

# Chaotic Collisionless Evolution in Galaxies and Charged-Particle Beams

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**ABSTRACT:** Both galaxies and charged particle beams can exhibit collisionless evolution on surprisingly short time scales. This can be attributed to the dynamics of chaotic orbits. The chaos is often triggered by resonances caused by time dependence in the bulk potential, which acts almost identically for attractive gravitational forces and repulsive electrostatic forces. The similarity suggests that many physical processes at work in galaxies, while inaccessible to direct controlled experiments, can be tested indirectly via controlled experiments with charged-particle beams such as those envisioned for the University of Maryland Electron Ring currently nearing completion.

**KEYWORDS:** Chaos,  $N$ -body problem, nonlinear dynamics, collisionless

## PREAMBLE

Henry Kandrup and Court Bohn had independently realized that there were important parallels between the collisionless evolution of charged-particle beams and large stellar systems. Both desired to pursue this matter explicitly by way of direct experimentation with beams. Also independently, Martin Reiser obtained funding to build the University of Maryland Electron Ring (UMER) for the expressed purpose of doing controlled experiments to measure the dynamical consequences and evolutionary time scales associated with internal Coulomb forces, i.e., space charge. All of these circumstances led to a strong collaboration. Henry had been eagerly anticipating the completion of UMER and experiments that the collaboration was planning.

We all endeavored to introduce the notion of an analogy between the dynamics of beams and galaxies to a broad spectrum of investigators. Before Henry passed away, we had completed a paper, one that excited Henry immensely, to review the pertinent literature and introduce this idea. Feedback from referees was generally negative toward publication but positive toward pursuit of the idea. Loosely translated, the referee reports stated that we have a nice proposal, e.g., to submit to a funding agency, but we should finish some new experiments prior to journal publication.

The paper has evolved considerably since Henry's passing, but it retains much of his language, particularly as concerns galactic dynamics. We, his colleagues, hereby offer this paper as part of the Symposium that honors Henry. What follows is a version that incorporates all referee comments and that is edited to mesh with other related Symposium contributions, but that retains the original flavor and Henry's unique touch. It would surely have his imprimatur.

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## I. INTRODUCTION

Many-body systems whose constituents interact via long-range inverse-square-law “Coulomb” forces, both gravitational and electrostatic, can exhibit macroscopic relaxation and loss of coherence on time scales much shorter than might be expected on dimensional grounds. This process moves the system toward a long-lived ‘metaequilibrium’ state, a state that differs from true thermal equilibrium (which, in the case of galaxies, cannot be accessed dynamically). When a galaxy has a sizeable gaseous component, the gas will interact with the stellar component and thereby enhance its relaxation. However, observations and simulations agree that even a relatively gas-poor (and thus presumably nearly dissipation-free) elliptical galaxy displaced from a metaequilibrium state as a result of an encounter with another galaxy can readjust itself towards a new metaequilibrium state within a few hundred million years (i.e., within  $\sim 10\%$  of the age of the galaxy) although the nominal relaxation time  $t_R$  associated with ‘collisions’ is orders of magnitude longer than the age of the Universe. And similarly, charged-particle beams, which would be expected to maintain coherence while traveling some 100 km or more through an accelerator, can lose coherence and disperse significantly within distances as short as 10 m.

Because collisions would cause relatively slow relaxation, any rapid relaxation must be due to collisionless, i.e., collective, processes. More specifically, the collective behavior must be connected with mixing, i.e., the tendency of initially localized clumps of orbits to disperse. Mixing is *much* more efficient in a chaotic system than in a system in which the bulk coarse-grained potential is integrable or near-integrable. An initially localized clump of regular, i.e., non-chaotic, orbits will typically disperse secularly, i.e., as a power law in time; a clump of chaotic orbits will instead disperse exponentially.

Allowing for a bulk potential that is strongly chaotic, thereby supporting “chaotic mixing”, would enable one to understand how a galaxy can ‘relax’ toward a metaequilibrium state on a comparatively short time scale. Such an understanding is of *practical* importance in regard to charged-particle beams. There, rapid collisionless relaxation places strong constraints on ‘emittance compensation’, i.e., processes designed to confine the constituent particles to a compact volume of phase space, as is required for high-brightness beams.

Theoretical considerations and detailed numerical simulations suggest that, in this setting, the origin of the chaos that drives the evolution is largely irrelevant. In particular, whether the two-body forces are attractive or repulsive should not be crucial. What *is* important is that the long-range scalings of gravitational and electrostatic forces are identical and that, in both cases, the early stages of evolution should be driven by long-range, collective interactions (acting ‘globally’) as opposed to short-range collisional encounters (acting ‘locally’). All that seems to matter is whether the bulk potential associated with the many-body system admits a large measure of chaotic orbits<sup>1</sup>.

A complete understanding of these phenomena requires a synthesis of theory, simulations, and experiments. Performing experiments on self-gravitating systems like galaxies is impossible. However, controlled experiments *can* be performed with charged-particle beams, and combining the results of such experiments with simulations and theory should lead to a clear picture of the role of chaotic phase mixing in beams. Moreover, as we will exemplify in Sec. II below, the physics should not depend crucially on whether the force between particles is attractive or repulsive, and one would thus expect that many results about beams should translate more or less directly into detailed predictions about the structure and evolution of galaxies. Indeed, one can go one step further and argue that, in a real sense, carefully constructed experiments involving charged-particle beams can be used as semi-direct probes of the physics of self-gravitating systems like galaxies.

## II. THE BEAM-GALAXY ANALOGY: THEORETICAL CONSIDERATIONS

That collisional relaxation should be largely irrelevant in many settings involving galaxies and beams is easily seen. Viewing such effects as an incoherent sum of binary encounters, one computes, respectively, for galaxies<sup>2</sup> and (in gaussian units) for charged particle beams<sup>3,4</sup> relaxation times

$$t_R \sim \frac{v^3}{(Gm)^2 n \log \Lambda} \quad \text{and} \quad t_R \sim \frac{v^3}{q^2 n \log \Lambda}. \quad (1)$$

Here  $v$  is a typical speed associated with random motions;  $G$  the gravitational constant;  $m$  and  $q$  the typical stellar mass and particle charge, respectively;  $n$  a characteristic number density, and  $\log \Lambda$  the so-called Coulomb logarithm\*, which scales as a positive power of the number of constituent ‘particles’  $N$ .

