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DIFFUSION DUE TO BEAM-BEAM RESONANCES IN HADRON COLLIDERS*

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Abstract

Beam-beam tune spread in hadron colliders usually is small enough to avoid most dangerous low-order betatron resonances. However, even weak high-order resonances can be detrimental due to cooperative effect of the external noise. Mechanisms of such cooperation are considered, simple analytical estimates of the diffusion rate being verified with numerical simulations. The developed theory is used to evaluate the beam-beam resonance contribution to the emittance growth in the Tevatron.

INTRODUCTION

Experience with the Tevatron collider shows that the beam-beam interaction affects the lifetime of both proton and antiproton beams and under certain conditions may cause the antiproton beam emittance blow-up [1].

The classical mechanism of dynamical stochasticity due to resonance overlap [2] (including the overlap of the synchrotron sidebands for particles with large synchrotron amplitudes [3]) can not explain the effect of high order (and correspondingly weak) resonances since their width is not sufficient to overlap.

There are external sources of noise (ground motion, microphonics due to helium flow etc.), as well as scattering processes (residual gas, IBS) which can provide particle transport between the resonance islands (Fig. 1).

Since the motion inside the resonance islands is much faster than the characteristic time of the *extrinsic* diffusion, for a crude estimate of the average diffusion rate the area of these islands can be simply excluded from the available phase space, leading to *a multiplicative diffusion enhancement* [4]. A refined theory of such enhancement was presented in Ref. [5]. However, since the resonance width is relatively small this effect is not very significant.

In the discussion of the resonance overlap it was tacitly assumed that the betatron oscillations keep their phase long enough for the synchrotron sidebands to develop, which may be not true in the presence of sufficiently strong external noise.

Since the synchrotron tune in hadron colliders is low, the external noise can destroy the betatron phase correlation between subsequent resonance crossings. In the result all phase space area of betatron oscillations which is swept by the resonance in the course of synchrotron oscillations becomes the region of enhanced diffusion.

In the present report analytical estimates are given of the diffusion rate due to *multiple crossing* of an isolated resonance [6] in different regimes. But before that, the dynamic diffusion across the immobile resonance islands is studied and shown to substantially exceed the naive estimates.

Finally, the obtained formulas for the diffusion coefficient are used to compute emittance growth in the colliding beams.

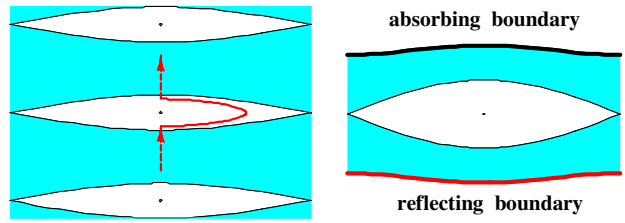

Figure 1. Schematic picture of the diffusion enhancement by immobile resonances (left) and configuration for the numerical experiment (right)

BASIC RELATIONS

Assuming the synchrotron tune Q_s to be low we may consider the longitudinal coordinate z and relative momentum deviation δ_p as slow varying parameters. Now let a particle with action variable values I_{x0}, I_{y0}, cross a 2D betatron resonance at some value of δ_p:

$$\underline{m}\cdot\underline{Q} \equiv \underline{m}\cdot(\underline{Q}_0 + \underline{C}\,\delta_p) = n, \tag{1}$$

$\underline{Q}_0 = \{Q_{x0}, Q_{y0}\}$, $\underline{C} = \{C_x, C_y\}$ being vectors of on-momentum tunes and chromaticities, $\underline{m}$ being the resonance vector.

Introducing new variables

$$I_1 = \frac{m_x\Delta I_x + m_y\Delta I_y}{m_x^2+m_y^2},\ I_2 = \frac{m_y\Delta I_x - m_x\Delta I_y}{m_x^2+m_y^2} = \text{const}, \tag{2}$$

$$\psi_1 = m_x\psi_x + m_y\psi_y - n\theta,\quad \psi_2 = m_y\psi_x - m_x\psi_y,$$

where $\theta = s/R$ is generalized azimuth and $\Delta I_{x,y} = I_{x,y} - I_{x,y0}$ we may write for the Hamiltonian

$$H = \frac{1}{2}\Lambda_{\underline{m}} I_1^2 + R_{\underline{m}}\cos(\psi_1 - \psi_{10}) \tag{3}$$

where

$$\Lambda_{\underline{m}} = m_x^2\frac{\partial Q_{x0}}{\partial I_x} + 2m_x m_y\frac{\partial Q_{x0}}{\partial I_y} + m_y^2\frac{\partial Q_{y0}}{\partial I_y} \tag{4}$$

and R_m is the resonance driving term (RDT) which may depend on the instant value of z due to finite crossing angle and the betatron phase variation over the interaction length and on δ_p due to finite dispesion.

