FERMILAB-PUB-04-283-A

Cosmological influence of super-Hubble perturbations

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(Dated: February 2, 2008)

arXiv:astro-ph/0410541v2 10 Sep 2005

The existence of cosmological perturbations of wavelength larger than the Hubble radius is a generic prediction of the inflationary paradigm. We provide the derivation beyond perturbation theory of a conserved quantity which generalizes the linear comoving curvature perturbation. As a by-product, we show that super-Hubble-radius (super-Hubble) perturbations have no physical influence on local observables (*e.g.,* the local expansion rate) if cosmological perturbations are of the adiabatic type.

PACS numbers: 98.80.cq

Inflation is an elegant explanation for the flatness, horizon, and monopole problems of the standard big-bang cosmology [1]. But perhaps the most compelling feature of inflation is a theory for the origin of density perturbations (the seeds for the large-scale structure of the Universe) and anisotropies in the cosmic microwave background (CMB) [2]. Density and gravitational-wave perturbations are created during inflation from quantum fluctuations and "redshifted" to sizes larger than the Hubble radius ($R_H \equiv H^{-1}$). They are then "frozen" until sometime after inflation when they once again come within the Hubble radius. The last and most impressive confirmation of this idea has been provided by the data of the Wilkinson Microwave Anisotropy Probe [3].

A general feature of inflation is the existence of scalar perturbations of wavelength larger than the Hubble radius. During inflation a small region of size less than the Hubble radius grows large enough to encompass easily the comoving volume of the entire presently observable Universe. This requires a minimum number of e-foldings, $N \gtrsim 60$, where N measures the logarithmic growth of the scale factor during inflation. However, most models of inflation predict a number of e-foldings that is, by far, much larger than 60 [2]. This amounts to saying that there is a huge phase space for super-Hubble perturbations. These super-Hubble perturbations will re-cross the Hubble radius only in the very far future.

This paper deals with the issue of the physical influence of the infrared modes produced during inflation on cosmological observables such as the local Hubble expansion rate. The question is, can we see beyond the horizon and learn the nature of the very long-wavelength perturbations. This question has already been posed in the literature with contradictory answers [4, 5, 6, 7, 8, 9]. In this note we demonstrate that if adiabaticity holds, super-Hubble perturbations do not have an impact on local physical observables, for instance, the local Hubble expansion rate. Our results are valid at any order in perturbation theory.

Let us first review some generalities which will turn out to be useful. Since we are interested in very long-wavelength perturbations, from now on we will neglect spatial gradients. Our starting point is the Arnowitt-Deser-Misner (ADM) formalism, with metric

$$ds^2 = -N^2\, dt^2 + N_i\, dt\, dx^i + \gamma_{ij}\, dx^i\, dx^j\,.$$

The three-metric γ_{ij}, and the lapse and the shift functions N and N_i, describe the evolution of the timelike hypersurfaces. The extrinsic curvature three-tensor is

$$K_{ij} = \frac{1}{2N}\left(N_{i|j} + N_{j|i} - \frac{\partial \gamma_{ij}}{\partial t}\right),$$

where three-space covariant derivatives with connection coefficients determined from γ_{ij} are indicated by a vertical bar. The trace $K = K^i{}_i$ is the generalization of the Hubble parameter of homogeneous isotropic cosmologies. The traceless part of the tensor is denoted by an overbar, $\overline{K}_{ij} = K_{ij} - \frac{1}{3}\, K\, \gamma_{ij}$. In the ADM formalism the equations simplify considerably if we set $N^i = 0$.

It is convenient to express the spatial metric as [10]

$$\gamma_{ij} = \exp\left[2\alpha(t, x^i)\right]\; h_{ij}(x^i),$$

where the conformal factor $\exp[\alpha(t, x^i)]$ may be interpreted as the spatial-dependent scale factor. The time-independent three metric $h_{ij}(x^i)$ (with determinant equal to unity) describes the three geometry of the conformally transformed space. Within linear perturbation theory, $\alpha(t, x^i)$ would contribute only to scalar perturbations, whereas h_{ij} contains another scalar, as well as vector and tensor perturbations. Vector and tensor perturbations are necessary to satisfy Einstein equations, however in the long wavelength limit they do not affect the equations for the scalar perturbations. On super-Hubble scales, $(\partial \overline{K}^i{}_j/\partial t) = NK\overline{K}^i{}_j$ and $K = -(3/2N)\dot{\alpha}$. This implies that $\overline{K}^i{}_j$ decays with time and can therefore be safely set to zero. Since $\exp[\alpha(t, x^i)]$ is interpreted as a scale factor, we can use the Hubble parameter $H(t, x^i) = \dot{\alpha}(t, x^i)/N(t, x^i) \equiv -K(t, x^i)/3$ in place of the trace K of the extrinsic curvature.

