Second Order Geodesic Corrections to Cosmic Shear

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We consider the impact of second order corrections to the geodesic equation governing gravitational lensing. We start from the full second order metric, including scalar, vector and tensor perturbations, and retain all relevant contributions to the cosmic shear corrections that are second order in the gravitational potential. The relevant terms are: the nonlinear evolution of the scalar gravitational potential, the Born correction, and lens-lens coupling. No other second order terms contribute appreciably to the lensing signal. Since ray-tracing algorithims currently include these three effects, this derivation serves as rigorous justification for the numerical predictions.

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I. INTRODUCTION

Gravitational lensing of background galaxies by large scale structure offers an excellent way to study the distribution of matter in the universe [1, 2, 3, 4]. Measurements of the cosmic shear are already enabling us to constrain the dark matter abundance [5, 6]. In the future, large surveys may well uncover properties of dark energy, such as its abundance and equation of state [7, 8, 9, 10] and of neutrinos [11]. This program will be successful only if we can make very accurate predictions about what theories predict [12].

Predicting the cosmic shear signal is more difficult than making predictions for the cosmic microwave background but much more straightforward than for the galaxy distribution. The former rely predominantly on linear perturbation theory so are extremely robust. The latter require not only nonlinear evolution but also an understanding of how the galaxy distribution is related to the mass. For lensing, we need to account for nonlinearities, i.e. move beyond linear perturbation theory, but, since the light trajectories depend only on the mass, we do not need to worry about how and where galaxies form[1].

The obvious way to treat nonlinearities is to perform numerical simulations of structure formation. Since dark matter is much more abundant than baryonic gas, only N-Body simulations are needed, not[2] the much more computational costly hydrodynamical simulations. A number of groups have run such simulations and extracted predictions by *ray-tracing* photon paths through the simulated universe [15, 16]. In principle, this proceedure accounts for some second order effects, such as the second order perturbations to the potential and the fact that the path of the photons is not simply a straight line. However, the ray-tracing algorithims use the first-order geodesic equation to find the distortions in each slice of the universe. Therefore, again in principle, the algorithms may be missing relevant terms. A fully consistent second order calculation is warranted.

In this paper, we start from the second order metric, proceed to obtain the second order geodesic equation, and then compute the cosmic shear consistently retaining all second order terms. The rest of the paper is organized as follows: §II reviews the first order calculation, and §III presents an order of magnitude estimate that will prove useful when wading through a host of second order terms. §IV presents the full second order geodesic equation, borrowing heavily from previous work. In it, we winnow out the terms that contribute negligibly, keeping only those terms that are responsible for observable deflections.

II. FIRST ORDER SHEAR

Consider the first order deflection of light due to inhomogeneities along the line of sight. We work in a flat universe; then the geodesic equation may be integrated to give

$$x_\perp^{(1)i}(\vec{\theta}) = \int_0^{\chi_s} d\chi\,(\chi_s - \chi) f_\perp^{(1)i}(\chi, \vec{\theta}). \tag{1}$$

[1] We refer here to cosmic shear. Other lensing phenomena, such as galaxy-galaxy lensing, do involve the interplay of light and mass.
[2] This ceases to be true on very small scales where the effects of baryons can no longer be neglected [13, 14].

Here $x_\perp$ is the perpendicular deflection of a ray starting at comoving distance away χ_s and traveling towards us at $\chi = 0$ detected at angular position $\vec{\theta}$ with respect to a fixed z-axis. The first superscript on $x_\perp$ denotes the order of the perturbation, the second is a space-time index; we are interested only in the two components perpendicular to the line of sight. The first order distortion is

$$f_\perp^{(1)i}(\chi, \vec{\theta}) = -\Gamma^{(1)i}_{\alpha\beta} p^{(0)\alpha} p^{(0)\beta}. \tag{2}$$

Here $\Gamma^{(1)}$ is the first order Christoffel symbol and the zero-order space-time vector $p^{(0)\alpha} = (1, -e^i) \simeq (1, -\theta^i, -1)$, where e^i is the spatial direction vector. The square of $\hat{e}$ is unity, and θ^i is the transverse displacement from the z-axis. We work in the small angle approximation throughout, so θ is assumed small, and the z-component of $\hat{e}$ is approximately unity.

