FERMILAB-Pub-03/323-A

Inflationary potentials yielding constant scalar perturbation spectral indices

Alberto Vallinotto
Physics Department, The University of Chicago, Chicago, Illinois 60637-1433, USA,
and Dipartimento di Fisica, Politecnico di Torino,
C.so Duca degli Abruzzi 24, 10129 Torino, Italy.

Edmund J. Copeland
Department of Physics and Astronomy, University of Sussex, Brighton BN1 9QH

Edward W. Kolb
Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, Illinois, 60510-0500, USA,
and Department of Astronomy and Astrophysics, Enrico Fermi Institute,
University of Chicago, Chicago, Illinois 60637-1433, USA

Andrew R. Liddle
Astronomy Centre, University of Sussex, Brighton BN1 9QH
(Dated: October 31, 2003)

We explore the types of slow-roll inflationary potentials that result in scalar perturbations with a *constant* spectral index, *i.e.,* perturbations that may be described by a single power-law spectrum over all observable scales. We devote particular attention to the type of potentials that result in the Harrison–Zel'dovich spectrum.

PACS numbers: 98.80.Cq

I. INTRODUCTION

Inflation, a cornerstone of our modern framework for understanding the development of the early universe [1, 2], predicts the initial conditions for the formation of structure, as well as the Cosmic Microwave Background (CMB) temperature anisotropies. Primordial scalar (density) and tensor (gravitational wave) fluctuations are generated by quantum fluctuations, 'redshifted' beyond the Hubble radius during inflation and become frozen as perturbations in the background metric [3, 4, 5, 6, 7]. However the number of inflation models that have been proposed in the literature is quite large [2], even if we restrict ourselves to models with only one scalar field, the *inflaton*. In principle, determination of the properties of the scalar perturbations and tensor perturbations from CMB and large-scale structure observations allows one to place interesting constraints on the space of possible inflation models [8, 9, 10, 11, 12, 13, 14].

It is often adequate to characterize the inflationary perturbations in terms of four numbers: the amplitudes of the scalar and tensor spectral indices at some conveniently chosen scale, the scalar spectral index n and the tensor spectral index.[1] Successful inflation models predict a scalar spectral index close to $n = 1$, the so-called Harrison–Zel'dovich model. Typically, inflation models predict a rather small scale dependence of the spectral index; *i.e.,* the scalar perturbation spectrum should be very nearly a power-law, and it is fair to say that a *constant* spectral index remains a viable possibility, as does the $n = 1$ Harrison–Zel'dovich value.

It is known that within the slow-roll approximation [15] (discussed in the next section) inflaton potentials of the form $V(\phi) = \exp(-\alpha\phi)$ for constant $\alpha^2 < 2$ lead to perturbation spectra that are exact power laws [16]. However there has not as yet been a systematic analysis of the types of inflaton potentials that yield exact power-law scalar perturbations or the Harrison–Zel'dovich spectrum. This paper is meant to be the first step in that direction, classifying those potentials within the framework of the slow-roll approximation.

In the next section we review the basic results employed to calculate the properties of the perturbation spectrum using the slow-roll parameterization of the inflaton potential. In Sec. III.A we derive the properties of potentials that to lowest order in slow-roll parameters produce the Harrison–Zel'dovich ($n = 1$) spectrum. In Sec. III B we perform a similar analysis for potentials that to lowest order in slow-roll parameters produce a perturbation spectrum that is a pure power law with n close to, but not equal to, unity. Section IV is a discussion of the flow of ϵ, whose understanding will allow the reader to connect the present results with power-law inflation and to understand the number of solutions that arise. The development of Sec. V parallels that of Sec. III A, except in Sec. V the calculations are to next order in slow-roll parameters, while the calculations of Sec. III are to lowest order. The conclusions are contained in Section VI. Appendix A is devoted to an outline of the next-order approximation equivalent of Sec. III B.

[1] In this paper we will be concerned with the value of the *scalar* spectral index. Unless explicitly indicated, when we refer to the "spectral index," we mean the scalar spectral index.