For head-on colliding round beams RDT can be factorized [7],

$$R_m = T_m(I_x, I_y)\cdot L_m(z), \tag{5}$$

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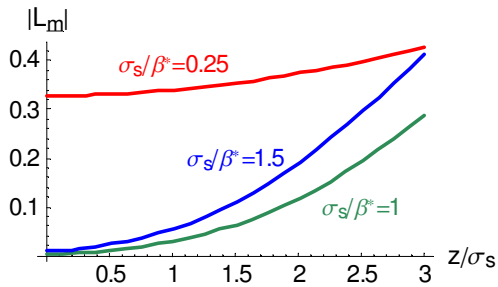

Figure 2. Longitudinal factor for 12th order resonances in head-on colliding round beams.

$L_m(z)\equiv 1$ for short bunches ($\sigma_s/\beta^* \ll 1$). Fig.2 shows that the longitudinal factor for long bunches can be very small suppressing the resonance excitation.

DIFFUSION OVER RESONANCE ISLAND

First let us consider on-momentum particles, $\delta_p=0$. If the tune of small-amplitude libration around the stable fixed point (*the island tune*) is low

$$\nu_{isl} = |R_{\underline{m}}\Lambda_{\underline{m}}|^{1/2} \ll 1 \tag{6}$$

the action variable increment per turn is $\delta I_1 = 2\pi |R_m| \cos\psi_1$. Assuming action to change at this pace for $n_{\rm corr}$ turns we get for the diffusion rate

$$D_{\rm naive} = \frac{1}{2n_{\rm corr}}\left\langle (n_{\rm corr}\delta I_1)^2 \right\rangle = \pi^2 \, |R_{\underline{m}}|^2 \, n_{\rm corr} \tag{7}$$

This estimate (with $n_{\rm corr}=1$) is widely used; to be correct it requires the motion to be chaotic which is certainly not true for hadron colliders. In absence of external noise the motion described by Hamiltonian (3) is regular, there is no diffusion at all.

If there is some noise, it can drive particles from coasting trajectories into the separatrix (Fig.1 left) and, a few libration half-periods later (number of turns $n_{isl} \sim 1/\nu_{isl}$), out of the separatrix on the other side. In this process the action invariant $J_1 = \int I_1 d\psi_1/2\pi$ changes by the island full width:

$$\Delta J_1 = \frac{8}{\pi}\,|R_{\underline{m}}/\Lambda_{\underline{m}}|^{1/2}, \tag{8}$$

so the effective diffusion rate (within the island) scales as

$$D_{\underline{m}0} = \frac{1}{2n_{isl}}\left\langle \Delta J_1^2 \right\rangle = \kappa\,|R_{\underline{m}}|^{3/2}/|\Lambda_{\underline{m}}|^{1/2} \tag{9}$$

Constant κ was determined in a numerical experiment. 10^4 particles were placed at the reflecting boundary (Fig.1 right) and tracked for 10^5 turns at different levels of external noise. The resonance contribution to the diffusion rate was inferred from the excess in the loss rate. For ν_{isl} = 0.0025-0.01 (resonance strength varying by 16 times) and the noise level varying in a wide range the constant was found to be $\kappa = 0.65\pm0.1$.

There is an important difference between the two estimates. Since both R_m and Λ_m are proportional to the intensity of the strong beam $N_{\rm st}$, $D_{m0} \sim N_{\rm st}$ whereas $D_{\rm naive} \sim N_{\rm st}^2$.

