ky_0$ and $T_L = \tanh k(L - y_0)$.

For convenience we define $v_i = k\lambda_i = kM_i^2/M^3$ and $w_i = U_i/k = V_i/(2kM^3)$ and solve (5.63) and (5.64) for T_0 and T_L respectively to get

$$\begin{aligned} T_i &= \frac{U_i}{12k} + \frac{\lambda_i}{2k}\frac{H^2\cosh^2 ky_0}{\cosh^2 k(y_i - y_0)} = \frac{w_i}{12} + \frac{v_i}{2}(1 - T_i^2) \\ &\rightarrow \quad T_i^{\pm} = \frac{1}{v_i}\Big(-1 \pm \sqrt{1 + \frac{1}{6}w_i v_i + v_i^2}\Big)\,. \end{aligned} \tag{5.65}$$

Given any input values for the brane tensions V_i and brane Planck constants M_i, we can find a background solution by solving for the 4d curvature parameter H and the brane separation L. Equivalently, we can specify w_0, w_L, v_0 and v_L as inputs and solve for T_0 and T_L using (5.65). For example, if $w_0 = -w_L = 12$, $v_0 > -1$ and $v_L < 1$, then $H = 0$ (*i.e.* the branes are flat), the value of L is undetermined, and $T_0 = -T_L = 1$. This special case becomes the original Randall-Sundrum model when we take v_0, $v_L \rightarrow 0$.

Recall that we are only considering the case where the 4d curvature is AdS-like, *i.e.* the bulk space is approximately AdS_5/AdS_4. This means that $H^2 > 0$, and the T_i are real and satisfy $|T_i| < 1$. Choices of input parameters which do not satisfy these conditions do not give AdS_5/AdS_4 solutions. Solving $-1 < T_i^+ < 1$, we get

$$\begin{aligned} &\Big(v_i \geq 0 \ \cap \ w_i \geq -6v_i - \frac{6}{v_i} \\ &\qquad \cap \Big((v_i \geq 1 \ \cap \ w_i \leq 12) \ \cup \ (v_i < 1 \ \cap \ -12 \leq w_i \leq 12)\Big)\Big) \\ &\cup \Big(v_i < 0 \ \cap \ w_i \leq -6v_i - \frac{6}{v_i} \\ &\qquad \cap \Big((v_i < -1 \ \cap \ w_i \geq -12) \ \cup \ (v_i \geq -1 \ \cap \ -12 \leq w_i \leq 12)\Big)\Big)\,. \end{aligned} \tag{5.66}$$

The results for T_i^- are similar. Note that there are solutions for both positive and negative brane tensions, and for both positive and negative brane Planck constants.

5.4 Gauge fixing

Having determined the general background solution, we have to deal with the metric fluctuations, h_{MN}, as given in (5.48). We will perform a complete gauge-fixing, starting with the straight gauge implied by the background solution. All indices will be raised and lowered using the background metric $G^{\mathbf{0}}_{MN}$, but to reduce clutter we will omit the superscript $^{\mathbf{0}}$ on $g_{\mu\nu}$.

Under a linearized 5d general coordinate transformation $x^M \rightarrow x^M + \xi^M$ the metric fluctuations transform as follows:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - g_{\mu\nu}\frac{2a'}{a}\xi^4 - \tilde{\nabla}_\mu\xi_\nu - \tilde{\nabla}_\nu\xi_\mu\,, \tag{5.67}$$

$$h_{\mu 4} \rightarrow h_{\mu 4} - g_{\mu\nu}\xi^{\nu\prime} - \partial_\mu\xi^4\,, \tag{5.68}$$

$$h_{44} \rightarrow h_{44} - 2\xi^{4\prime}\,. \tag{5.69}$$

We start with a partial gauge-fixing to exhibit the radion, letting $\xi^{(\mathrm{I})\mu} = 0$ and

$$\xi^{(\mathrm{I})4} = \frac{1}{2}\int^y h_{44} dy - \frac{1}{2}\int^y F(y)\psi(x) dy \ , \tag{5.70}$$

with $F(y)$ a fixed but arbitrary function of y. This transforms an arbitrary h_{44} into

$$h_{44} = F(y)\psi(x) \, . \tag{5.71}$$

Since we want to be in a straight gauge, we must require that $\xi^{(\mathrm{I})4}$ vanishes at the locations of the branes. On the interval $0 < y < L$, this fixes the y-independent part of (5.70):

$$\xi^{(\mathrm{I})4} = \frac{1}{2}\int_0^y h_{44} \, dy - \frac{1}{2}\int_0^y F(y)\psi(x) \, dy \ , \tag{5.72}$$

and fixes a relation between the radion field $\psi(x)$, $F(y)$ and the original metric fluctuation $h_{44}(x, y)$:

$$\psi(x) = \frac{\int_0^L h_{44} \, dy}{\int_0^L F(y) \, dy} \ . \tag{5.73}$$

From (5.73) we see that $F(y)$, though arbitrary, must be nonzero. More precisely, the straight gauge condition requires:

$$\int_0^L F(y) dy \neq 0 \ . \tag{5.74}$$

Note that for a general metric fluctuation $h_{\mu 4}(x, y)$, we are not yet in a straight gauge since $G^{(i)}_{\mu 4} \neq 0$. So our next step is to fix to a straight gauge, by a partial gauge-fixing which eliminates $h_{\mu 4}(x, y)$ altogether. Choose $\xi^{(\mathrm{II})4} = 0$ and

$$\xi^{(\mathrm{II})\mu} = \int^y h^{\mu 4} dy \ . \tag{5.75}$$

Then $h_{\mu\nu}$ is still arbitrary, h_{44} is unaffected, and

$$h_{\mu 4} = 0 \ . \tag{5.76}$$

Given the straight gauge conditions and the gauge choices (5.71) and (5.76), the residual gauge freedom is generated by

$$\xi^4 = 0 \, , \quad \xi^\mu = \xi^\mu(x) \ . \tag{5.77}$$

Note that what actually appears in the general coordinate transformation for $h_{\mu\nu}$ is $\tilde{\nabla}_\mu \xi_\nu(x) + \tilde{\nabla}_\nu \xi_\mu(x)$, which picks up a nontrivial y dependence, $a^2(y)$, from lowering the vector index.