II. REVIEW OF BASIC CONCEPTS

A. The Hamilton–Jacobi approach to inflationary dynamics

The dynamics of the standard Friedmann–Robertson–Walker (FRW) universe driven by the potential energy of a single scalar field is usually expressed by the Friedmann equation for flat spatial sections,

$$H^2 = \frac{8\pi}{3M_p^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right], \tag{1}$$

and by the conservation of energy equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{2}$$

where ϕ is the scalar field driving inflation, $V(\phi)$ is its potential, $M_p = G^{-1/2}$ is the Planck mass and $H = \dot{a}/a$ is the Hubble expansion parameter.

An alternative approach to the inflationary dynamics is given by the Hamilton–Jacobi formulation [17]. In this approach the Hubble expansion parameter $H(\phi)$ is considered the fundamental quantity to be specified. The corresponding potential may be expressed by a reformulation of the Friedmann equation as

$$\left[\frac{dH(\phi)}{d\phi}\right]^2 - \frac{12\pi}{M_p^2}H^2(\phi) = -\frac{32\pi^2}{M_p^4}V(\phi). \tag{3}$$

The dynamics of the inflaton field is then obtained by solving the differential equation

$$\dot{\phi} = -\frac{M_p^2}{4\pi}\frac{dH(\phi)}{d\phi}. \tag{4}$$

Finally, once the dynamics of the inflaton field is obtained, the time evolution of the scale factor can be computed by integrating the Hubble expansion parameter $H[\phi(t)]$. Without loss of generality we will always take $\dot{\phi} > 0$.

In the Hamilton–Jacobi formulation of inflationary dynamics, the slow-roll parameters ϵ, η and ξ^2 are defined as [18]

$$\epsilon(\phi) \equiv \frac{3\dot{\phi}^2}{2}\left[V(\phi) + \frac{\dot{\phi}^2}{2}\right]^{-1} = \frac{M_p^2}{4\pi}\left[\frac{H'(\phi)}{H(\phi)}\right]^2, \tag{5}$$

$$\eta(\phi) \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{M_p^2}{4\pi}\frac{H''(\phi)}{H(\phi)}, \tag{6}$$

$$\xi^2(\phi) \equiv \frac{M_p^4}{16\pi^2}\frac{H'(\phi)H'''(\phi)}{H^2(\phi)}. \tag{7}$$

The last two parameters are the first members of an infinite hierarchy of slow-roll parameters, whose l-th member is defined by

$$\lambda_H^l(\phi) \equiv \left(\frac{M_p^2}{4\pi}\right)^l \frac{(H')^{l-1}}{H^l}\frac{d^{(l+1)}H(\phi)}{d\phi^{(l+1)}}. \tag{8}$$

The potential and its derivatives can be expressed as *exact* functions of the slow-roll parameters defined above:

$$V(\phi) = \frac{M_p^2}{8\pi}H^2(3-\epsilon), \tag{9}$$

$$\frac{dV(\phi)}{d\phi} = -\frac{M_p}{2\sqrt{\pi}}H^2\sqrt{\epsilon}(3-\eta), \tag{10}$$

$$\frac{d^2V(\phi)}{d\phi^2} = H^2\left[3\epsilon + 3\eta - (\eta^2 + \xi^2)\right]. \tag{11}$$

B. A Hierarchy of Approximation Orders

As reported by Stewart and Lyth [19], the expressions for the power spectra and for the spectral indices depend on the approximation order assumed. This can be formalized by saying that each set of expressions for the power spectra and the spectral indices refers to a specific order l_0 in the parameter expansion and to a specific *set* of slow-roll parameters $\{\epsilon, \lambda_H^l,\ l = 1, 2, ..l_0\}$ consistent with such an order. For instance, the *lowest-order* approximation [20] corresponds to $l_0 = 1$ and it refers to the set of parameters $\{\epsilon, \eta \equiv \lambda_H^1\}$. This also implies that the equations for the power spectra and the spectral indices will only involve terms up to an overall first degree in these parameters. The *next-order* approximation, on the other hand, corresponds to $l_0 = 2$ and it is associated with the enlarged set of parameters $\{\epsilon, \eta, \xi^2 \equiv \lambda_H^2\}$. In this case, terms up to an overall second degree are retained in the expressions for the power spectra and the spectral indices. It is important to point out that in general the λ_H^l parameter is considered to give a contribution of degree l to the factor wherein it appears. As a consequence, whenever an approximate expression is inserted into an exact relation, *order consistency* requires that such a relation is expanded in a power series of slow-roll parameters and that terms are retained only up to an overall degree consistent with the level of approximation assumed.