In either case, assuming the bulk random kinetic and potential energies are comparable in magnitude implies that  $t_R \sim (N/\log \Lambda)t_D$ , with  $t_D \sim R/v$  denoting the ‘dynamical time’, a characteristic orbital time scale defined in terms of the ‘size’  $R$  of the system. For large  $N$  (typically  $N \sim 10^9$ - $10^{12}$  in realistic, large stellar and particle-beam systems) the relaxation time  $t_R$  is clearly orders of magnitude longer than the dynamical time  $t_D$ ; collisional relaxation is slow. By contrast, mixing of chaotic orbits, i.e., ‘chaotic mixing’, can proceed extremely fast; the  $e$ -folding time associated with the dispersal of an initially localized ‘clump’ of particles, given as the inverse of the largest positive Lyapunov exponent relative to that clump<sup>6</sup>, is typically comparable in magnitude to  $t_D$ . This is, for example, the case for the systems illustrated in Figs. 2 and 4 discussed below in Secs. III.A.3 and III.B.3, respectively

Presently there is no known generic algorithm permitting accurate analytic or quasi-analytic estimates of the largest Lyapunov exponent in three-dimensional bulk potentials. However, recent work<sup>7-9</sup> has shown that, in many cases, an analytic technique developed for systems with many degrees of freedom<sup>10</sup> can be adapted to provide reasonable estimates for lower-dimensional systems, the breakdown of that approach reflecting typically systems in which autocorrelation functions for properties of representative orbits have long time tails<sup>11</sup>. It is therefore relevant to recall the analytic results for the largest Lyapunov exponent  $\chi$  in a three-dimensional time-independent bulk potential, for this then becomes a quantitative measure of the rate of collisionless relaxation by way of chaotic mixing:

$$\chi(\xi) \approx \frac{1}{\sqrt{3}} \frac{L^2(\xi) - 1}{L(\xi)} \sqrt{\kappa}; \quad L(\xi) = \left[ T(\xi) + \sqrt{1 + T^2(\xi)} \right]^{1/3}; \quad T(\xi) = \frac{3\pi\sqrt{3}}{8} \frac{\xi^2}{2\sqrt{1+\xi} + \pi\xi}. \quad (2)$$

The auxiliary quantities  $\kappa$  and  $\xi$  are determined from the potential  $V(\mathbf{x})$ :

$$\kappa = \frac{1}{2} \langle \nabla^2 V \rangle; \quad \xi = \frac{1}{\kappa\sqrt{2}} \sqrt{\left\langle \left( \nabla^2 V \right)^2 \right\rangle - \left\langle \nabla^2 V \right\rangle^2}; \quad (3)$$

wherein the averages are taken over the microcanonical ensemble in the manner

\*If one assumes that collisions act as a source of Brownian motion,  $t_R$  can be related to the time integral of the quantity  $N\langle \mathbf{F}(0) \cdot \mathbf{F}(t) \rangle$ , where  $\langle \mathbf{F}(0) \cdot \mathbf{F}(t) \rangle$  is the autocorrelation function for the test ‘particle’ interacting with a single field ‘particle’.<sup>5</sup>

$$\langle A \rangle \equiv \frac{\int d\mathbf{x} \int d\mathbf{p} A(\mathbf{x}) \delta[H(\mathbf{x}, \mathbf{p}) - E]}{\int d\mathbf{x} \int d\mathbf{p} \delta[H(\mathbf{x}, \mathbf{p}) - E]}. \quad (4)$$

Here,  $E$  denotes the total particle energy. Upon invoking Poisson's equation, we see immediately that the auxiliary quantities are determined from the density distribution. For a *galaxy*, we have

$$\nabla^2 V = 4\pi G\rho; \quad \kappa = 2\pi G\rho(0)\langle \zeta \rangle; \quad \xi = \frac{\sqrt{2}}{\langle \zeta \rangle} \sqrt{\langle \zeta^2 \rangle - \langle \zeta \rangle^2}, \quad (5)$$

where  $\zeta = \rho(\mathbf{x})/\rho(0)$ ,  $\rho(\mathbf{x})$  denoting the mass density. For a *beam*, we take the external focusing potential  $V_f$  to be quadratic in the coordinates  $\mathbf{x}$  comoving with the bunch, i.e.,  $V_f(\mathbf{x}) = (\boldsymbol{\omega} \cdot \mathbf{x})^2/2$ , wherein  $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$  corresponds to the focusing strength; the total potential is  $V = V_f + V_s$ . Then we have

$$\nabla^2 V_s = -4\pi\rho; \quad \kappa = \frac{\omega_{p0}^2}{2} [(\omega/\omega_{p0})^2 - \langle \zeta \rangle], \quad \xi = \frac{\sqrt{2}}{(\omega/\omega_{p0})^2 - \langle \zeta \rangle} \sqrt{\langle \zeta^2 \rangle - \langle \zeta \rangle^2}, \quad (6)$$

where  $\omega_{p0}$  is the plasma frequency at the bunch centroid and  $\rho(\mathbf{x})$  now refers to charge density. Now, the time scale for chaotic mixing is  $t_m = 1/\chi \propto f(\xi)\kappa^{-1/2}$ . The analogy between chaotic mixing in beams versus galaxies becomes apparent: for both classes of systems, the dynamical time is  $t_D \sim \kappa^{-1/2}$ , the auxiliary quantity  $\xi$  involves a ratio of the dispersion in the density profile to the square of the dynamical time, and  $f(\xi)$  is the same function for both systems. For beams, space charge is a repulsive collective force that acts to lengthen the dynamical time by weakening the net focusing force acting on a particle (resulting in what is called the 'space-charge-depressed period'). For galaxies no such weakening appears; gravity is strictly attractive.

To do a computational test of this result, one chooses an energy  $E$  and integrates a large number (say, 2,000) tightly localized initial conditions corresponding to an energy very close to  $E$ . These trajectories then spread, and one can calculate moments, such as  $\langle x^2(t) \rangle$ , of the corresponding distribution of trajectories versus time and assess whether they grow exponentially. If they do, then one can extract the  $e$ -folding time and compare it to the analytic estimate. Examples of such comparisons in galactic and beam systems appear in Fig. 1. The galactic system is a uniform-density ellipsoid containing a supermassive black hole at its centroid<sup>8</sup>. The beam system is a configuration of thermal equilibrium having triaxial symmetry<sup>9</sup>.

The preceding analytic results follow from a geometric treatment of scleronomous Hamiltonian systems in the spirit of Pettini and his collaborators<sup>10</sup>. It does not apply to time-dependent systems, and thus it is not presently possible to point to an unambiguous analogy between the dynamics of beams and galaxies involving rheonomous Hamiltonians. A geometric treatment of the latter would be based on a Finsler metric, i.e., a metric that incorporates velocities and time, but it becomes unclear how to define an invariant measure to use in place of the microcanonical ensemble for evaluating phase-space averages, particularly when one considers that resonances between orbital frequencies and the frequency spectrum of the time-dependent potential come into play. Nonetheless, a reasonable ansatz is that a successful geometric treatment of rheonomous systems would result in a connection between beams and galaxies analogous to that of time-independent systems. The underlying reason is that both systems involve an inverse-square long-range force, and this force is what drives chaotic mixing.