The general coordinate transformation generated by

$$\xi^4 = \xi^4(x) \equiv \epsilon(x) \ , \quad \xi^\mu = -\frac{a^2}{H^2}\frac{a'}{a}\tilde{\nabla}^\mu \epsilon(x) \ , \tag{5.78}$$

respects (5.71) and (5.76) but takes us out of the straight gauge. The scalar $\epsilon(x)$ is the putative brane-bending mode. Since the equations of motion are covariant, even under a brane-bending transformation generated by $\epsilon(x)$, this mode is pure gauge.

The full linearized bulk equations of motion are given by:

$$\begin{aligned}
\mu\nu \text{ part}: & \nabla_P\nabla_\mu h^P_\nu + \nabla_P\nabla_\nu h^P_\mu - \nabla^2 h_{\mu\nu} - \nabla_\mu\nabla_\nu h \\
& -g_{\mu\nu}(\nabla_M\nabla_N h^{MN} - \nabla^2 h) - 4k^2 g_{\mu\nu} h + 8k^2 h_{\mu\nu} = 0\,, \qquad (5.79)\\
\mu 4 \text{ part}: & \nabla_P\nabla_\mu h^P_4 + \nabla_P\nabla_4 h^P_\mu - \nabla^2 h_{\mu 4} - \nabla_\mu\nabla_4 h = 0\,, \qquad (5.80)\\
44 \text{ part}: & 2\nabla_P\nabla_4 h^P_4 - \nabla^2 h_{44} - \nabla_4\nabla_4 h \\
& -\nabla_M\nabla_N h^{MN} + \nabla^2 h - 4k^2 h + 8k^2 h_{44} = 0\,, \qquad (5.81)
\end{aligned}$$

where $h = G^{MN}h_{MN}$. In our background the above EOM can be expanded using the following identities, which hold for any 5-vector T^M:

$$\nabla_\mu T^\nu = \tilde\nabla_\mu T^\nu + \frac{a'}{a}\delta^\nu_\mu T^4\,, \qquad (5.82)$$

$$\nabla_\mu T^4 = \tilde\nabla_\mu T^4 - \frac{a'}{a}T_\mu\,. \qquad (5.83)$$

Using these and our partial gauge-fixings, (5.71) and (5.76), we obtain

$$\begin{aligned}
0 = {} & \tilde\nabla_\rho\tilde\nabla_\mu h^\rho_\nu + \tilde\nabla_\rho\tilde\nabla_\nu h^\rho_\mu - \tilde\nabla^2 h_{\mu\nu} - \tilde\nabla_\mu\tilde\nabla_\nu\tilde h - g_{\mu\nu}(\tilde\nabla_\rho\tilde\nabla_\sigma h^{\rho\sigma} - \tilde\nabla^2\tilde h) \\
& -h''_{\mu\nu} + g_{\mu\nu}\tilde h'' + \frac{4a'}{a}g_{\mu\nu}\tilde h' + \frac{8H^2+4a'^2}{a^2}h_{\mu\nu} - \frac{3H^2}{a^2}g_{\mu\nu}\tilde h \\
& -F\tilde\nabla_\mu\tilde\nabla_\nu\psi + g_{\mu\nu}F\tilde\nabla^2\psi - \frac{3a'}{a}g_{\mu\nu}F'\psi - \frac{6H^2+12a'^2}{a^2}g_{\mu\nu}F\psi\,, \qquad (5.84)\\
0 = {} & (\tilde\nabla_\nu h^\nu_\mu)' - \partial_\mu\tilde h' + \frac{3a'}{a}F\partial_\mu\psi\,, \qquad (5.85)\\
0 = {} & -\tilde\nabla_\mu\tilde\nabla_\nu h^{\mu\nu} + \tilde\nabla^2\tilde h + \frac{3a'}{a}\tilde h' - \frac{3H^2}{a^2}\tilde h - \frac{12a'^2}{a^2}F\psi\,, \qquad (5.86)
\end{aligned}$$

with $\tilde h = g^{\mu\nu}h_{\mu\nu}$. Also twice (5.86) subtracted from the trace of (5.84) gives the auxiliary EOM:

$$0 = \frac{(a^2\tilde h')'}{a^2} + F\tilde\nabla^2\psi - \frac{4a'}{a}F'\psi - 8k^2F\psi\,. \qquad (5.87)$$

By a similar procedure the brane-boundary equations become

$$\begin{aligned}
0 = {} & \Big[\theta_i({h_{\mu\nu}}' - g_{\mu\nu}\tilde h') + \Big(\frac{3\lambda_i H^2}{a^2} - 2kT_i\Big)h_{\mu\nu} - \frac{3\lambda_i H^2}{2a^2}g_{\mu\nu}\tilde h + 3kT_i g_{\mu\nu}F\psi \\
& +\frac{\lambda_i}{2}(\tilde\nabla_\rho\tilde\nabla_\mu h^\rho_\nu + \tilde\nabla_\rho\tilde\nabla_\nu h^\rho_\mu - \tilde\nabla^2 h_{\mu\nu} - \tilde\nabla_\mu\tilde\nabla_\nu\tilde h) \\
& -\frac{\lambda_i}{2}g_{\mu\nu}(\tilde\nabla_\rho\tilde\nabla_\sigma h^{\rho\sigma} - \tilde\nabla^2\tilde h)\Big]_{y=y_i}\,. \qquad (5.88)
\end{aligned}$$

5.4.1 "massive" case

We can generalize (1) of Appendix (and shuffle ϕ_1 and ϕ_2) to get

$$h_{\mu\nu} = b_{\mu\nu} + \tilde{\nabla}_\mu V_\nu + \tilde{\nabla}_\nu V_\mu + a^2\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{1}{4}g_{\mu\nu}\tilde{\nabla}^2\Big)\phi_1 + g_{\mu\nu}\phi_2\,, \tag{5.89}$$

with

$$\tilde{\nabla}^\mu b_{\mu\nu} = 0,\ \tilde{b} = 0\,, \tag{5.90}$$

$$\tilde{\nabla}^\mu V_\mu = 0\,. \tag{5.91}$$

After y-integration, (5.87) gives the first equation for ϕ_1 and ϕ_2:

$$\tilde{h}' = 4\phi_2' = \frac{f_1(x)}{a^2} - \mathcal{F}\,\mathcal{D}_4\psi(x) + \frac{4a'}{a}F\,\psi(x)\,, \tag{5.92}$$

where $\mathcal{D}_n = \tilde{\nabla}^2 - \frac{nH^2}{a^2}$, $\mathcal{F}'(y) = F(y)$ and a new field, $f_1(x)$, is introduced as an integration "constant" for $\tilde{h}'$. Of course, there should have been other generic integration constants arising from integrating $F(y)$ and $a(y)$. But all of them can be absorbed into $\mathcal{F}$ and f_1.