Recalling Lidsey *et al.* [20], we can therefore think of an infinite hierarchy of expressions for the perturbation spectra and for the spectral indices. It is unfortunate that, due to the complexity of the problem, only the first two approximation orders, known as *lowest-order* and *next-order*, are currently available in the general case.

Following Stewart and Lyth [19], the fundamental quantities arising from the approximate solution of the quantum fluctuation differential equations are the scalar and tensor perturbation power spectra, denoted by $\mathcal{P}_{\mathcal{R}}^{1/2}$ and $\mathcal{P}_g^{1/2}$ respectively. The *rescaled* scalar and tensor power spectra $A_S(k)$ and $A_T(k)$ are defined in terms of the above quantities by

$$A_S(k) \equiv \frac{2}{5}\mathcal{P}_{\mathcal{R}}^{1/2}(k), \tag{12}$$

$$A_T(k) \equiv \frac{1}{10}\mathcal{P}_g^{1/2}(k). \tag{13}$$

The scalar and tensor perturbation spectral indices, denoted by $n(k)$ and $n_T(k)$ are then defined in terms of the corresponding rescaled power spectra by

$$n(k) - 1 \equiv \frac{d \ln A_S^2(k)}{d \ln k}, \tag{14}$$

$$n_T(k) \equiv \frac{d \ln A_T^2(k)}{d \ln k}. \tag{15}$$

Depending on the approximation order assumed, different expressions for $\mathcal{P}_{\mathcal{R}}^{1/2}$ and $\mathcal{P}_g^{1/2}$ arise, which then produce more and more accurate expressions for the quantities defined by Eqs. (12-15). To *next to lowest-order* ($l_0 = 2$), the scalar and tensor rescaled power spectra are given by

$$A_S(k) \simeq \frac{4}{5}[1 - \{(2C+1)\epsilon + C\eta\}] \left.\frac{H}{M_p}\frac{H}{M_p|H'|}\right|_{k=aH} \tag{16}$$

$$A_T(k) \simeq \frac{2}{5}\frac{1}{\sqrt{\pi}}[1 - \{(C+1)\epsilon\}] \left.\frac{H}{M_p}\right|_{k=aH}, \tag{17}$$

where $C \simeq -0.73$ is a constant which enters in the derivation of $\mathcal{P}_{\mathcal{R}}^{1/2}$ and $\mathcal{P}_g^{1/2}$ due to the expansion of a factor involving the function $\Gamma(\epsilon, \eta)$ (cf. [19]). As in Lidsey *et al.* [20], throughout this work we will use the symbol "$\simeq$" to stress the fact that the relation holds true *to the order of approximation currently assumed.* The corresponding expressions for the scalar (n) and tensor (n_T) spectral indices to *next to lowest-order* are

$$\begin{aligned} n(k) - 1 &\simeq -4\epsilon + 2\eta - \{8(C+1)\epsilon^2 \\ &\quad + (6+10C)\epsilon\eta - 2C\xi^2\}, \end{aligned} \tag{18}$$

$$n_T(k) \simeq -2\epsilon - \{2\epsilon^2(3+2C) + 4(1+C)\epsilon\eta\}. \tag{19}$$

To recover the *lowest-order* corresponding results it is sufficient to set the terms within the $\{\cdots\}$ brackets to zero.

C. The Parametrization Method

In general only the knowledge of the infinite set of slow-roll parameters of Eq. (8) allows the determination of $H(\phi)$, and hence of the potential $V(\phi)$. The case of constant $n(k)$ is remarkable because, to any order of approximation l_0, requiring the scalar spectral index to be k-independent endows the problem with an additional set of $(l_0 - 1)$ relations, stemming from the fact that $d^{(i)}n(k)/d(\ln k)^{(i)} = 0$ to any order i, which allows the expression of *all* the slow-roll parameters as functions of a single one, which is chosen to be ϵ. It is then possible to exploit this fact to determine the potential as functions of ϵ and the field ϕ.

The method outlined above is implemented through three phases. First, the conditions $n(k) = const.$ and $d^{(i)}n(k)/d(\ln k)^i = 0$, $i = 1, 2, ..., (l_0 - 1)$ may be used to express all the slow-roll parameters as functions of ϵ. Second, an expression for the potential as a function of ϵ consistent with the order of approximation assumed is derived. Third, a differential equation connecting ϕ and ϵ is integrated to relate the field to the slow-roll parameter.