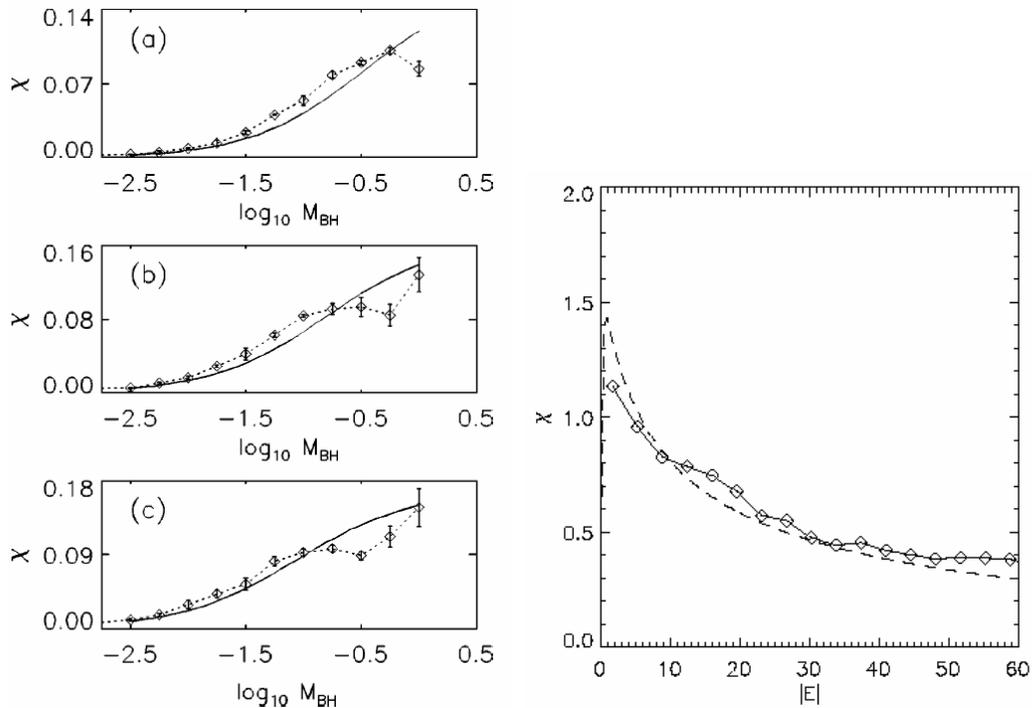


Figure 1. (left) Numerical results (diamonds) and analytic (solid line) estimates of the mixing rate for chaotic orbits evolved in a triaxial galactic potential as a function of black-hole mass  $M_{BH}$  and for total particle energy  $E = 1.0$ , (b)  $E = 0.6$ , (c)  $E = 0.4$ . (right) Numerical (diamonds) and analytic estimates (dashed line) of the mixing rate for chaotic orbits in a triaxial thermal-equilibrium beam as a function of  $E$ . In both figures the unit of  $\chi$  is  $t_D^{-1}$ . Further details can be found in Refs. [8,9] from which these figures are reprinted by permission of the American Physical Society.

### III. REGULAR VS. CHAOTIC ORBITS: A TORTURED HISTORY

Chaos has been largely ignored until comparatively recently in both the galactic and accelerator-dynamics communities. For example, although the famous Hénon-Heiles model<sup>12</sup> arose originally in attempts to understand meridional motions in axisymmetric galaxies, as recently as 15 years ago the potential role of chaos in galaxy structure and evolution was almost completely neglected (with the exception of a handful of groups in Europe). Only with the advent of high-resolution photometry, facilitated in part by the Hubble Space Telescope, did many galactic astronomers begin to recognize that the bulk potentials associated with realistically shaped galaxies are likely to admit significant measures of chaotic orbits.

#### A. Galaxies

It has been long recognized that the dominant mechanism for relaxation in galaxies cannot be ‘collisional’. For example, in the 1940s Chandrasekhar<sup>2</sup> showed that the relaxation time scale  $t_R$  on which binary encounters between individual stars could significantly alter the trajectories of stars in the Milky Way must be  $\sim 10^{12}$  years or more. Shorter-time relaxation must somehow involve collective effects. Two decades later, Lynden-Bell<sup>13</sup> proposed a theory of ‘violent relaxation’ which argued, *inter alia*, that regular (i.e., nonchaotic) phase mixing associated with a time-dependent potential might explain such collective effects. Substantial evidence for rapid relaxation accumulated over the next twenty years derived both from numerical simulations of many-body systems and from the interpretation of observations of galaxies that have been involved in

collisions with other galaxies<sup>14</sup>. Despite this, however, when subjected to closer scrutiny, it seemed that, at least in its simplest guise where the orbits that phase mix are assumed to be regular, violent relaxation could not explain why relaxation was as fast as it appears to be. Some ingredient seemed to be missing. Today there is good reason to think that the missing ingredient is *chaos*.

In the early 1990s Kandrup and Mahon<sup>15</sup> recognized that, because of their exponentially sensitive dependence on initial conditions, chaotic orbits should mix far more rapidly than regular orbits, in fact exponentially fast. In the astronomical community this phenomenon, now termed ‘chaotic mixing’<sup>16</sup>, led to speculations that chaos could play a critical role in violent relaxation. However, chaotic mixing in itself does not constitute a complete and satisfactory explanation. It cannot drive collective relaxation unless many/most of the orbits are chaotic, a prerequisite whose fulfillment is far from obvious. However, a few years later, motivated in part by the work of accelerator physicists<sup>17-19</sup>, astronomers<sup>20</sup> recognized that time-dependent pulsations in the bulk potential of a galaxy readjusting toward a metaequilibrium state could, via resonant couplings, make many/most of the orbits in a galaxy chaotic with large finite-time Lyapunov exponents<sup>\*21,22</sup>, and that the resulting ‘resonant phase mixing’ might be sufficiently strong and ubiquitous to explain violent relaxation.

## B. Charged-Particle Beams

Concerns about collisionless relaxation in charged-particle beams have arisen with recent advances in technology for the production of high-brightness beams, wherein the collective Coulomb self-force, i.e., the space-charge force, becomes important. In the laboratory frame this force decreases inversely as the square of the beam energy<sup>24</sup>. For the transverse component, this is due to the partial cancellation between the self-magnetic and self-electrostatic forces; for the longitudinal component, it is due to Lorentz contraction. Nonetheless, there are still many situations involving high-brightness beams where space charge is important. Examples include both low-to-medium-energy hadron accelerators such as those envisioned to drive spallation-neutron sources or heavy-ion fusion or that serve as boosters for high-energy machines, as well as low-energy lepton, (e.g., electron) accelerators such as photoinjectors<sup>25</sup>.

One of the earliest papers to treat space charge in beams was by Kapchinskij and Vladimirskij<sup>26</sup>, who considered a direct-current beam with uniform charge density and elliptical cross section confined by linear external focusing forces, and derived the equations governing the motion of the beam envelope. The corresponding distribution function in the four-dimensional transverse phase space of a single charge, commonly called the ‘KV distribution’, is a hyperellipsoidal shell. A decade later, Sacherer<sup>27</sup> noted that these results can readily be generalized to three-dimensional bunched beams (i.e., to six-dimensional phase space) so as to include the influence of space charge on bunch length and energy spread. These two papers, regarded as classics by the accelerator community, set the stage for much of the subsequent investigations concerning space charge, from which evolved now-conventional design strategies for high-brightness accelerators, strategies based on controlling root-mean-square (rms) properties of the beam.

However, the past decade has brought the realization that, albeit necessary, controlling the rms

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\*Finite-time Lyapunov exponents probe the average exponential instability of orbit segments over finite time intervals.

Formally, they satisfy  $\chi(t) = \frac{1}{t} \lim_{\delta Z(0) \rightarrow 0} \left\{ \ln \left| \frac{\delta Z(t)}{\delta Z(0)} \right| \right\}$ , where  $|\delta Z|^2 = |\delta \mathbf{r}|^2 + |\delta \mathbf{v}|^2$ . The largest such exponent can be estimated numerically using an algorithm introduced by Benettin, Galgani, and Strelcyn<sup>23</sup>.

properties of a beam is not sufficient. Perhaps the most prominent example concerns beam halos, i.e., particles that reach large orbital amplitudes due to a time-dependent space-charge potential arising because irregularities in the beamline prevent the beam from reaching a long-lived equilibrium state<sup>28</sup>. The concern is that a tiny impingement of particles on the beamline,  $\sim 1$  W/m for beam energies exceeding  $\sim 20$  MeV, can generate sufficient radioactivation to preclude routine, hands-on maintenance. Efforts to push our understanding of space charge beyond that required for computing bulk moments of the beam brought, as a spin-off, the realization that early-time dynamics in fully self-consistent charged-particle systems resembled that of violent relaxation in stellar systems<sup>18</sup>. However, that resemblance was explored no further – until now.