Then (5.86) and (5.85) become

$$\frac{a^2}{4}\mathcal{D}_4\tilde{\nabla}^2\phi_1 = \mathcal{D}_4\Big(\phi_2 - \frac{a'}{a}\mathcal{F}\psi\Big) + \frac{a'}{a}\frac{f_1}{a^2}\,, \tag{5.93}$$

$$(\mathcal{D}_3 V_\mu)' = 3\tilde{\nabla}_\mu\Big(\phi_2' - \frac{a'}{a}F\psi - \frac{a^2}{4}\mathcal{D}_4\phi_1'\Big)\,. \tag{5.94}$$

As in the flat or RSI case, (5.89) breaks down when $\tilde{\nabla}_\mu V_\nu + \tilde{\nabla}_\nu V_\mu$ and $\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{1}{4}g_{\mu\nu}\tilde{\nabla}^2\Big)\phi_1$ become transverse, *i.e.* when

$$\tilde{\nabla}^\nu(\tilde{\nabla}_\mu V_\nu + \tilde{\nabla}_\nu V_\mu) = \mathcal{D}_3 V_\mu = 0\,, \tag{5.95}$$

$$\tilde{\nabla}^\nu\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{1}{4}g_{\mu\nu}\tilde{\nabla}^2\Big)\phi_1 = \frac{3}{4}\tilde{\nabla}_\mu\mathcal{D}_4\phi_1 = 0\,. \tag{5.96}$$

So our "massive" tensor decomposition (5.89) is valid for modes such that:

i. scalar modes are not annihilated by $\mathcal{D}_4$, and

ii. vector modes are not annihilated by $\mathcal{D}_3$.

Due to the condition i, we can safely rewrite f_1 as

$$f_1(x) = a^2\mathcal{D}_4\sigma(x)\,. \tag{5.97}$$

We remove $\mathcal{D}_4$ from (5.93) to get

$$\frac{a^2}{4}\tilde{\nabla}^2\phi_1 = \phi_2 + \frac{a'}{a}(\sigma - \mathcal{F}\psi)\,. \tag{5.98}$$

Taking a y-derivative of it and using (5.92),

$$\tilde{\nabla}^2(a^2\phi_1' - \sigma + \mathcal{F}\psi) = 0\,. \tag{5.99}$$

In AdS_4, the eigenvalue of $\tilde{\nabla}^2$ acting on a scalar is bounded below by $4H^2/a^2$, that is, $\tilde{\nabla}^2$ cannot kill a scalar [37]. Then (5.99) gives

$$a^2\phi_1' = \sigma - \mathcal{F}\psi\,. \tag{5.100}$$

Plugging this and (5.92) into (5.94), we get

$$(\mathcal{D}_3 V_\mu)' = 0\,. \tag{5.101}$$

Noting condition ii, we see that the y-dependence of V_μ should be a^2. This allows us to eliminate V_μ using the transverse part of the residual gauge freedom.

Now (5.84) boils down to

$$\begin{aligned}
0 =& -\tilde{\nabla}^2 b_{\mu\nu} - b_{\mu\nu}'' + \frac{4a'^2}{a^2} b_{\mu\nu} \\
&+(g_{\mu\nu}\tilde{\nabla}^2 - \tilde{\nabla}_\mu\tilde{\nabla}_\nu)(a^2\phi_1'' + 4aa'\phi_1' - \frac{a^2}{2}\tilde{\nabla}^2\phi_1 + 2\phi_2 + F\psi) \\
&+g_{\mu\nu}\Big(-\frac{3}{4}a^2\tilde{\nabla}^2\phi_1'' - 3aa'\tilde{\nabla}^2\phi_1' + \frac{3H^2}{2}\tilde{\nabla}^2\phi_1 \\
&\qquad +3\phi_2'' + \frac{12a'}{a}\phi_2' - \frac{6H^2}{a^2}\phi_2 - \frac{3a'}{a}F'\psi - \frac{6H^2+12a'^2}{a^2}F\psi\Big) \\
=& -\tilde{\nabla}^2 b_{\mu\nu} - b_{\mu\nu}'' + \frac{4a'^2}{a^2} b_{\mu\nu}\,,
\end{aligned} \tag{5.102}$$

and (5.88) becomes

$$\begin{aligned}
0 =& \Big[\theta_i b_{\mu\nu}' - \Big(\frac{\lambda_i H^2}{a^2} + 2kT_i\Big) b_{\mu\nu} - \frac{\lambda_i}{2}\tilde{\nabla}^2 b_{\mu\nu} \\
&+(g_{\mu\nu}\tilde{\nabla}^2 - \tilde{\nabla}_\mu\tilde{\nabla}_\nu)(-\theta_i a^2\phi_1' - \frac{\lambda_i}{4}a^2\tilde{\nabla}^2\phi_1 + \lambda_i\phi_2) \\
&+g_{\mu\nu}\Big(\frac{3\theta_i}{4}a^2\tilde{\nabla}^2\phi_1' + \frac{3\lambda_i H^2}{4}\tilde{\nabla}^2\phi_1 - \frac{3\lambda_i H^2}{a^2}\phi_2 - 3\theta_i\phi_2' + 3kT_i F\psi\Big)\Big]_{y=y_i} \\
=& \Big[\theta_i b_{\mu\nu}' - \Big(\frac{\lambda_i H^2}{a^2} + 2kT_i\Big) b_{\mu\nu} - \frac{\lambda_i}{2}\tilde{\nabla}^2 b_{\mu\nu} \\
&+\theta_i(1+k\lambda_i T_i)\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - g_{\mu\nu}\tilde{\nabla}^2 + \frac{3H^2}{a^2}g_{\mu\nu}\Big)(\sigma - \mathcal{F}\psi)\Big]_{y=y_i}\,.
\end{aligned} \tag{5.103}$$