III. LOWEST-ORDER ANALYSIS

A. Harrison–Zel'dovich: $n(k) = 1$

Imposing $n = 1$ to lowest order in Eq. (18) implies $-4\epsilon + 2\eta = 0$, which relates η to ϵ:

$$\eta(\epsilon) = 2\epsilon. \tag{20}$$

To obtain $V(\epsilon)$, note that Eq. (17) can be inverted in order to express H as a function of $A_T(k)$:

$$H^2 \simeq \frac{25 M_p^2 \pi}{4} A_T^2. \tag{21}$$

From Eq. (14), it is then straightforward to note that the $n(k) = 1$ constraint also implies

$$A_S^2(k) = A_S^2(k_0). \tag{22}$$

Furthermore, to *lowest-order* approximation the rescaled power spectra ratio can be related to ϵ by [20]

$$\epsilon \simeq \frac{A_T^2}{A_S^2}. \tag{23}$$

We can then use Eq. (23) together with Eqs. (21, 22) to express H^2 as a linear function of ϵ:

$$H^2(\epsilon) \simeq \frac{25\pi M_p^2}{4} A_S^2(k_0)\epsilon. \tag{24}$$

As we will see in Sec. V, the *next-order* expression analogous to Eq. (24) will acquire a term proportional to ϵ^2.[2] Substituting Eq. (24) into the exact form for the potential, Eq. (9), and expanding in the slow-roll parameter we obtain

$$\begin{aligned} V(\epsilon) &= \frac{25 M_p^4 A_S^2(k_0)}{32}\,\epsilon(3-\epsilon) \\ &\simeq \frac{25 M_p^4 A_S^2(k_0)}{32}\,3\epsilon. \end{aligned} \tag{25}$$

Again, this expression is not general since it relies on Eqs. (17, 23), which take different form depending on the level of approximation assumed. Furthermore, order consistency requires that terms of degree ϵ^2 and higher are neglected.

[2] This equation is also expressing the fact that the quantity that needs to be expressed as power series of the slow-roll parameters is H^2 and *not* H.

It is now important to find a relation between ϵ and ϕ. From Eq.(5), we have both

$$\frac{d\epsilon}{d\phi} = \frac{2M_p^2}{4\pi}\left[\frac{H'H''}{H^2} - \left(\frac{H'}{H}\right)^3\right]. \quad (26)$$

and

$$\frac{H'(\phi)}{H(\phi)} = -\left(\frac{4\pi}{M_p^2}\right)^{1/2}\sqrt{\epsilon}, \quad (27)$$

where the minus sign is justified by the assumption that $\dot{\phi} > 0$ implies $H'(\phi) < 0$. Similarly from the definition of η, Eq. (6), we also have

$$\frac{H''(\phi)}{H(\phi)} = \frac{4\pi}{M_p^2}\eta. \quad (28)$$

Inserting Eqs. (27) and (28) into Eq. (26), we obtain the the *exact* expression

$$\frac{d\epsilon}{d\phi} = \frac{4\sqrt{\pi}}{M_p}\sqrt{\epsilon}(\epsilon - \eta). \quad (29)$$

Imposing the constraint $n(k) = 1$ in the form $\eta = 2\epsilon$ finally yields

$$\frac{d\epsilon}{d\phi} \simeq -\frac{4\sqrt{\pi}}{M_p}\epsilon^{3/2}. \quad (30)$$

This equation can be integrated; letting $\alpha = 4\sqrt{\pi}/M_p$ for convenience,

$$\int d\phi = -\frac{1}{\alpha}\int \epsilon^{-3/2}d\epsilon, \Rightarrow \phi(\epsilon) = \phi(\epsilon_0) + \frac{M_p}{2\sqrt{\pi}}\frac{1}{\sqrt{\epsilon}}. \quad (31)$$

Choosing $\phi(\epsilon_0) = 0$ we then find

$$\phi(\epsilon) \simeq \frac{M_p}{2\sqrt{\pi\epsilon}}. \quad (32)$$

Having expressed the potential and the field as functions of the slow-roll parameter ϵ, it is then possible to solve for the potential as a function of the field. Inserting Eq. (32) into Eq. (25) we obtain

$$V(\phi) \simeq \frac{25M_p^4 A_S^2(k_0)}{32}\frac{3M_p^2}{4\pi\phi^2}. \quad (33)$$