In the contraction of the Christoffel symbol with the two factors of p, the only terms which contribute are those with $\alpha, \beta = 0$ or 3. Using standard results [4], we can write the contraction as

$$\Gamma^{(1)i}_{\alpha\beta} p^{(0)\alpha} p^{(0)\beta} = 2 \frac{\partial \phi^{(1)}(\vec{x}^{(0)}(\theta, \chi))}{\partial x^i} \tag{3}$$

where $\phi^{(1)}$ is the first-order gravitational potential evaluated at the unperturbed[3] position of the light: $\vec{x}^{(0)}(\theta, \chi) = \chi[\vec{\theta}, 1]$. So,

$$x_\perp^{(1)i} = -2 \int_0^{\chi_s} d\chi \, (\chi_s - \chi) \frac{\partial \phi^{(1)}(\vec{x}^{(0)}(\theta, \chi))}{\partial x^i} \tag{4}$$

The shear matrix is defined as the derivative of this perpendicular deflection with respect to observed angle $\vec{\theta}$:

$$\psi_{ij}(\vec{\theta}, \chi_s) \equiv \frac{1}{\chi_s} \frac{\partial x_\perp^i}{\partial \theta^j}. \tag{5}$$

To lowest order of course $x_\perp^{(0)i} = \chi_s \theta^i$, so the zeroth order term in the shear matrix is simply the identity. From Eq. (4), we see that the first order term in the shear matrix is

$$\begin{aligned} \psi_{ij}^{(1)}(\vec{\theta}, \chi_s) &= -2 \int_0^{\chi_s} d\chi \, \frac{\chi_s - \chi}{\chi_s} \frac{\partial^2 \phi^{(1)}(\vec{x}^{(0)}(\theta, \chi))}{\partial x^i \partial \theta^j} \\ &= -2 \int_0^{\chi_s} d\chi \, \chi \frac{\chi_s - \chi}{\chi_s} \frac{\partial^2 \phi^{(1)}(\vec{x}^{(0)}(\theta, \chi))}{\partial x^i \partial x^j}. \end{aligned} \tag{6}$$

Again the second equality here follows because the potential is evaluated along the unperturbed path of the light.

III. ORDER OF MAGNITUDE ESTIMATE

In order to wade through the second order terms and extract the most relevant ones, we need a way of doing order of magnitude estimates. From Eq. (6), the shear induced by a perturbation with wavelength λ is of order $(r_H/\lambda)^2 \phi$ since cosmological distances χ are of order the Hubble radius, $r_H = 3000\mathrm{h}^{-1}$ Mpc. The mean of the shear of course vanishes, but its rms we would expect to be of order $(r_H/\lambda_{\max})^2 \phi_{\mathrm{rms}}$, where $\lambda_{\max}$ is the wavelength near which perturbations contribute the most to the deflections. In currently popular models, $\lambda_{\max} \sim 30\mathrm{h}^{-1}$ Mpc. The rms amplitude of the gravitational potential is of order 10^{-5}, so one would naively expect the rms shear to be about ten percent. The rms is actually smaller than this because not all Fourier modes contribute to the variance. Perturbations which vary rapidly in the z-direction lead to little total distortion since regions of positive and negative overdensity along the line of sight cancel each other. Thus, in the 3D Fourier space only modes with very small k_3 ($\leq r_H^{-1}$) contribute. The fraction of modes which satisfy this constraint is (λ/r_H) for a given λ [1, 4]; the variance is therefore smaller than the naive estimate by this fraction, and the rms by its square root. The rms amplitude is therefore of order $(r_H/\lambda_{\max})^{3/2} \phi_{\mathrm{rms}}$, less than a percent.

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[3] Evaluating it at the perturbed position leads to a second order term, and for now we are considering only first order terms.

There are two lessons we learn from this order of magnitude estimate which will be useful when we evaluate second order terms. First, we need consider only those terms in $f^{(2)}$ which vary little along the line of sight. Terms with partial derivatives with respect to $z \equiv x^3$ can be neglected. Second, the shear is much larger than the depth of the typical potential well, $\phi_{\rm rms}$. It is the rapid changes in the potential which lead to large deflection; i.e., $(\partial\phi/\partial x^i) r_H \gg \phi$ (as long the derivatives are in one of the transverse directions, $i = 1, 2$). When considering second order terms, therefore, we will be most impressed by those with the most derivatives.