The trace of (5.103) is

$$\mathcal{D}_4\Big(\sigma - \mathcal{F}(0)\psi\Big) = 0\,,\ \ \mathcal{D}_4\Big(\sigma - \mathcal{F}(L)\psi\Big) = 0\,, \tag{5.104}$$

from which, considering condition i and $\mathcal{F}(0) \neq \mathcal{F}(L)$, it follows that

$$\sigma = 0\,,\ \ \psi = 0\,. \tag{5.105}$$

Then, from (5.92), (5.98) and (5.100):

$$\phi_2' = 0\,,\ \ \frac{a^2}{4}\tilde{\nabla}^2\phi_1 = \phi_2\,,\ \ \phi_1 = f_2(x)\,, \tag{5.106}$$

and $h_{\mu\nu}$ is

$$h_{\mu\nu} = b_{\mu\nu} + a^2 \tilde{\nabla}_\mu \tilde{\nabla}_\nu f_2 \,. \tag{5.107}$$

Since $a^2 \tilde{\nabla}_\mu \tilde{\nabla}_\nu f_2$ has the correct y-dependence and form, it is removed by the longitudinal component of the residual gauge freedom, leaving only $b_{\mu\nu}$.

To get the spectrum of $b_{\mu\nu}$, first we solve (5.102). Using the EOM for a transverse-traceless spin-2 field of mass $m \neq 0$ in an AdS_4 background [8]:

$$\tilde{\nabla}^2 b_{\mu\nu} + \frac{2H^2 - m^2}{a^2} b_{\mu\nu} = 0 \,, \tag{5.108}$$

and substituting $z = \tanh k(y - y_0)$, it becomes

$$(1 - z^2)\frac{d^2 b_{\mu\nu}}{dz^2} - 2z\frac{db_{\mu\nu}}{dz} + \Big(2 + \frac{m^2}{H^2} - \frac{4}{1 - z^2}\Big) b_{\mu\nu} = 0 \,, \tag{5.109}$$

and its solution is

$$b_{\mu\nu} = A_{\mu\nu}\, P(l, 2, z) + B_{\mu\nu}\, Q(l, 2, z) \,, \tag{5.110}$$

where P and Q are associated Legendre functions of the 1st and 2nd kind respectively and $l = \frac{1}{2}(-1 + \sqrt{9 + \frac{4m^2}{H^2}}\,)$.

With (5.105), (5.103) gives the boundary conditions:

$$\Big[2k(1 - z^2)\frac{db_{\mu\nu}}{dz} + \Big(4kT_0 + \lambda_0 k^2 \frac{m^2}{H^2}(1 - z^2)\Big) b_{\mu\nu}\Big]_{y=0} = 0 \,, \tag{5.111}$$

$$\Big[-2k(1 - z^2)\frac{db_{\mu\nu}}{dz} + \Big(4kT_L + \lambda_L k^2 \frac{m^2}{H^2}(1 - z^2)\Big) b_{\mu\nu}\Big]_{y=L} = 0 \,. \tag{5.112}$$

Plugging (5.110) into (5.111) and (5.112), we get

$$\begin{pmatrix} a_0 & b_0 \\ a_L & b_L \end{pmatrix} \begin{pmatrix} A_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \,, \tag{5.113}$$

where

$$\begin{aligned}
a_0 &= (k\lambda_0 q(1 - T_0^2) + (3 + \sqrt{9 + 4q})T_0)\, P(\frac{1}{2}(-1 + \sqrt{9 + 4q}), 2, -T_0) \\
&\quad + (3 + \sqrt{9 + 4q})\, P(\frac{1}{2}(-3 + \sqrt{9 + 4q}), 2, -T_0) \,, \\
b_0 &= (k\lambda_0 q(1 - T_0^2) + (3 + \sqrt{9 + 4q})T_0)\, Q(\frac{1}{2}(-1 + \sqrt{9 + 4q}), 2, -T_0) \\
&\quad + (3 + \sqrt{9 + 4q})\, Q(\frac{1}{2}(-3 + \sqrt{9 + 4q}), 2, -T_0) \,, \\
a_L &= (k\lambda_L q(1 - T_L^2) + (3 + \sqrt{9 + 4q})T_L)\, P(\frac{1}{2}(-1 + \sqrt{9 + 4q}), 2, T_L) \\
&\quad - (3 + \sqrt{9 + 4q})\, P(\frac{1}{2}(-3 + \sqrt{9 + 4q}), 2, T_L) \,, \\
b_L &= (k\lambda_L q(1 - T_L^2) + (3 + \sqrt{9 + 4q})T_L)\, Q(\frac{1}{2}(-1 + \sqrt{9 + 4q}), 2, T_L) \\
&\quad - (3 + \sqrt{9 + 4q})\, Q(\frac{1}{2}(-3 + \sqrt{9 + 4q}), 2, T_L) \,,
\end{aligned} \tag{5.114}$$

with $q = \frac{m^2}{H^2}$. The condition

$$0 = a_0 b_L - a_L b_0 \,, \tag{5.115}$$

determines the mass spectrum of the massive graviton. Up to an overall normalization (5.110) can now be written as

$$b_{\mu\nu} = B_{\mu\nu}\{b_0 P(l,2,z) - a_0 Q(l,2,z)\}\,. \tag{5.116}$$

It seems that (5.115) can be solved by $q = -2$, which implies the emergence of a tachyon. But actually this is just an artifact of (5.115): when $m^2/H^2 = -2$, *i.e.*, $l = 0$, $P(0,2,z)$ identically vanishes, and we need another independent solution. Solving (5.109) with $m^2/H^2 = -2$, we get

$$b_{\mu\nu}^{(q=-2)} = A_{\mu\nu}^{(q=-2)} \frac{1+z^2}{1-z^2} + B_{\mu\nu}^{(q=-2)} \frac{z}{1-z^2}\,, \tag{5.117}$$

which can satisfy (5.111-5.112) only by $A_{\mu\nu}^{(q=-2)} = B_{\mu\nu}^{(q=-2)} = 0$. That is, $b_{\mu\nu}^{(q=-2)} = 0$ and therefore there is no tachyon.