## IV. EVIDENCE FOR CHAOS AND CHAOTIC MIXING

### A. Chaos in Galaxies

#### 1. *The inevitability of chaos*

High-resolution observations of galaxies over the past decade or so have provided compelling evidence that many galaxies are more irregularly shaped than had been assumed as recently as 15 years ago; and attempts to model such irregularly shaped objects have led many galactic dynamicists to conclude that the bulk potentials associated with realistic galaxies admit large measures of chaotic orbits. It has been argued<sup>29</sup> that nonaxisymmetric elliptical galaxies containing central density cusps of the form inferred from observations<sup>30</sup> are very likely to admit large numbers of chaotic orbits. And similarly, models of rotating barred spiral galaxies suggest<sup>31-33</sup> that breaking axisymmetry with even a comparatively weak bar can trigger large numbers of chaotic orbits, especially near certain resonances. More generally, as first stressed by Udry and Pfenniger<sup>34</sup>, making a galaxy less symmetric, e.g., by deforming it from axisymmetric to triaxial or by introducing ‘local’ asymmetries, tends generically to increase both the relative measure of chaotic orbits and the size of the largest Lyapunov exponents. Although it is possible to contrive models of cuspy, nonaxisymmetric galaxies that are integrable or near-integrable<sup>35-37</sup>, they are not generic. Instead, there has emerged a general sense in much of the galactic dynamics community that ‘generic’ irregularly shaped galaxies might be expected to contain large numbers of strongly chaotic orbits.

#### 2. *Are galaxies really ‘in equilibrium’?*

One intriguing possibility is that, perhaps because of the presence of chaos, evolving galaxies will find it difficult, if not impossible, to approach a true equilibrium. Rather, it may well be that, at the time of formation, a galaxy settles down toward a long-lived ‘metaequilibrium’ rather than a true equilibrium; and subsequently, in response to, e.g., external irregularities associated with a densely populated galactic cluster, exhibits a slow, secular evolution<sup>29,38</sup>. To the extent that this be true, a basic question is whether a galaxy originally in a nonaxisymmetric metaequilibrium will evolve toward a more nearly axisymmetric state<sup>29</sup>; or whether instead a galaxy originally containing large numbers of strongly chaotic orbits might evolve toward other metaequilibria, not necessarily more nearly axisymmetric, which contain smaller numbers of chaotic orbits<sup>39</sup>. In any event, it is generally accepted that a robust, stable metaequilibrium must contain large measures of

regular<sup>40</sup> and/or sticky chaotic<sup>41</sup> orbits to provide the ‘skeleton’ (i.e., foundation) of the interesting configurations that support chaotic orbits in the first place.

### 3. *The role of a time-dependent bulk potential*

There is also emerging evidence that chaos should be even more ubiquitous in systems that feel a strongly time-dependent bulk potential, especially a time dependence involving roughly periodic oscillations. Nonlinear dynamicists argue that chaos typically arises via resonance overlaps<sup>6</sup>, and this time-dependent chaos is simply another example thereof. When the time dependence influencing stellar orbits in a galaxy has power at frequencies sufficiently close to (multiples of) the frequencies at which the orbits themselves have power, the orbits and the time dependence can resonate with the result that the orbits become strongly chaotic. If the time dependence is weak, such resonances may require a near-perfect frequency match, but for stronger time dependence it often suffices for the pulsation and orbital time scales to agree within an order of magnitude<sup>20</sup>. However, in a nearly collisionless system like a galaxy, dimensionally there is only one natural time scale, the dynamical time  $t_D \sim (G\rho)^{-1/2}$ , with  $G$  the gravitational constant and  $\rho$  a characteristic density.\* Consequently the pulsation and orbital times are likely to be comparable in magnitude throughout much of the galaxy, thus rendering chaos extremely common. Simple models suggest that galaxies subjected to damped oscillations could (i) become almost completely mixed *and* (ii) settle down towards a nearly integrable metaequilibrium within a time as short as  $\sim 10 t_D$ . Analogous effects can also be triggered by other nearly periodic phenomena such as localized, nonstationary collective modes, or a supermassive black hole binary orbiting near the center of a galaxy<sup>42</sup>. Indeed, such a binary could produce anomalous ‘dips’ observed in the surface-brightness profiles of galaxies like NGC 3706 or NGC 4406 which suggest their respective mass densities do not decrease monotonically with distance from the center<sup>43</sup>.

An example of such resonant phase mixing is illustrated in Fig. 2. It tracks three initially localized clumps of test stars evolved in a galactic potential with periodic driving that damps as a power law in time. The left and center panels exhibit the  $(x,y)$ -coordinates at several different times; the right panel exhibits the exponential growth of components of an *emittance*-like quantity  $\varepsilon_i$  ( $i=x,y,z$ ), which measures the area of the occupied phase-space planes corresponding to the coordinate  $r_i$ .<sup>†</sup> Here, e.g.,

$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle v_x^2 \rangle - \langle xv_x \rangle^2}, \quad (7)$$

where  $\langle \dots \rangle$  denotes an average over the clump. As was argued in Sec. II, initially localized clumps of regular orbits typically diverge secularly, whereas clumps of chaotic orbits diverge exponentially at a rate set by the typical value of the largest finite-time Lyapunov exponent  $\chi$ :

$$\varepsilon_i \propto (t/t_D)^p \quad (\text{regular orbits}) \quad \text{and} \quad \varepsilon_i \propto e^{\chi t} \quad (\text{chaotic orbits}), \quad (8)$$

with  $p$  a constant of order unity.

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\*Assuming the bulk kinetic and potential energies are comparable in magnitude, then  $(G\rho)^{-1/2} \sim R/v$ , which is the time scale of a typical orbit.

<sup>†</sup>In the context of charged-particle beams, emittance is given a more precise definition, which will be described in Sec. IV.B.