Armed with this information, let's return to Eq. (3) for $f_\perp^{(1)i}$ and consider the correction incurred when we account for the fact that the photon does not travel along a straight line so the argument of ϕ might reasonably be taken as $\vec{x}^{(0)} + \vec{x}^{(1)}$. Expanding about the zero order path then leads to one possible second order term

$$\frac{\partial^2\phi}{\partial x_i \partial x_j}\Big|_{x=x^{(0)}} x^{(1)j} \tag{7}$$

This term is known as the *Born correction*[17, 18]. We will soon see that it emerges as one of many second order terms. To derive it here, we have cheated since the argument of ϕ in Eq. (3) is the *zero order* path. We introduce this term now only because we know it will show up in the second order zoo, and we want to estimate its order of magnitude. From Eq. (4), $x^{(1)}$ is of order $r_H^2 \partial\phi$, so the Born correction is of order $(\partial^2\phi) r_H^2 \partial\phi$. Thus, we need keep second order terms in $f_\perp^{(2)}$ only if they are of order $r_H^2 \partial^3 \phi^2$.

IV. SECOND ORDER GEODESIC CORRECTIONS

The perpendicular deflection to second order is given by the analogue of Eq. (1), with the superscripts changed from $^{(1)}$ to $^{(2)}$, and [19]

$$\begin{aligned} f_\perp^{(2)i} &= -\Gamma^{(0)i}{}_{\alpha\beta} p^{(1)\alpha} p^{(1)\beta} - 2\Gamma^{(1)i}{}_{\alpha\beta} p^{(0)\alpha} p^{(1)\beta} - 2\partial_\sigma \Gamma^{(0)i}{}_{\alpha\beta} x^{(1)\sigma} p^{(0)\alpha} p^{(1)\beta} \\ &- \partial_\sigma \Gamma^{(1)i}{}_{\alpha\beta} x^{(1)\sigma} p^{(0)\alpha} p^{(0)\beta} - \frac{1}{2}\partial_\sigma \partial_\tau \Gamma^{(0)i}{}_{\alpha\beta} x^{(1)\sigma} x^{(1)\tau} p^{(0)\alpha} p^{(0)\beta} - \Gamma^{(2)i}{}_{\alpha\beta} p^{(0)\alpha} p^{(0)\beta}. \end{aligned} \tag{8}$$

The zero order and first order Christoffel symbols are well-known [4, 20], while the second order $\Gamma^{(2)}$ has been computed by Bartolo et al. [21]. The zero order direction vector $p^{(0)}$ is given after Eq. (2), while the time and space components of the first order direction vector are [19]

$$\begin{aligned} p^{(1)0}(\chi) &= p^{(1)0}(\chi = 0) - \int_0^\chi f^{(1)0}(\chi') d\chi' \ , \\ p^{(1)i}(\chi) &= -\int_0^\chi f^{(1)i}(\chi') d\chi' \end{aligned} \tag{9}$$

with $f^{(1)\alpha} = -\Gamma^{(1)\alpha}{}_{\mu\nu} p^{(0)\mu} p^{(0)\nu}$. To compute the first and second order Christoffel symbols, we need to specify the metric, thereby choosing a gauge. Following the formalism of [21], we set

$$\begin{aligned} g_{00} &= -a^2 \left[1 + 2\phi^{(1)} + \phi^{(2)} \right] \\ g_{0i} &= a^2 \left[\frac{1}{2}\partial_i \omega^{(2)} + \frac{1}{2}\omega_i^{(2)} \right] \\ g_{ij} &= a^2 \Big[\left(1 - 2\psi^{(1)} - \psi^{(2)} \right) \delta_{ij} + \frac{1}{2} D_{ij} \chi^{(2)} + \frac{1}{2} \left(\partial_i \chi_j^{(2)} + \partial_j \chi_i^{(2)} + \chi_{ij}^{(2)} \right) \Big]. \end{aligned} \tag{10}$$

Scalar perturbations are described by ϕ, ψ, χ and ω; vector perturbations by $\omega_i^{(2)}$ and $\chi_i^{(2)}$; and tensors by $\chi_{ij}^{(2)}$. Finally

$$D_{ij} \equiv \partial_i \partial_j - \frac{\delta_{ij}}{3} \nabla^2. \tag{11}$$

Note that we assume that there are no first order vector or tensor perturbations. Even without these, scalar, vectors, and tensors mix at second order, so $\chi_{ij}^{(2)}$ and $\omega_i^{(2)}$ and $\chi_i^{(2)}$ are generally non-zero, quadratic in first order scalar perturbations.