The final result is that the physical degrees of freedom in the massive sector consist of a Kaluza-Klein tower of massive gravitons from $b_{\mu\nu}(x,y)$, with 5 DOF each.

5.4.2 "massless" case

For modes which do not satisfy the "massive" conditions i and ii, we should use the curved space version of (7):

$$\begin{aligned} h_{\mu\nu} &= \beta_{\mu\nu} + \tilde{\nabla}_\mu v_\nu + \tilde{\nabla}_\nu v_\mu + a^2\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{1}{4} g_{\mu\nu}\tilde{\nabla}^2\Big)\varphi_1 \\ &\quad + \tilde{\nabla}_\mu n_\nu + \tilde{\nabla}_\nu n_\mu + c_{\mu\nu} + g_{\mu\nu}\varphi_2\,. \end{aligned} \tag{5.118}$$

In this decomposition, vector and scalar modes are annihilated by $\mathcal{D}_3$ and $\mathcal{D}_4$, respectively, while tensor modes (see (5.108)) are annihilated by $\mathcal{D}_{-2}$.

Equation (5.87) gives

$$2\tilde{\nabla}_\mu n^{\mu\prime} + 4\varphi_2' = \frac{f_1}{a^2} + \frac{4a'}{a} F\,\psi\,, \tag{5.119}$$

so (5.86) and (5.85) become

$$\tilde{\nabla}_\mu\tilde{\nabla}_\nu c^{\mu\nu} = \frac{3a'}{a}\frac{f_1}{a^2}\,, \tag{5.120}$$

$$\tilde{\nabla}_\nu c_\mu^{\nu\prime} = \partial_\mu\Big(\frac{f_1}{a^2} + \frac{a'}{a} F\,\psi - \tilde{\nabla}_\nu n^{\nu\prime} - \varphi_2'\Big)\,. \tag{5.121}$$

Since

$$\tilde{\nabla}^2\tilde{\nabla}^\nu c_{\mu\nu} = \tilde{\nabla}^\nu\tilde{\nabla}^2 c_{\mu\nu} + \frac{5H^2}{a^2}\tilde{\nabla}^\nu c_{\mu\nu} = \frac{3H^2}{a^2}\tilde{\nabla}^\nu c_{\mu\nu}\,, \tag{5.122}$$

$$\tilde{\nabla}^2\tilde{\nabla}_\mu\tilde{\nabla}_\nu n^\nu = \tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\nabla}^2 n^\nu = \frac{3H^2}{a^2}\tilde{\nabla}_\mu\tilde{\nabla}_\nu n^\nu\,, \tag{5.123}$$

$$\tilde{\nabla}^2\tilde{\nabla}_\mu(\text{scalar}) = \Big(\tilde{\nabla}_\mu\tilde{\nabla}^2 - \frac{3H^2}{a^2}\tilde{\nabla}_\mu\Big)(\text{scalar}) = \frac{H^2}{a^2}\tilde{\nabla}_\mu(\text{scalar})\,, \tag{5.124}$$

acting $\mathcal{D}_3$ on (5.121) reduces it into

$$0 = \frac{H^2}{a^2}\partial_\mu\Big(\frac{f_1}{a^2} + \frac{a'}{a}F\,\psi - \varphi_2'\Big)\,, \tag{5.125}$$

or

$$\varphi_2' = \frac{f_1}{a^2} + \frac{a'}{a}F\,\psi\,. \tag{5.126}$$

Then (5.119) gives

$$\tilde{\nabla}_\mu n^{\mu\prime} = -\frac{3f_1}{2a^2}\,. \tag{5.127}$$

The trace of the brane-boundary equation (5.88) is

$$\begin{aligned} 0 &= \Big[\,-3\theta_i\tilde{h}' - \frac{3\lambda_i H^2}{a^2}\tilde{h} + 12kT_iF\psi - \lambda_i(\tilde{\nabla}_\rho\tilde{\nabla}_\sigma h^{\rho\sigma} - \tilde{\nabla}^2\tilde{h})\Big]_{y=y_i} \\ &= -3\theta_i(1+k\lambda_iT_i)\frac{f_1(x)}{a^2}\Big|_{y=y_i}\,, \end{aligned} \tag{5.128}$$

i.e.,

$$f_1(x) = 0\,. \tag{5.129}$$

Since $\tilde{\nabla}^\mu c_{\mu\nu} \neq 0$ and $\tilde{\nabla}^\mu n_\mu \neq 0$, (5.120) and (5.127) give

$$c_{\mu\nu} = 0\,,\quad n^{\mu\prime} = 0\,. \tag{5.130}$$

Now that n^μ has the same y-dependence as ξ^μ, it is gauged away. Also from (5.126) we get

$$\varphi_2 = f_2(x) + \Big(\frac{a'}{a}\mathcal{F} - H^2\mathfrak{F}\Big)\psi\,, \tag{5.131}$$

where $\mathfrak{F}'(y) = \mathcal{F}/a^2$.