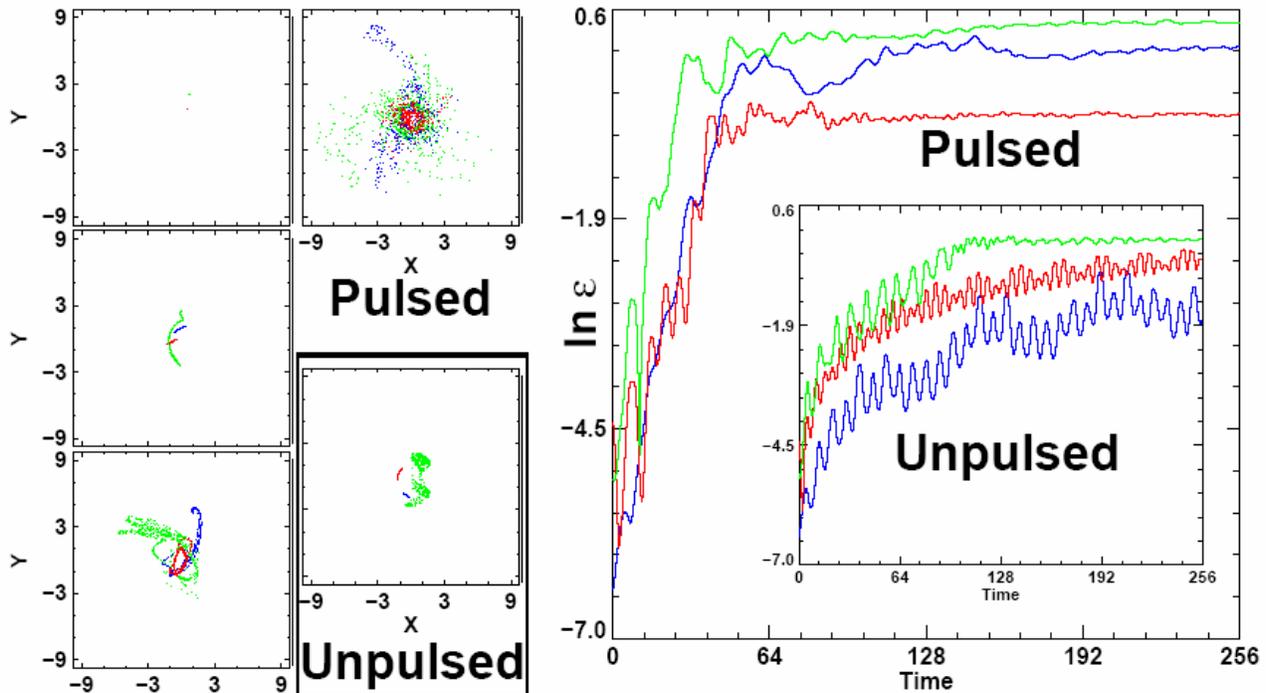


Figure 2. Left-hand and top middle panels:  $x$  and  $y$  coordinates for three different clumps of 1600 test stars evolved in the time-dependent galactic potential  $V(t)=-A(t)/(1+x^2+y^2+z^2)^{1/2}$ , with  $A(t)=1+(a\sin\omega t)/(1+t/t_0)^2$ , for  $a=0.5$ ,  $\omega=1.25$ , and  $t_0=100$ . From top to bottom, the snapshots are at times  $t=0$ , 32, 64, and, for the top middle panel,  $t=128$ . The dynamical time  $t_D \sim 20$ . The clumps had initial size  $\delta x = \delta y = 0.04$ . Bottom middle panel: A snapshot at  $t=128$  for the same clumps evolved in a time-independent potential with  $A=1$ . Right-hand panel: The quantity  $\varepsilon = (\varepsilon_x^2 + \varepsilon_y^2)^{1/2}$  for the time-dependent clumps where, in terms of velocity components, the emittances  $\varepsilon_x$  and  $\varepsilon_y$  satisfy, e.g.,  $\varepsilon_x^2 = \langle x^2 \rangle \langle v_x^2 \rangle - \langle xv_x \rangle^2$ . Here the angular brackets represent an average over the 1600 stars. Right-hand panel (inset): Same for  $A=1$ .

#### 4. Experimental evidence for chaos in galaxies

There can, of course, be no direct experimental evidence for chaos in galaxies. However, careful analysis of observable velocity fields in suitably oriented galaxies provides compelling evidence that the gas flows in such spirals as NGC 3632 could be chaotic, especially near various resonances<sup>44</sup>.

### B. Chaos in charged-particle beams

Intense charged-particle beams are, like galaxies, typically collisionless Hamiltonian systems wherein the density distribution self-consistently governs the dynamics via Poisson's equation. Transients in the beam distribution often arise as the accelerator manipulates the beam, whereby questions of equilibration, damping, and reversibility become fundamentally important to establishing and preserving the desired phase-space properties of the beam. For example, equipartitioning of anisotropic beams involves nonlinear energy transfer and evolution towards an isotropic metaequilibrium state<sup>45</sup>. As will be shown, this is a consequence of chaotic mixing. Strictly speaking, chaotic mixing is a reversible process in that it is governed by Vlasov's equation. However, an essential question for the accelerator designer is whether this process is *operationally*

reversible. While it may be possible in principle to compensate operationally against phase-space dilution<sup>46</sup>, this compensation must be completed before any mixing has smeared a significant number of particles through global regions of phase space<sup>9,47</sup>. The question then becomes one of time scales. It arises regarding any process for manipulating a beam with space charge, be this changing the beam's transverse geometry (flat-to-round or round-to-flat transformations<sup>48</sup>), its longitudinal geometry (bunch compression<sup>49</sup>), or controlling the beam through sudden changes or imperfections in the beamline<sup>50</sup>.

### 1. Emittance and space charge

Consider, for simplicity, an infinitely long, i.e., direct-current, beam that coasts without acceleration in the  $z$ -direction while confined by an external transverse focusing force. It is then natural to compute particle dynamics in a reference plane that comoves with the beam and is oriented transversely with respect to the beam motion. The particle velocities may in general exhibit both a systematic and a random component. Regarding the former, the  $(x,y)$ -coordinates are then measured from the beam centroid. Regarding the latter, an average kinetic temperature can then be defined for each transverse  $(x,y)$ -axis. Roughly speaking, the product of this temperature and the rms beam size is defined as the 'rms emittance' of the beam, and this quantity is conserved for the special case that the  $(x,y)$ -components of the total transverse force (focusing plus space charge) are decoupled, linear, and time-independent in the reference frame comoving with the beam. More precisely, the rms emittance for the  $x$ -direction is calculated as

$$\varepsilon_x = \frac{1}{\beta\gamma mc} \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle xp_x \rangle^2}, \quad (9)$$

where the averages involve moments defined with respect to the single-particle distribution function  $f(x,p_x,y,p_y)$ . Here  $(p_x,p_y)$  are the components of the transverse particle momentum with respect to the reference trajectory,  $\beta=v_z/c$ ,  $\gamma=(1-\beta^2)^{-1/2}$ ,  $m$  is the particle rest mass, and  $c$  is the speed of light. The 'effective emittance', often called simply 'emittance', is  $4\varepsilon_x$ , a quantity that corresponds to the area of the  $(x,p_x)$  phase space subtended by the beam and which has units of length. The respective emittances in the  $y$ -direction are calculated analogously.

Suppose the external transverse focusing force is linear, axisymmetric, and time-independent. For a beam with a small number of particles, the individual particles will oscillate harmonically (they execute what accelerator physicists call 'betatron oscillations') at the 'undepressed' betatron frequency,  $\omega_{\beta_0}$ , determined solely by the external force. The amplitude of the oscillation for each particle depends on its total energy, and this is determined completely by the initial position and velocity of that particle. Now, as the beam current is increased, the superimposed electric field generated by the particles themselves becomes non-negligible, a phenomenon known as space charge. Space charge alters the net force seen by the individual particles in a way that is generally nonlinear and dependent on the density distribution of the beam. One can quantify space charge using a single parameter: the dimensionless intensity parameter  $\mu$  defined as the ratio of the average space charge force to the external focusing force at the beam edge.

Since space charge is repulsive, it lowers the frequency of the betatron oscillations, resulting in a 'depressed' betatron frequency  $\omega_\beta < \omega_{\beta_0}$ . The average 'tune depression', defined as  $\eta \equiv \omega_\beta/\omega_{\beta_0}$ , is related to the intensity parameter by the formula  $\eta = (1 - \mu)^{1/2}$ . Another important effect of space charge is the tendency to induce waves in the beam, a collective effect. These waves are characterized by the plasma frequency  $\omega_p$ , which in turn relates to the intensity parameter  $\mu$  as  $\omega_p =$

$(2\mu)^{1/2}$ . Thus, in the limit of zero space charge ( $\mu = 0$ ), the plasma frequency is zero, the tune depression is unity, and the particles behave as individual particles that only see the external focusing force. At the opposite end, the space charge limit ( $\mu = 1$ ), the tune depression is zero, and the plasma frequency reaches its maximum value, meaning that collective oscillations dominate over the individual particles' betatron motion. At intermediate values of  $\mu$ , excepting certain contrived theoretical distributions, e.g., the KV distribution<sup>26</sup>, the net force acting on individual particles is typically nonlinear. Inasmuch as real beams are commonly out of equilibrium and subject to time-dependent focusing, the net force is often time-dependent as well.