Diffusion over the synchrotron satellites

If motion of an off-momentum particle is correlated over many synchrotron periods the multiplet resonance structure develops. Eq. 9 holds for an isolated synchrotron satellite as well with an appropriate modification of the longitudinal factor in R_m. For short bunches it is accomplished with the replacement

$$R_{\underline{m}} \to T_{\underline{m}} J_{ms}(\nu_M/Q_s), \quad \nu_M = \underline{m}\cdot\underline{C}\,\delta_{p0}, \tag{10}$$

where $J_m(x)$ is the Bessel function, δ_{p0} is the synchrotron amplitude. The general formula can be found in [7].

Eqs. 8-10 permit to calculate the rate of diffusion across the resonance islands (or, more, precisely, across the resonance layers) and, in combination with the extrinsic diffusion between the layers, to find the average diffusion rate over the multiplet.

The question is if there is enough time for the multiplet structure to emerge. The strongest and first to develop (in a few synchrotron periods) are the satellites of order $m_s = \pm[\nu_M/Q_s]$. But in a real situation the correlation may be lost even sooner.

DIFFUSION DUE TO MULTIPLE CROSSING OF A RESONANCE

Analytical estimates can be made for particles crossing the resonance at momentum deviation δ_{pr} far from the amplitude value δ_{p0} and for particles which reach the resonance near the turning point, $\delta_{pr} \approx \delta_{p0}$.

Crossing at a constant rate

In the first case we may assume the detuning from the resonance to be a linear function of time:

$$\underline{m}\cdot\underline{Q} - n \approx \underline{m}\cdot\underline{C}\,\frac{d\delta_p}{d\theta}\theta = \pm\,\underline{m}\cdot\underline{C}\,Q_s\sqrt{\delta_{p0}^2 - \delta_{pr}^2}\;\theta$$

and find variation of I_1 over period $\theta \in (-\infty, \infty)$ in absence of noise.

Invoking the noise only to destroy correlation between successive crossings we find for the diffusion rate

$$D_{\underline{m}}^{(\rm cv)} = \frac{|R_{\underline{m}}|\,Q_s}{2\,|\Lambda_{\underline{m}}|}\Phi(\mathrm{v}) \tag{11}$$

where function $\Phi(\mathrm{v})$ of the normalized crossing velocity

$$\mathrm{v} = \left|\frac{\underline{m}\cdot\underline{C}\,Q_s}{2R_{\underline{m}}\Lambda_{\underline{m}}}\right|\sqrt{\delta_{p0}^2 - \delta_{pr}^2} \tag{12}$$

is presented in Fig.3.

Let us note that R_m in Eqs. 11, 12 is that of Eq. 5 with values of longitudinal variables

$$|z| = \frac{R\alpha_c}{Q_s}\sqrt{\delta_{p0}^2 - \delta_{pr}^2}, \quad \delta_p = \delta_{pr}, \tag{13}$$

α_c being the momentum compaction factor and R the machine mean radius.

Function $\Phi(\mathrm{v})$ has a pronounced peak at $\mathrm{v}=1$ corresponding to what may be called *catching up with* (but not trapping into) *the moving separatrix*. For fast

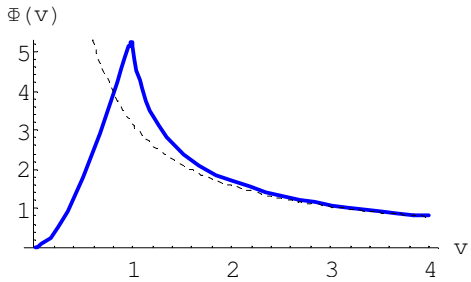

Figure 3. The crossing velocity factor.

crossing (v >> 1) $\Phi(v) \approx \pi/v$ [6], for v < 1 numerical fit gives $\Phi(v) \approx 5.64v^{1.683}$.

Diffusion vanishes in the limit v<<1 since all particles receive the same kick δI_1 (one crossing up, the other down) independently of the initial phase.

Comparison with quasi-linear diffusion rate

Averaging rate (12) over the range $\delta_{pr} \in (-\delta_{p0}, \delta_{p0})$ in the limit of fast crossing (v >> 1) we obtain

$$\overline{D_{\underline{m}}^{(\mathrm{cv})}} = \frac{\pi^2 R_{\underline{m}}^2}{2\nu_M} = \frac{\pi}{4} D_{QL} \qquad (14)$$

where D_{QL} is the average diffusion rate over the multiplet of overlapping synchrotron sidebands [3]. This shows that it is not important what destroys phase correlation: external noise or interference with other resonances, but in the case of noise there is no threshold intensity.