Now consider a Universe filled with some fluid(s) described by an energy-momentum tensor of the form $T_{\mu\nu} = (\rho + P)\; u_\mu u_\nu + P\, g_{\mu\nu}$, where ρ and P are the energy density and pressure, respectively. The four-velocity vector can be chosen to be $u^\mu = (1, \vec{0})$. In the case of one fluid, this amounts to saying that a volume measured by a local observer is comoving with the energy flow of the fluid. In the multi-fluid case, the volume comoving with the total energy density ρ is not the same as the individual volumes comoving with the energy density components. However, one can show that on super-Hubble scales all comoving volumes become equivalent [11], and therefore our choice of the four velocity is well justified.

The relevant set of equations is provided by the continuity equation, $N^{-1}\partial\rho/\partial t = -3H(\rho + P)$, and by $N^{-1}\dot{\alpha} = H$. Eliminating the lapse function N from the set of equations, we derive

$$\dot{\alpha} + \frac{1}{3}\frac{\dot{\rho}}{(\rho + P)} = 0. \tag{1}$$

The quantity

$$\mathcal{F} \equiv \alpha + \frac{1}{3}\int^{\rho} \frac{d\rho'}{(\rho' + P')} \tag{2}$$

is therefore conserved in time at any order in perturbation theory. To understand its physical significance, consider an adiabatic fluid for which the pressure is a unique function of the energy density $P = P(\rho)$. If we set

$$\exp\left[\alpha(t, x^i)\right] = a(t)\; \exp\left[-\psi(t, x^i)\right] \tag{3}$$

and expand Eq. (2) to first order in perturbation theory, we find that the perturbation $\delta_1\mathcal{F}$ coincides with the comoving curvature perturbation

$$\zeta_1 = -\psi_1 - H\frac{\delta\rho_1}{\dot{\rho}_0},$$

where ρ_0 is the background value of the energy density. Similarly, expanding Eq. (2) to second order in perturbation theory, one finds that the perturbation $\delta_2\mathcal{F}$ coincides with the definition of the second-order comoving curvature perturbation ζ_2 [12] (up to a constant piece ζ_1^2)

$$\begin{aligned}\zeta_2 \;=\; & -\psi_2 - H\frac{\delta\rho_2}{\dot{\rho}_0} - 2\frac{H}{\dot{\rho}_0^2}\delta\dot{\rho}_1\delta\rho_1 - 2\frac{\delta\rho_1}{\dot{\rho}_0}\left(\dot{\psi}_1 + 2H\psi_1\right) \\ & + \left(\frac{\delta\rho_1}{\dot{\rho}_0}\right)^2\left(H\frac{\ddot{\rho}_0}{\dot{\rho}_0} - \dot{H} - 2H^2\right).\end{aligned}$$

We conclude that $\delta\mathcal{F}$ is the nonlinear generalization of the comoving curvature perturbation on super-Hubble scales for adiabatic fluids. Equivalently, ψ plays the role of the nonlinear generalization of the comoving linear gravitational potential.

For a generic nonadiabatic fluid, the perturbation of the pressure energy density δP can not be expressed in terms of the perturbation of the energy density $\delta\rho$. Perturbing Eq. (2) to first order, one recovers the nonconservation of the comoving curvature perturbation

$$\dot{\zeta}_1 = -\frac{H}{(\rho + P)}\,\delta P_1^{\rm nad},$$

where $\delta P_1^{\rm nad} = \delta P_1 - (\dot{P}/\dot{\rho})\,\delta\rho_1$ is the first-order nonadiabatic pressure perturbation.