With $t_{\mu\nu} = \beta_{\mu\nu} + \tilde{\nabla}_\mu v_\nu + \tilde{\nabla}_\nu v_\mu$, (5.84) and (5.88) become

$$\begin{aligned} 0 &= -t_{\mu\nu}'' + \frac{2H^2+4a'^2}{a^2}t_{\mu\nu} \\ &\quad -\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{H^2}{a^2}g_{\mu\nu}\Big)\Big\{a^2\varphi_1'' + 4aa'\varphi_1' - 2H^2\varphi_1 + 2f_2 + \Big(F + \frac{2a'}{a}\mathcal{F} - 2H^2\mathfrak{F}\Big)\psi\Big\}\,, \end{aligned} \tag{5.132}$$

$$\begin{aligned} 0 &= \Big[\,\theta_i t_{\mu\nu}' - 2kT_it_{\mu\nu} \\ &\quad +\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{H^2}{a^2}g_{\mu\nu}\Big)\Big\{\theta_i a^2\varphi_1' + \lambda_iH^2\varphi_1 - \lambda_if_2(x) - \lambda_i\Big(\frac{a'}{a}\mathcal{F} - H^2\mathfrak{F}\Big)\psi\Big\}\Big]_{y=y_i}\,. \end{aligned} \tag{5.133}$$

By construction $\mathcal{D}_{-2}$ kills $\beta_{\mu\nu}$, and since

$$\tilde{\nabla}^2\tilde{\nabla}_{(\mu}v_{\nu)} = \tilde{\nabla}_{(\mu}\mathcal{D}_3v_{\nu)} - \frac{2H^2}{a^2}\tilde{\nabla}_{(\mu}v_{\nu)} = -\frac{2H^2}{a^2}\tilde{\nabla}_{(\mu}v_{\nu)}\,, \tag{5.134}$$

$\tilde{\nabla}_\mu v_\nu + \tilde{\nabla}_\nu v_\mu$ is also annihilated by $\mathcal{D}_{-2}$. But, acting on a scalar,

$$\mathcal{D}_{-2}\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{H^2}{a^2}g_{\mu\nu}\Big) = \Big(-\frac{4H^2}{a^2} + \frac{2H^2}{a^2}\Big)\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{H^2}{a^2}g_{\mu\nu}\Big)\,. \tag{5.135}$$

Therefore, by applying $\mathcal{D}_{-2}$ to (5.132) and (5.133) we separate scalar parts from the remaining. The separated scalar parts have the form

$$\tilde{\nabla}_\mu\tilde{\nabla}_\nu(\text{scalars}) = \frac{H^2}{a^2}g_{\mu\nu}(\text{scalars})\,, \tag{5.136}$$

which can only be solved by (scalars) = 0. Thus (5.132) gives

$$0 = -t_{\mu\nu}{}'' + \frac{2H^2 + 4a'^2}{a^2}t_{\mu\nu}\,, \tag{5.137}$$

$$0 = a^2\varphi_1'' + 4aa'\varphi_1' - 2H^2\varphi_1 + 2f_2 + \Big(F + \frac{2a'}{a}\mathcal{F} - 2H^2\mathfrak{F}\Big)\psi\,, \tag{5.138}$$

while from (5.133) we get

$$0 = \Big[\,\theta_i t_{\mu\nu}{}' - 2kT_i t_{\mu\nu}\Big]_{y=y_i}\,, \tag{5.139}$$

$$0 = \Big[\,\theta_i a^2\varphi_1' + \lambda_i H^2\varphi_1 - \lambda_i f_2(x) - \lambda_i\Big(\frac{a'}{a}\mathcal{F} - H^2\mathfrak{F}\Big)\psi\Big\}\,\Big]_{y=y_i}\,. \tag{5.140}$$

Introducing $z = \tanh k(y - y_0)$, the most general solution of (5.137) is

$$t_{\mu\nu}(x,y) = A_{\mu\nu}(x)\frac{z - \frac{z^3}{3}}{1-z^2} + B_{\mu\nu}(x)\frac{1}{1-z^2}\,. \tag{5.141}$$

The boundary conditions provided by (5.139) requires $A_{\mu\nu} = 0$. Thus,

$$t_{\mu\nu} = B_{\mu\nu}(x)\frac{1}{1-z^2}\,. \tag{5.142}$$

Since $1/(1-z^2) = \cosh^2 k(y-y_0) = a^2(y)\cosh^2 ky_0$, (5.142) is up to overall normalization

$$t_{\mu\nu}(x,y) = a^2(y)B_{\mu\nu}(x)\,. \tag{5.143}$$

Then v_μ has the correct y-dependence to be gauged away, leaving only $\beta_{\mu\nu}$.

(5.138) has a general solution

$$\varphi_1(x,y) = \frac{f_2(x)}{H^2} + (1-z)^2C(x) + zD(x) - \mathfrak{F}(y)\psi(x)\,, \tag{5.144}$$

and $h_{\mu\nu}$ becomes

$$\begin{aligned} h_{\mu\nu} &= \beta_{\mu\nu} + a^2\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{H^2}{a^2}g_{\mu\nu}\Big)\frac{f_2(x)}{H^2} + g_{\mu\nu}\Big\{f_2(x) + \Big(\frac{a'}{a}\mathcal{F} - H^2\mathfrak{F}\Big)\psi\Big\} \\ &\quad + a^2\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{H^2}{a^2}g_{\mu\nu}\Big)\Big((1-z)^2C(x) + zD(x) - \mathfrak{F}(y)\psi(x)\Big) \\ &= \beta_{\mu\nu} + a^2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\frac{f_2}{H^2} - \Big(a^2\mathfrak{F}\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{a'}{a}\mathcal{F}g_{\mu\nu}\Big)\psi \\ &\quad + a^2\Big(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \frac{H^2}{a^2}g_{\mu\nu}\Big)\Big((1-z)^2C(x) + zD(x)\Big)\,. \end{aligned} \tag{5.145}$$

Then we can see that f_2 can be gauged away.