We now work through the terms in Eq. (8) explicitly, retaining only those of order $r_H^2 \partial^3 \phi^2$ or higher.

- First Term: $-\Gamma^{(0)i}{}_{\alpha\beta}p^{(1)\alpha}p^{(1)\beta}$

  The zero order Christoffel symbol is proportional to the Hubble rate and is non-zero only when one of the lower indices is equal to i and the other equal to zero. Thus this first term reduces to

$$-\Gamma^{(0)i}{}_{\alpha\beta}p^{(1)\alpha}p^{(1)\beta} = -2r_H^{-1}p^{(1)0}p^{(1)i}. \tag{12}$$

  Forgetting about the boundary term in the first of Eq. (9), and remembering that $\chi \sim r_H$, we see that $p^{(1)0} \sim r_H f^{(1)0}$ and $p^{(1)i} \sim r_H f^{(1)i}$. So this first term in $f^{(2)i}$ is of order $r_H f^{(1)0} f^{(1)i}$. We showed in §II that $f^{(1)i} = 2\phi_{,i}$ so this first term is of order $r_H \partial\phi f^{(1)0}$; it contributes appreciably only if $f^{(1)0}$ is of order $r_H\partial^2\phi$ or higher. But

$$f^{(1)0} = -\Gamma^{(1)0}_{\alpha\beta}p^{(0)\alpha}p^{(0)\beta}. \tag{13}$$

  Recall that $p^{(0)\alpha}$ is negligible unless $\alpha = 0$ or 3. If both α and β are zero, then the Christoffel symbol is $\dot{\phi} \sim r_H^{-1}\phi \ll r_H\partial^2\phi$. If both indices are equal to 3, the Christoffel symbol again is of order ϕ/r_H, i.e. negligible. If one of the indices is zero and the other equal to three, the Christoffel symbol is equal to $\partial_3\phi$, which is negligible for all perturbations of interest, i.e. for all perturbations which vary little along the line of sight. Thus this first term does not contribute.

- Second Term: $-2\Gamma^{(1)i}_{\alpha\beta}p^{(0)\alpha}p^{(1)\beta}$

  Since only the $\alpha = 0$ or 3 components of $p^{(0)}$ are non-negligible,

$$-2\Gamma^{(1)i}_{\alpha\beta}p^{(0)\alpha}p^{(1)\beta} \to -2\Gamma^{(1)i}_{0\beta}p^{(1)\beta} + 2\Gamma^{(1)i}_{3\beta}p^{(1)\beta} \tag{14}$$

  Recall from the previous paragraph that

$$\begin{aligned} p^{(1)0} &\sim \phi \\ p^{(1)i} &\sim r_H\partial_i\phi \end{aligned} \tag{15}$$

  So the biggest contribution will come from $\beta = j$, with $j = 1$ or 2, one of the transverse directions. The contribution is of order $r_H\partial\phi\Gamma$. But $\Gamma^{(1)i}_{0j} = -\dot{\psi}\delta_{ij} \sim r_H^{-1}\phi$, so the first term on the right in Eq. (14) is of order $\partial\phi^2$ and can be neglected. The second term in Eq. (14), $2\Gamma^{(1)i}_{3j}p^{(1)j}$, is of order $r_H\partial\phi\Gamma^{(1)i}_{3j}$; recall [4] that this Christoffel symbol sets rotating pairs of its indices equal to each other, with the other index applying to the derivative of the potential. We certainly then do not want the index $_3$ to be the derivative since ∂_3 is very small. But both i and j are transverse indices so cannot be equal to 3: all terms here vanish.