## 2. Chaos and equipartitioning

Anisotropy in a beam can be caused by essentially any anisotropic external influence, such as anisotropic focusing. In addition, a recent computational study provided strong evidence that chaotic mixing due to nonlinear forces from space-charge waves is intimately connected with equipartitioning, i.e., the tendency of the velocity ellipsoid (or equivalently, the temperature) to isotropize rapidly<sup>51</sup>. These computations were done using the '2+1/2'-dimensional\* version of the particle-in-cell code WARP<sup>52</sup>, which tracks macroparticles with prespecified initial conditions through external electric and magnetic fields while including the self-consistently computed self-fields. The work concerned a highly space-charge-dominated, direct-current, cylindrical beam in which the initial momentum space reflected a temperature anisotropy. Accordingly, the initial rms emittances  $\varepsilon_x$  and  $\varepsilon_y$  were unequal, but the external focusing was axisymmetric. As the beam evolved, the temperature isotropized rapidly. Full equipartitioning occurred within just  $\sim 5$  m, after which the temperature exhibited anisotropic oscillations that largely damped by  $\sim 50$  m. The equipartitioning time scales were found to correlate with the evolution of initially localized clumps of globally chaotic particles. These clumps dispersed exponentially with an  $e$ -folding 'time'  $\sim 2$  m (roughly two plasma periods) and filled their accessible phase spaces in  $\sim 50$  m.

This first study concerned a form of 'symmetry breaking', with the broken symmetry appearing in momentum space rather than configuration space. The beam began in a nonequilibrium state and evolved toward a metaequilibrium in which the particle orbits filled an invariant measure of phase space. The transient dynamics reflected an intricate, evolving network of space-charge waves that set up a complicated time-dependent potential in which a substantial population of particle orbits became globally chaotic. By contrast, an analogously evolved symmetric, isotropic system exhibited a near-static potential that was essentially integrable, so that the orbits were essentially regular. The character of the orbits is evident in Fig. 3, which shows trajectories of 20 test particles randomly selected from a given initially localized clump in both isotropic and anisotropic systems. Progressively reducing the area of the phase space initially spanned by the clump, as would be done in a calculation of finite-time Lyapunov exponents, reveals that the test-particle orbits are regular in the isotropic beam. However, the orbits are clearly chaotic in the anisotropic beam, this reflecting the complicated network of space-charge waves that arise in the presence of anisotropy. Equipartitioning did not lead to a significant halo because the rms properties of the beam were 'matched' to the strength of the focusing forces, thereby minimizing large-scale time-dependent oscillations\*.

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\*Distance down the accelerator is viewed as a 'time' coordinate; hence the appellation '2+1/2'-dimensional.

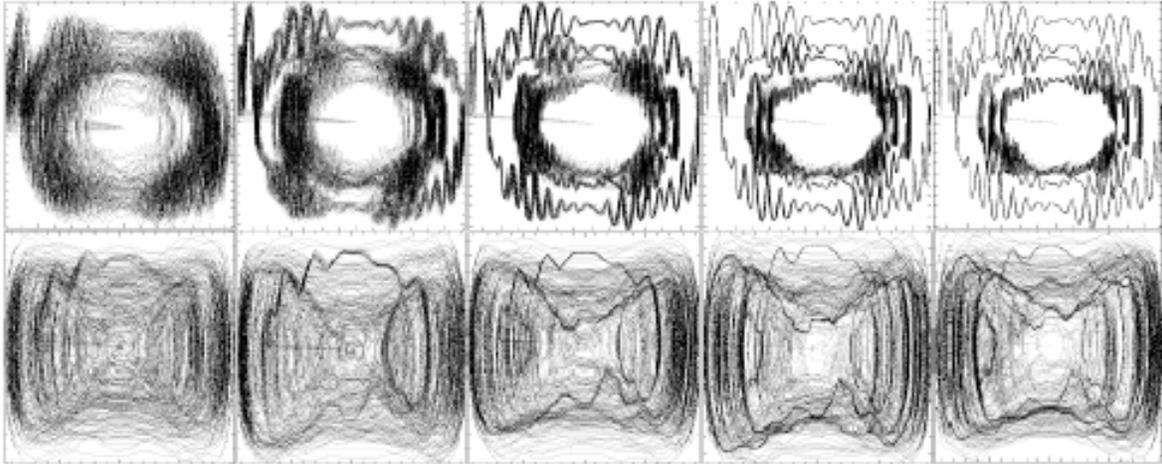


Figure 3. Trajectories of 20 test particles in  $(x,y)$ -space for the isotropic beam (top); and the anisotropic beam (bottom). The initial clump ‘emittance’ decreases progressively by factors of 10 from left to right, thereby exposing the chaotic behavior of orbits in the anisotropic beam.

### 3. Merging beamlets: analysis of a real laboratory experiment

A unique laboratory experiment concerning violent relaxation in charged-particle beams, conducted in the early 1990s, involved the propagation and merging of five beamlets in a periodic solenoidal (hence axisymmetric) transport channel of length slightly more than 5 m<sup>53-55</sup>. The beamlets were initially oriented in a quincunx pattern and were close enough to each other that mutual interactions were important. The beam was nonrelativistic and subject to considerable time-dependent space charge. Given such a highly anisotropic initial density distribution and isotropic focusing, and considering that the time scale for collisional relaxation is orders of magnitude longer than the transport channel, one might naively expect the beamlets to merge (hence, ‘disappear’) and reappear periodically. However, the beamlets were observed to reappear *only once*, at a point  $\sim 1$  m from the source, regardless how well (or poorly) the rms beam properties were matched to the transport channel. Moreover, rms-mismatched beams led to the formation of an extended halo, with the density of the halo increasing with the degree of mismatch. Detailed simulations with a particle-in-cell code successfully reproduced the measurements<sup>55</sup>. The failure of the beamlets to reappear again would seem to reflect a collisionless process that, in effect, causes the particle orbits to lose memory of their initial conditions.

To explore to what extent chaotic mixing influences the dynamics of such a manifestly nonequilibrium beam, we redid the simulations using WARP. Our new simulations differ slightly from the experiment in that we considered a simpler transport channel, one that imparts a constant, linear external focusing force, whereas in the experiment the channel constituted a periodic solenoidal focusing lattice. We used a total of  $4 \times 10^6$  particles distributed equally between each of the five beamlets. The idea was to generate a reasonably smooth potential. Our results correlated

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\*A beam is said to be ‘matched’ to the transport channel if its transverse density profile is stationary over the length of the channel. Otherwise, the density profile evolves. Consider the rms transverse radius of an evolving density profile. If the rms radius is stationary (equaling that of the matched beam), then the beam is ‘rms-matched’; otherwise, it is ‘rms-mismatched’. The density profile, hence space-charge potential, is normally more weakly time-dependent in a rms-matched beam than in a rms-mismatched beam. Only in the case of the strictly matched beam will the space-charge potential be stationary.

well with measurements of the density profile versus position along the beamline.