In the limit of fast crossing, v >> 1, (which implies limitation on the strong beam intensity) $D_m^{(\mathrm{c.v.})} \sim N_{\mathrm{st}}^2$. But the phase space area swept by the resonance is

$$\Delta I_1 \approx 2\nu_M / |\Lambda_{\underline{m}}| \sim 1/N_{\mathrm{st}}, \qquad (15)$$

so the effect on lifetime may even decrease with N_{st}.

Crossing in the turning point neighborhood

When the resonance crossing occurs in the vicinity of the turning point of synchrotron oscillations, $\delta_{pr} \approx \delta_{p0}$, the assumption of constant velocity does not hold.

For a weak resonance, following the path which led to Eq. 7 and assuming loss of phase correlation between successive "touches" of the resonance, we obtain at $\delta_{pr} = \delta_{p0}$:

$$D_{\underline{m}}^{(\mathrm{tp})} = \pi^2 |R_{\underline{m}}|^2 \mathbf{J}_\mu^2(\mu)/Q_s, \quad \mu = \nu_M / Q_s, \qquad (16)$$

where $\mathbf{J}_\mu(x)$ is the Anger function (which coincides with $J_\mu(x)$ for integer μ)

In the case of a strong resonance ($\nu_{isl} > Q_s$) the so-called *adiabatic trapping* occurs. To describe it we may use Eqs. 8-10 for the strongest satellite, $m_s = \mu$..

SOLUTION OF THE DIFFUSION EQUATION

Since the diffusion in the longitudinal phase space is usually slow, it can be ignored on the transverse diffusion time scale. In the result the problem is reduced to 2D diffusion equation for a given synchrotron amplitude δ_{p0}:

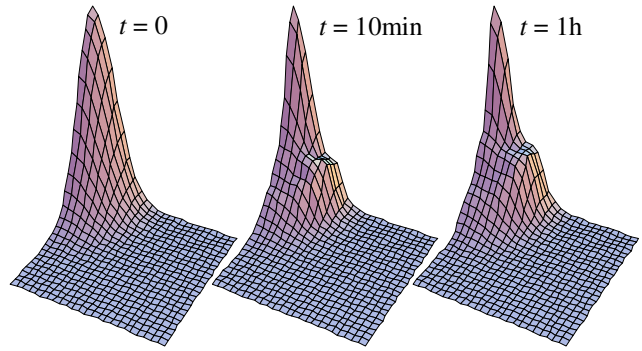

Figure 4. Evolution of distribution in coordinates $I_x^{1/2}$, $I_y^{1/2}$

$$\frac{\partial}{\partial t} F = \hat{D} F, \qquad (17)$$

where the diffusion operator was introduced ($\partial_{x,y} \equiv \partial/\partial I_{x,y}$):

$$\hat{D} = \partial_x (D_{xx}\partial_x + D_{xy}\partial_y) + \partial_y (D_{yx}\partial_x + D_{yy}\partial_y). \qquad (18)$$

Inverting Eqs.2 we obtain for one resonance contribution

$$D_{xx} = m_x^2 D_{\underline{m}}, \quad D_{xy} = D_{yx} = m_x m_y D_{\underline{m}}, \quad D_{yy} = m_y^2 D_{\underline{m}}$$

Due to symmetry and the fact that D_m=0 if I_x=0 or I_y=0 operator (18) is self-adjoint on functions vanishing at the absorbing boundary (or at infinity in its absence). Therefore it has an orthogonal set of eigenfunctions with non-positive eigenvalues since $D_{xx}D_{yy} - D_{xy}D_{yx} \geq 0$.

Fig. 4 shows evolution of the antiproton distribution (bunch #12) with initial emittances 16π mm·mrad and 9π mm·mrad, lattice tunes $\underline{Q}$={20.576, 20.575}, chromaticities $\underline{C}$={20, 10} and total beam-beam tuneshifts 0.021, 0.031. Five points in δ_{p0} were taken. One can see fast emittance growth (~10% vertically) in the first minutes of collisions due to 5th order resonances.

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