Now (5.140) gives

$$-\alpha_0 C + \beta_0 D = \frac{k}{H^2}\beta_0 \mathcal{F}(0)\,\psi\,,$$
$$-\alpha_L C + \beta_L D = \frac{k}{H^2}\beta_L \mathcal{F}(L)\,\psi\,,$$

where

$$\alpha_0 = 2(1+T_0) + k\lambda_0(1+T_0)^2\,,\quad \beta_0 = 1 + k\lambda_0 T_0\,,$$
$$\alpha_L = 2(1-T_L) - k\lambda_L(1-T_L)^2\,,\quad \beta_L = 1 + k\lambda_L T_L\,.$$

$C(x)$ and $D(x)$ can be solved;

$$C = \frac{k}{H^2}\frac{\beta_0\beta_L}{\alpha_0\beta_L - \alpha_L\beta_0}\Big(\mathcal{F}(L) - \mathcal{F}(0)\Big)\psi\,, \tag{5.146}$$
$$D = \frac{k}{H^2}\frac{\alpha_0\beta_L\mathcal{F}(L) - \alpha_L\beta_0\mathcal{F}(0)}{\alpha_0\beta_L - \alpha_L\beta_0}\psi\,. \tag{5.147}$$

We can use the gauge freedom of $F(y)$ to simplify C and D. For example, choosing

$$k\mathcal{F}(y) = -\frac{y}{L}\Big(\frac{\alpha_0}{\beta_0} - \frac{\alpha_L}{\beta_L}\Big) + \frac{\alpha_0}{\beta_0}\,, \tag{5.148}$$

gives

$$C = -\frac{1}{H^2}\psi\,,\quad D = 0\,. \tag{5.149}$$

All the scalars are written in terms of $\psi(x)$.

In summary, the physical degrees of freedom of the massless sector consist of a massless graviton from $\beta_{\mu\nu}(x)$ with two on-shell degrees of freedom, and a massless radion $\psi(x)$.

6. Conclusion

In this paper we have developed a detailed methodology for analyzing models of braneworld gravity. We have used the interval picture, in which braneworld gravity has a well-defined action principle. The key result is equation (5.24), which gives the full variation of the braneworld gravity action with respect to an arbitrary metric variation. From this, we obtain the usual bulk Einstein equations, supplemented by additional constraints which we call "brane-boundary" equations.

The brane-boundary equations are generally covariant, even for coordinate transformations that change the boundary. An immediate consequence of our result is that there are no physical "brane-bending" modes of the 5d metric in braneworld gravity, as one would expect if general covariance were partially broken. This is important since scalar modes can lead to strong coupling behavior and kinetic ghosts. In the general class of models

considered in this paper, the radion and the KK gravitons are the only possible sources of such pathologies.

We have introduced the concept of straight gauges, and showed how it is always possible to reach a straight gauge starting from an arbitrary bulk coordinate system. Then we showed how the analysis of linearized metric fluctuations and their equations of motion simplify in a straight gauge. The equations of motion for metric fluctuations of higher dimensional gravity have previously been analyzed in axial, harmonic, de Donder, or Gaussian normal gauges. However, for braneworld setups with more than one brane, none of these gauge choices corresponds to a straight gauge in a single coordinate patch.

In §3, §4, and §5, we have explicitly gauge-fixed and solved the equations of motions for setups with two branes, and 5d backgrounds that are flat, warped Randall-Sundrum, or general warped AdS_5/AdS_4. In all three cases we define a family of straight gauges. The straight gauges are parametrized by a single function $F(y)$, that obeys the condition (5.74) but is otherwise arbitrary.

The greatest practical importance of our work is in applications to more complicated models and to more subtle issues. Since we start with a well-defined 5d generally covariant action, and gauge-fix it explicitly to an effective 4d action, there can be no arguments about the counting of physical degrees of freedom, the identification of kinetic ghosts, or the onset of strong coupling behavior (to the extent that such behavior can be accessed starting from a linearized theory). We intend to exploit these advantages in future work.

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Tensor decomposition

A massive symmetric tensor field $T_{\mu\nu}$ in flat 4d spacetime has the decomposition

$$T_{\mu\nu} = b_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu + \partial_\mu\partial_\nu\phi_1 + \eta_{\mu\nu}\phi_2\,, \tag{1}$$

where

$$b \equiv \eta^{\mu\nu} b_{\mu\nu} = 0\,, \quad \partial^\mu b_{\mu\nu} = 0\,, \tag{2}$$

$$\partial^\mu V_\mu = 0\,. \tag{3}$$

(2) provides $4+1$ conditions, and then $b_{\mu\nu}$ has only $10-5=5$ DOF. Similarly, V_μ has $4-1=3$ DOF due to 1 condition imposed by (3). Obviously, ϕ_1 and ϕ_2 have one DOF each.

When the 4d fields are massless, *i.e.*, $\partial^2(\text{fields}) = 0$, both $\partial_\mu V_\nu + \partial_\nu V_\mu$ and $\partial_\mu\partial_\nu\phi_1$ become transverse-traceless, and a simple decomposition like (1) breaks down. Let's derive the correct decomposition in the massless case.

First of all, we know that 4 out of 10 DOF of $T_{\mu\nu}$ should be expressed in a pure gauge form, $\partial_\mu V_\nu + \partial_\nu V_\mu$. The vector V_μ should have three transverse components, one of which can be written as the gradient of a massless scalar $\partial_\mu \varphi_1$. Let the transverse vector v_μ denote the other two transverse DOF, and n_μ denote the longitudinal component, so:

$$V_\mu = v_\mu + \partial_\mu \varphi_1 + n_\mu \,, \tag{4}$$

with $\partial_\mu v^\mu = 0$ and $\partial_\mu n^\mu \neq 0$.

The DOF of any symmetric tensor can be divided into the following:

$\mathcal{B}_{\mu\nu}$: transverse-traceless,

$\mathcal{C}_{\mu\nu}$: traceless but not transverse,

$\mathcal{D}_{\mu\nu}$: trace piece, which we can take to be proportional to $\eta_{\mu\nu}$.

Obviously $\mathcal{B}$ has $10 - 5 = 5$ DOF. We have already exhibited 3 of them; $\partial_\mu v_\nu + \partial_\nu v_\mu$ and $\partial_\mu \partial_\nu \varphi_1$. Therefore,

$$\mathcal{B}_{\mu\nu} = \beta_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu + \partial_\mu \partial_\nu \varphi_1 \,, \tag{5}$$

where $\beta_{\mu\nu}$ is a traceless-transverse tensor with 2 DOF.

The sum of the DOF of $\mathcal{B}$ and $\mathcal{C}$ is 9, so $\mathcal{C}$ has 4 DOF. One of these is the pure gauge DOF n_μ; we can write:

$$\mathcal{C}_{\mu\nu} = c_{\mu\nu} + \partial_\mu n_\nu + \partial_\nu n_\mu - \frac{1}{2}\eta_{\mu\nu}\partial_\rho n^\rho \,, \tag{6}$$

where $c_{\mu\nu}$ is a traceless but not transverse tensor with 3 DOF.