Let us now return to the physical influence of super-Hubble modes. We will consider the effect on the local expansion rate of the Universe (although an effect on the local expansion rate suggests that there will be effects on other cosmological measurables as well). We choose to work in the synchronous gauge for which $N = 1$ (together with $N^i = 0$). In this gauge the field equations look just like the familiar equations point by point. As mentioned, for a generic set of fluids we may safely take the four velocity to be $u^\mu = (1, \vec{0})$ on super-Hubble scales, from which we define the local expansion rate of tangential surfaces orthogonal to the fluid flow, *i.e.,* the local expansion rate, to be $\frac{1}{3}D_\mu u^\mu$. In the synchronous gauge this quantity and H coincide.

Using Eq. (3), we find that at any order in perturbation theory the local expansion rate is $H = \dot{a}/a - \dot{\psi}$. We immediately infer that the local physical expansion rate is influenced by long-wavelength perturbations only if the gravitational potential is time dependent.[1] Let us first analyze the case of an adiabatic fluid. From Einstein's equations for a globally flat space we deduce (on super-Hubble scales)

$$3H^2 = 8\pi G\rho = -8\pi G\,P + 2\dot{H}. \tag{4}$$

If the pressure P is a unique function of the energy density, we may expand $\rho = \rho_0 + \Delta\rho$, where ρ_0 is the energy density entering the homogeneous Einstein equations from which the homogeneous scale factor $a(t)$ is computed. Inserting this expansion into Eq. (4) and eliminating the pressure in favor of the energy density $\rho \propto H^2$, we find a differential equation of the generic form $\mathcal{G}\left(\ddot{\psi}, \dot{\psi}\right) = 0$, which does not contain any term proportional to ψ. Therefore, $\psi = \psi(x^i)$ is a solution of Einstein equations at any order in perturbation theory and $\dot{\psi} = 0$ holds on super-Hubble scales. This result reproduces the familiar result that the gravitational potential does not depend upon time for super-Hubble modes if there is only one fluid in the Universe. As a by product of Eq. (2), we see that the perturbation of the energy density vanishes on super-Hubble scales in the synchronous gauge at any order in perturbation theory.

From this generic argument we conclude that there is no influence of very long wavelength modes on the physical local expansion rate of the Universe if adiabaticity holds (again, we have ignored gradients in ψ). In the adiabatic case, the influence of infrared modes is not locally measurable. There is a simple explanation of this result. When adiabaticity holds, the pressure is a well defined function of the energy density. This means that the Hubble rate is only a function of the unique available physical clock, the energy density. Indeed, suppose that the local expansion rate is $H(t, x^i) = H(\rho(t, x^i), t)$. This leads to

$$\begin{aligned}\left(\frac{\partial H}{\partial t}\right)_{x^i} &= \left(\frac{\partial \rho}{\partial t}\right)_{x^i}\left(\frac{\partial H}{\partial \rho}\right)_t + \left(\frac{\partial t}{\partial t}\right)_{x^i}\left(\frac{\partial H}{\partial t}\right)_\rho \\ &= -4\pi G(\rho + P) + \left(\frac{\partial H}{\partial t}\right)_\rho,\end{aligned} \tag{5}$$

where we have made use of Eq. (1) evaluated in the synchronous gauge, $3H = -\dot{\rho}/(\rho + P)$. Comparing Eq. (5) with $\dot{H} = -4\pi G(\rho + P)$, we see that $(\partial H/\partial t)_\rho = 0$, and hence $H(t, x^i) = H(\rho(t, x^i))$. The dependence of the local expansion rate of the Universe on the clock time takes the same form as in the unperturbed Universe when evaluated at a fixed value of the only clock available, the energy density ρ. Infrared modes do not have any locally measurable effect on the expansion rate of the Universe. This result applies, in particular, during inflation if the energy density is dominated by a single inflaton field.

Our findings suggest that adiabatic super-Hubble modes cannot modify cosmological observables like the Hubble rate.

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[1] Recall that we have ignored spatial gradients in ψ. A perturbative calculation of the effect of spatial gradients on the expansion rate was considered in [13].

Acknowledgments

E.W.K. is supported in part by NASA grant NAG5-10842 and by the Department of Energy.

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