- Third Term: $-2\partial_\sigma\Gamma^{(0)i}{}_{\alpha\beta}x^{(1)\sigma}p^{(0)\alpha}p^{(1)\beta}$

  After invoking some of the approximations used in the previous paragraphs, we get

$$-2\partial_\sigma\Gamma^{(0)i}{}_{\alpha\beta}x^{(1)\sigma}p^{(0)\alpha}p^{(1)\beta} \to -2\partial_\sigma\Gamma^{(0)i}{}_{0j}x^{(1)\sigma}p^{(1)j}. \tag{16}$$

  The derivative here acts only on the Christoffel symbol, which depends only on time. Thus $\sigma = 0$, and $\partial\Gamma \sim r_H^{-2}$. The first order perturbation in $x^{(1)0}$ is of order $r_H\phi$, so this term is of order $\partial\phi^2$, far too small to contribute.

- Fourth Term: $-\partial_\sigma\Gamma^{(1)i}{}_{\alpha\beta}x^{(1)\sigma}p^{(0)\alpha}p^{(0)\beta}$

  This is the only term in Eq. (8) quadratic in the first order variables (i.e., the only one of the first five terms) which contributes. It gives the Born correction and lens-lens coupling, a very nice result emerging naturally from this second order formalism. To see this, note that the only terms in the implicit sum which contribute are those with $\alpha = \beta = 0$ or $\alpha = \beta = 3$. Both of these contribute identically, so this term is

$$-2\left[\partial_\sigma(\partial_i\phi)\right]x^{(1)\sigma} = -4\frac{\partial^2\phi}{\partial x^i\partial x^j}\int_0^\chi d\chi'(\chi-\chi')\frac{\partial\phi}{\partial x^j}. \tag{17}$$

- Fifth Term: $-(1/2)\partial_\sigma\partial_\tau\Gamma^{(0)i}{}_{\alpha\beta}x^{(1)\sigma}x^{(1)\tau}p^{(0)\alpha}p^{(0)\beta}$

  This term vanishes since the zero order direction vectors are non-negligible only if the indices α and β are set to 0 or 3, and $\Gamma^{(0)i}{}_{00} = \Gamma^{(0)i}{}_{03} = \Gamma^{(0)i}{}_{33} = 0$.

- Sixth Term: $-\Gamma^{(2)i}{}_{\alpha\beta}p^{(0)\alpha}p^{(0)\beta}$

  The only remaining term is the one explicitly second order in Γ. Before delving into this term, we can localize it further by noticing that the indices α and β must be either 0 or 3. Thus, this last term is

$$-\left[\Gamma^{(2)i}{}_{00}+\Gamma^{(2)i}{}_{33}-2\Gamma^{(2)i}{}_{03}\right]. \tag{18}$$

  To compute $\Gamma^{(2)}$, we lift the results from the Appendix of Ref. [21]. There, they compute the Christoffel symbols to second order. There are a number of simplifications we can make. First, we work in a gauge (the so-called Poisson gauge) in which the scalars $\omega^{(2)}$ and $\chi^{(2)}$ as well as the vector $\chi_i^{(2)}$ are set to zero, so vector perturbations are described only by $\omega^{(2)i}$, henceforth simply called ω. Similarly, we assume there are no tensor perturbations at first order, and the second order perturbations are described solely by the traceless and divergenceless tensor χ_{ij} [spatial indices are raised and lowered with the Euclidean metric, so we take no care in distinguishing upper from lower indices]. Within this framework,

$$\Gamma^{(2)i}_{00}+\Gamma^{(2)i}_{33}-2\Gamma^{(2)i}_{03}=\frac{1}{2}\partial^i\left[\phi^{(2)}+\psi^{(2)}\right]+4\phi^{(1)}\partial^i\left[\phi^{(1)}\right]+\frac{1}{2}\left[\omega^{i\prime}-\partial_3\omega^i+\partial_i\omega^3\right]+\frac{1}{4}\left[2\partial_3\chi_{i3}-\partial_i\chi_{33}-2\chi'_{i3}\right]. \tag{19}$$

  Here prime denotes differentiation with respect to conformal time.