One might expect the strongly time-dependent space-charge potential to drive a large population of globally chaotic orbits. That this is the case is illustrated in Fig. 4, which shows that clumps of representative test particles initially localized in phase space grew exponentially to fill much of their accessible phase-space regions. In each case, an initial extremely fast growth rate subsequently gave way to a slower rate, the transition occurring after a distance  $\sim 5$  m at which time the beamlets had lost their identity and the phase-space distribution had become rounder. This computational finding is completely in keeping with the experimental findings.

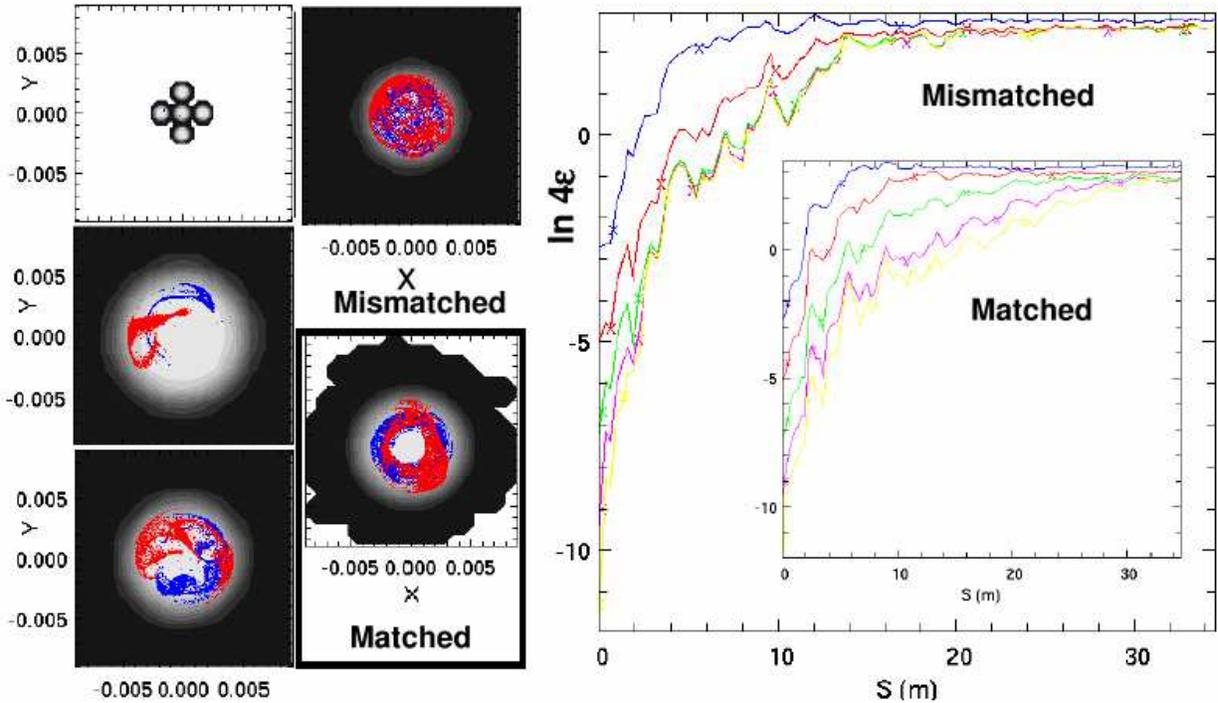


Figure 4. Left-hand and middle panels:  $(x,y)$  plots (unit = m) for two different initially localized clumps of 20,000 test particles evolved in the total potential self-consistently computed using WARP. The snapshots are taken from a simulation of an rms-mismatched beam, at locations  $s=0.0, 10.08, 14.4,$  and  $20.16$  m along the beamline, with the exception of the bottom middle snapshot, which is at  $s=31.68$  m and pertains to a simulation with the rms-matched beam. The plasma wavelength is  $0.47$  m and the betatron wavelength is  $2.0$  m. The initial emittance of each clump is  $\varepsilon_x=\varepsilon_y=6.48\times 10^{-10}$  m, which is  $10^{-5}$  the full beam emittance. Right-hand panel: Natural logarithm of the emittance  $4\varepsilon_x$  for 5 clumps, each sampling a progressively smaller portion of the ‘red’ clump on the left, hence progressively smaller initial emittances. Right-hand panel (inset): Same for the rms-matched beam.

#### 4. Halo formation

Los Alamos recently completed a laboratory experiment<sup>56</sup> involving the production and measurement of halo generated in a proton beam that was intentionally mismatched to a periodic focusing channel comprised of quadrupole magnets. The beam energy and current were  $6.7$  MeV and  $75$  mA, respectively, meaning the beam was nonrelativistic and space charge was strong. The length of the focusing channel spanned  $\sim 10$  mismatch oscillations. The principal inferences from this experiment and corresponding simulations<sup>57</sup> were that (i) the phase-space volume of the beam grew in conjunction with the conversion of free energy from mismatch into ‘thermal’ energy of the

beam, and (ii) parametric resonance was the principal mechanism driving halo formation. These inferences correspond to expectations from idealized theoretical models<sup>58,59</sup>. However, the quantitative data appeared to be sensitive to the exact phase-space distribution of the input beam, which could not be measured with precision, and the finite sensitivity of the diagnostics precluded characterization of the tenuous outermost wings of the halo profile. Moreover, the theoretical models provide no prediction of growth rates; the simulations were used to extract this information for comparison with the experiment.

As is documented in a companion paper in these Annals<sup>28</sup>, recent work has revealed that parametric resonance is not the whole story respecting halo formation<sup>60-62</sup>. The presence of colored noise (noise with a nonzero autocorrelation time), a real phenomenon associated with hardware imperfections and/or charge-density fluctuations, in combination with parametric resonance, can lead to much larger halos and remove the hard upper bound to the halo amplitude inferred from parametric resonance acting alone. Basically, this happens because the noise can keep a statistically small number of particles more in phase with low-order oscillatory modes of the beam. Here the fact that the orbits are chaotic is extremely important. Because chaotic orbits have power over a continuous set of frequencies, their coupling to both the modes and the noise can be significantly enhanced relative to the couplings which would arise for regular, multiply periodic orbits.

Recent theoretical studies indicate that precisely the same phenomenology applies to galaxies, as well. Bohn and Sideris<sup>60</sup> found that substantial halo appears in gravitational systems as well as in beams. Colored noise in galaxies arises from the ambient intergalactic environment through the influence of neighboring stellar systems and/or clumpy dark-matter halos, as well as from internal density fluctuations within the subject galaxy. Subsequent work indicated that galactic halo formation is insensitive to the details of the bulk potential; generally all that is required are collective modes and noise<sup>63</sup>. This finding raises interesting questions: Are the observed light profiles of real galaxies primarily the product of violent relaxation at early epochs? Or can remanent oscillations act over a Hubble time to alter substantially the product of violent relaxation alone? An effort toward answering these questions is underway, and a preliminary study indicates that long-time evolution and its associated halo formation can indeed influence observational properties of large galaxies<sup>64</sup>.

### 5. *The smooth-potential approximation*

The foregoing discussion implies that, viewed ‘on the whole’, discrete systems of stars or charged particles, if sufficiently large, can be approximated by a continuous density distribution and a smooth bulk potential. As pointed out in the aforementioned companion paper<sup>28</sup>, this assumption has been questioned in both the galactic<sup>65</sup> and accelerator<sup>66,67</sup> communities. To what extent is it really true that there actually *is* a smooth continuous-density limit? And assuming this limit exists, how large must the system be before discreteness effects (i.e., granularity associated with finite particle number) can safely be ignored? Can one, e.g., treat a realistic beam bunch comprised of  $10^9$ - $10^{11}$  particles as a continuous charge distribution?