Collecting the pieces, we get the decomposition of a massless tensor:

$$T_{\mu\nu} = \beta_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu + \partial_\mu \partial_\nu \varphi_1 + c_{\mu\nu} + \partial_\mu n_\nu + \partial_\nu n_\mu + \eta_{\mu\nu}\varphi_2 \,. \tag{7}$$

Let's look at the massless decomposition in momentum space, *i.e.*, consider the decomposition of $\bar{T}_{\mu\nu}(p) = \int d^4x T_{\mu\nu} e^{ip\cdot x}$, with $p^2 = 0$. When p^μ is null, it is not possible to find three vectors which are mutually orthogonal and transverse to p^μ. Instead, we introduce the following explicit basis:

$$\begin{aligned}
&\epsilon^{(1)}_\mu \ : \ \text{parallel to } p_\mu (\text{helicity } +1), \ \ \epsilon^{(1)\mu}\epsilon^{(1)}_\mu = 0 \,; \\
&\epsilon^{(2)}_\mu \ : \ \text{antiparallel to } p_\mu (\text{helicity } -1), \ \ \epsilon^{(2)\mu}\epsilon^{(2)}_\mu = 0 \,, \ \epsilon^{(1)\mu}\epsilon^{(2)}_\mu \neq 0 \,; \\
&\epsilon^{(j)}_\mu \ (j = 3,\ 4) \ : \ \epsilon^{(1)\mu}\epsilon^{(j)}_\mu = \epsilon^{(2)\mu}\epsilon^{(j)}_\mu = 0 \,, \ \ \epsilon^{(j)\mu}\epsilon^{(k)}_\mu = \delta_{jk} \,,
\end{aligned} \tag{8}$$

from which we can build bases for second rank symmetric tensors:

$$
\begin{aligned}
&\varepsilon^{(1)}_{\mu\nu} = \epsilon^{(3)}_{(\mu}\epsilon^{(4)}_{\nu)}\,, \quad \varepsilon^{(2)}_{\mu\nu} = \epsilon^{(3)}_{\mu}\epsilon^{(3)}_{\nu} - \epsilon^{(4)}_{\mu}\epsilon^{(4)}_{\nu}\,,\\
&\varepsilon^{(3)}_{\mu\nu} = \epsilon^{(1)}_{(\mu}\epsilon^{(3)}_{\nu)}\,, \quad \varepsilon^{(4)}_{\mu\nu} = \epsilon^{(1)}_{(\mu}\epsilon^{(4)}_{\nu)}\,,\\
&\varepsilon^{(5)}_{\mu\nu} = \epsilon^{(1)}_{\mu}\epsilon^{(1)}_{\nu}\,,\\
&\varepsilon^{(6)}_{\mu\nu} = \epsilon^{(1)}_{(\mu}\epsilon^{(2)}_{\nu)}\,,\\
&\varepsilon^{(7)}_{\mu\nu} = \epsilon^{(2)}_{\mu}\epsilon^{(2)}_{\nu}\,, \quad \varepsilon^{(8)}_{\mu\nu} = \epsilon^{(2)}_{(\mu}\epsilon^{(3)}_{\nu)}\,, \quad \varepsilon^{(9)}_{\mu\nu} = \epsilon^{(2)}_{(\mu}\epsilon^{(4)}_{\nu)}\,,\\
&\varepsilon^{(10)}_{\mu\nu} = -\epsilon^{(1)}_{(\mu}\epsilon^{(2)}_{\nu)} + \epsilon^{(3)}_{\mu}\epsilon^{(3)}_{\nu} + \epsilon^{(4)}_{\mu}\epsilon^{(4)}_{\nu}\,.
\end{aligned}
$$

With (8), we can read off characteristics of each basis component:

$\varepsilon^{(1-2)}$: traceless-transverse, and transverse to $\epsilon^{(2)}_{\mu}$,

$\varepsilon^{(3-5)}$: traceless-transverse,

$\varepsilon^{(6)}$: neither traceless nor transverse,

$\varepsilon^{(7-9)}$: traceless but not transverse, and transverse to $\epsilon^{(2)}_{\mu}$,

$\varepsilon^{(10)} \propto \eta_{\mu\nu}$.

Then using the fact that $\epsilon^{(1)}_{\mu}$ is parallel to p_μ, we can get the decomposition

$$
\bar{T}_{\mu\nu} = \bar{\beta}_{\mu\nu} + ip_\mu\bar{v}_\nu + ip_\nu\bar{v}_\mu - p_\mu p_\nu\bar{\varphi}_1 + ip_\mu\bar{n}_\nu + ip_\nu\bar{n}_\mu + \bar{c}_{\mu\nu} + \eta_{\mu\nu}\bar{\varphi}_2\,, \tag{9}
$$

where $\bar{n}_\mu$ is proportional to $\epsilon^{(2)}_{\mu}$, and

$\bar{\beta}_{\mu\nu}$ ($\equiv \varepsilon^{(1-2)}$): traceless-transverse, and transverse to $\bar{n}_\mu$, 2 DOF;

$p_\mu\bar{v}_\nu + p_\nu\bar{v}_\mu$ ($\equiv \varepsilon^{(3-4)}$): traceless-transverse, where $\bar{v}_\mu$ is transverse to p^μ and to $\bar{n}_\mu$, 2 DOF;

$p_\mu p_\nu\bar{\varphi}_1$ ($\equiv \varepsilon^{(5)}$): traceless-transverse, 1 DOF;

$p_\mu\bar{n}_\nu + p_\nu\bar{n}_\mu$ ($\equiv \varepsilon^{(6)}$): 1 DOF. $p_\mu\bar{n}^\mu \neq 0$;

$\bar{c}_{\mu\nu}$ ($\equiv \varepsilon^{(7-9)}$): traceless but not transverse, and transverse to $\bar{n}_\mu$, 3 DOF;

$\eta_{\mu\nu}\bar{\varphi}_2$ ($\equiv \varepsilon^{(10)}$): 1 DOF.

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