  To weigh the relative importance of each of these terms, it is important to remember that the only terms that contribute are those which are of order $r_H^2\partial^3\phi^2$. This results in both vector and tensor modes being irrelevant. Let's work through the vector case explicitly; the tensor case is similar. The vector modes have been worked out in Ref. [22]. Eqs. (17-19) there give explicit expressions for the second order vector perturbations (their $V=-\omega/2$). By counting powers of $k\to\partial$, we see that $\omega^i\sim r_H\partial\phi^2$. Even if we forget about the constraint that ∂_3 must be small, this means that the second line is at most of order $r_H\partial^2\phi^2$, too small by a factor of order $r_H/\lambda_{\max}$ to contribute. A similar argument applied to the expression for tensor modes in Ref. [22] leads to a similar conclusion. Thus, the only relevant explicit second order terms are from scalar perturbations, those on the first line of Eq. (19). Using the same order of magnitude estimate, we can immediately eliminate the second scalar term on the first line, since it is of order $\partial\phi^2$, clearly too small to be relevant.

  The only relevant explicit second order terms therefore are

$$\Gamma^{(2)i}_{00}+\Gamma^{(2)i}_{33}-2\Gamma^{(2)i}_{03}=\frac{1}{2}\partial^i\left[\phi^{(2)}+\psi^{(2)}\right]\to\phi^{(2)}_{,i}. \tag{20}$$

  The final limit comes from recognizing that anisotropic stress, inducing a $\psi-\phi\neq 0$ term, is a post-Newtonian second-order contribution; this can only be relevant either on very large scales or on very small highly non-linear scales [23], so $\psi=\phi$ to a good approximation for the range of wavelenghts of interest here.

Collecting second order terms and adding to the first order term, we emerge with an expression for the perpendicular deflection accurate to second order in the gravitational potential:

$$\begin{aligned} x^i_\perp(\vec\theta) \;=\; -2\int_0^{\chi_s}d\chi(\chi_s-\chi)\Bigg[&\frac{\partial}{\partial x^i}\Big(\phi^{(1)}\big(\vec x^{(0)}(\chi,\vec\theta)\big)+\frac{1}{2}\phi^{(2)}\big(\vec x^{(0)}(\chi,\vec\theta)\big)\Big)\\ &+2\frac{\partial^2\phi^{(1)}\big(\vec x^{(0)}(\chi,\vec\theta)\big)}{\partial x^i\partial x^k}\int_0^\chi d\chi'(\chi-\chi')\frac{\partial\phi^{(1)}\big(\vec x^{(0)}(\chi',\vec\theta)\big)}{\partial x^k}\Bigg]. \end{aligned} \tag{21}$$

Note that the first set of terms on the right $\phi^{(1)}+\phi^{(2)}/2$ is simply the full nonlinear potential out to second order; i.e., the metric (Eq. (10)) contains this combination. So we might expect the full deflection to be sensitive to the fully nonlinear gravitational potential. Indeed this is the assumption built into all previous work. Our second order treatment justifies this assumption. Also note that the second term is indeed the Born correction alluded to in Eq. (7) since twice the inner integral is equal to $x^{(1)}$. In fact, the so-called lens-lens correction is also included in this term: when differentiating $x^i_\perp$ to get the shear, the derivative acting on ϕ'' here gives what is usually called the Born correction, while the derivative acting on the ϕ' term inside the inner integral gives the lens-lens correction. We can rewrite the perpendicular deflection as

$$x^i_\perp(\vec\theta)=-2\int_0^{\chi_s}d\chi(\chi_s-\chi)\frac{\partial}{\partial x^i}\phi\big(\vec x(\chi,\vec\theta)\big) \tag{22}$$

where ϕ and $\vec x$ now include all orders in perturbations: $\phi=\phi^{(1)}+(1/2)\phi^{(2)}+\ldots$ and similarly for $\vec x$. Eq. (22) is the usual starting point for ray-tracing simulations. We have now rigorously justified the standard ray-tracing approach, at least to second order in geodesic corrections.

V. CONCLUSIONS

We have jusitifed the standard ray tracing formula of gravitational lensing. This argument connects much of the formal work on second order perturbations in cosmology with the more phenomenological approach used in lensing studies. A similar result holds for the luminosity distance: the first order expression for the luminosity distance [24] can be used as the starting point for the second order result, without going back to the geodesic equation.

Our primary result, Eq. (22), holds only on sub-horizon scales. On very large scales, many of the corrections we were able to neglect are of the same order as the Born correction and the nonlinear evolution of the potential. However, on large scales, we expect all such corrections to be small, of order 10^{-5}, since they are not enhanced by spatial variations. Therefore, our results are valid in the regime in which corrections are measurable. The one possible caveat to this claim is if super-horizon perturbations can influence the local observables[25].

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