Numerical computations performed over the last several years, for both self-gravitating<sup>68</sup> and self-electrostatically interacting<sup>69</sup> Coulomb systems, suggest strongly that, viewed macroscopically, there is a well-defined continuum limit, and that discreteness effects can be extremely well modeled, even for individual orbits over comparatively short times, by Gaussian white noise in the context of a Fokker-Planck description. Indeed, one can estimate smooth-potential Lyapunov exponents from  $N$ -body simulations<sup>70</sup>.

That a Fokker-Planck description can be justified is nontrivial since the standard derivations<sup>4</sup> and most experimental tests focus on the long-time behavior of orbit ensembles. Even more interesting, however, is the fact that, when applied to chaotic systems, a Fokker-Planck description implies that discreteness effects can be important on time scales much shorter than the collisional relaxation time  $t_R$ ! Discreteness effects can dramatically accelerate diffusion through a complex phase space, both by facilitating transport along the Arnol'd web<sup>5</sup> and, in some cases, by transforming regular orbits into chaotic orbits and vice versa. Indeed, under certain circumstances, e.g., for systems with ‘lumps’ and/or asymmetries and/or pronounced density gradients, discreteness effects can be important on relatively short time scales even for  $N$  as large as  $\sim 10^{10}$ !<sup>69</sup>

It is important to stress that, even if discreteness effects become important over comparatively short time scales, their effect is *not* to induce collisional relaxation. Suppose, e.g., that, in the absence of discreteness effects, the bulk potential, albeit chaotic, is strictly time-independent and the energies of individual ‘particles’ are thereby conserved absolutely. Discreteness effects can then act to accelerate diffusion through a complex phase space, serving as a source of what nonlinear dynamicists call ‘extrinsic diffusion’<sup>5</sup>; but, over time scales much smaller than  $t_R$ , they do *not* induce significant changes in energy.

## 6. Summary

There is growing evidence that physical processes involving chaos act very similarly in galaxies and charged-particle beams. In both cases a time-dependent potential can trigger resonances which lead to large measures of strongly chaotic orbits with large Lyapunov exponents, even if, as for the model used to generate Fig. 2, the potential becomes integrable when the time dependence is ‘turned off.’ Manifestations of chaos can also be quite similar in time-independent systems. For example, a systematic investigation<sup>9</sup> of how the amount of chaos in a thermal-equilibrium beam<sup>71</sup> varies with the beam’s geometry yields results very similar to what is found<sup>72</sup> in triaxial generalizations of the Dehnen potentials of galactic dynamics<sup>73</sup> that have been proposed to model cuspy, triaxial galaxies<sup>74</sup>.

## V. PLANS FOR FUTURE EXPERIMENTS

Charged-particle beams differ from galaxies in that beams will adjust themselves to screen the external focusing force. The screening distance is the Debye length, and in a configuration of thermal equilibrium, the density profile in the outer region of the beam decreases to a low-density tail over a few Debye lengths as a result of screening the external focusing force<sup>9</sup>. By contrast, galaxies do not exhibit any analog of this Debye shielding\*. Consequently, with a beam, the bulk potential (focusing plus space charge) cannot generally be molded to match precisely that of an evolving stellar system. For example, structures mimicking the presence of central density cusps or black holes in galaxies cannot be preserved in a beam because space charge is repulsive. Nor can a beam mimic effects from space-time evolution over cosmological time scales. However, phenomenology inherent to time-dependent collective dynamics in galaxies *can* be mimicked with beams. Gravitational examples (and their beam analogs) include colliding and merging galaxies (merging beamlets), collapsing galaxies (rms-mismatched beams), or perturbed but comparatively quiescent galaxies (beams with evolving density inhomogeneities). Thus, the key dynamics

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\*Concerning gravitation, the length scale of interest is the Jeans length over which a gravitational instability arises, thereby leading, e.g., to the formation of galaxies.

underlying galactic systems can be studied in the laboratory.

As implied in Sec. IV.B, to date there have been no laboratory experiments designed explicitly to explore the role of chaotic phase mixing via Coulomb forces on the evolution of nonequilibrium beams. Our simulations and interpretation of the merging-beamlet experiment in Sec. IV.B.3 point, however, to the importance of chaotic dynamics in real beams. And the preponderance of simulations that we have highlighted herein suggests strongly that the combined effects of transient chaos and resonances are the keys toward a full understanding of violent relaxation in both beams and galaxies. Accordingly, we are planning a series of experiments to study phase mixing and attendant collisionless relaxation using the University of Maryland Electron Ring (UMER), a facility that is just now coming on line<sup>75</sup>. The ring is designed to transport the beam through many turns spanning over 1 km, a distance spanning some 500-1000 plasma periods, and the relative strength of the collective space-charge force is adjustable over a wide range,  $0.25 \leq \mu \leq 0.97$ .

The evolution and mixing of initial perturbations can be tracked using the comprehensive suite of diagnostics incorporated into UMER. As a whole, the diagnostic suite permits direct measurement of mixing time scales in units of the characteristic dynamical time, and the degree of mixing in both configuration space and in energy, by enabling the evolution of macroscopic features to be observed and quantified. It should thereby be possible to distinguish observationally between chaotic (i.e., exponential, global) phase mixing versus regular (i.e., secular, more local) phase mixing. We also plan, of course, to confirm our interpretations using simulation codes. We project an added benefit, as well: establishing the phenomenology of phase mixing in time-dependent beam potentials both experimentally and numerically should likewise provide an unambiguous mechanism for validating codes and simulation techniques in both beam physics and galactic dynamics.

## VI. CONCLUSIONS

It is clear that, in principle, chaotic mixing can account for rapid macroscopic dynamics, including collective ‘relaxation’ to a metaequilibrium state. Moreover, there is substantial numerical evidence that such mixing could play an important role in the evolution of both galaxies and charged-particle beams. While a portion of this numerical evidence arose historically as part of interpreting real laboratory experiments with beams, there is need for considerably more work. Our idea is to look for evidence of chaos and chaotic phase mixing in controlled laboratory experiments involving large Coulomb systems. Unfortunately and obviously, it is impossible to perform controlled experiments on self-gravitating systems like galaxies. However, in view of the strong indications, both theoretical and numerical, that the relevant physics is virtually identical in galaxies and charged-particle beams, it seems possible – and highly desirable – to use beamlines like UMER as laboratories in which to perform indirect tests of the predictions of galactic dynamics. The key quantities to be measured in such experiments are the evolutionary time scales attendant to charged-particle beams with well-diagnosed and freely adjustable initial conditions, as well as the efficacy of mixing in both configuration space and energy.

The suite of diagnostics on UMER is capable of detailed, time-resolved measurement of the distribution function in the six-dimensional phase space of a single beam particle. These diagnostics are designed to measure the same macroscopic observables and their respective evolutionary time scales as are generated in numerical simulations. Accordingly, UMER serves as a platform for a virtually unlimited range of experiments to explore nonlinear, transient dynamics

of Coulomb systems, and our overarching plan is to exploit this capability to access the physics of collisionless relaxation that large charged-particle and self-gravitating stellar systems share in common.

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