Let's now recall the work of Barrow and Liddle on *intermediate inflation* [21], in which the general dynamics of the scale factor is assumed to be

$$a(t) = \exp\left(At^f\right), \quad (34)$$

with $0 < f < 1$, $A > 0 =$ constant. They prove that this is an exact solution for the intermediate inflation potential

$$V(\phi) = \frac{8A^2}{(\beta+4)^2}\left[\frac{(2A\beta)^{1/2}}{\phi}\right]^\beta \left[6 - \frac{\beta^2}{\phi^2}\right], \quad (35)$$

where $\beta = 4(f^{-1} - 1)$, and is also a solution *in the slow-roll approximation* for the potential

$$V(\phi) = \frac{48A^2}{(\beta+4)^2}\left[\frac{(2A\beta)^{1/2}}{\phi}\right]^\beta. \quad (36)$$

To see how the present results relate to the ones of Barrow and Liddle [21], we first quote the expressions for the slow-roll parameters obtained in the intermediate inflation case:

$$\epsilon = \frac{\beta^2}{2\phi^2}, \quad (37)$$

$$\eta = \left(1 + \frac{\beta}{2}\right)\frac{\beta}{\phi^2}. \quad (38)$$

Exploiting Eqs. (37, 38), the equation for the exact intermediate inflation potential can be recast in the form

$$V(\phi) = \frac{16A^2}{(\beta+4)^2}\left[\frac{(2A\beta)^{1/2}}{\phi}\right]^\beta [3 - \epsilon(\phi)]. \quad (39)$$

Now, in the spirit of this work one can think of this expression as a function of the slow-roll parameter ϵ instead of the field ϕ. In this perspective, neglecting the ϵ in the $(3-\epsilon)$ factor is the same as saying that we are assuming *lowest-order* slow-roll approximation and that by order consistency one should retain only the terms linear in ϵ arising from the expansion of $\phi(\epsilon)^{-\beta}$. In other words, the ϵ appearing in the $(3-\epsilon)$ factor will generate terms of order ϵ^2 and higher, all of which can be consistently neglected in a lowest-order calculation.

Finally, note that imposing the $n(k) = 1$ condition in the form consistent with the lowest-order approximation (that is, $\eta = 2\epsilon$) and using Eqs. (37,38) yields $\beta = 2$ and $f = 2/3$. This is consistent with our calculation, since inserting this value of β into Eq. (36) produces an expression for the inflaton potential analogous to Eq. (33):

$$V(\phi) \sim \frac{3}{\phi^2}. \quad (40)$$

It is also possible to solve for the dynamics of the inflaton field as a function of cosmic time. Eq. (4) gives

$$\phi^2\dot{\phi} = \frac{5M_p^4 A_S(k_0)}{16\pi}. \quad (41)$$

Defining for convenience $\phi_0^3 t_0^{-1} = 15M_p^4 A_S(k_0)/16\pi$, the integration of Eq. (41) yields

$$\phi(t) = \phi_0\ (t/t_0)^{1/3}. \quad (42)$$

A graphical representation of the behavior is shown in Fig. 1. The slow-roll parameter is now

$$\epsilon(t) = \frac{M_p^2}{4\pi\phi_0^2}(t/t_0)^{-2/3}. \quad (43)$$

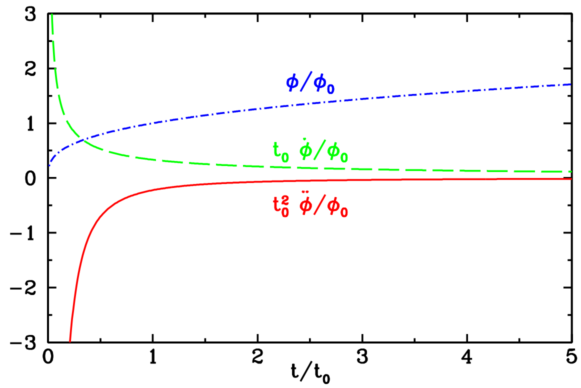

FIG. 1: Evolution of the inflaton field $\phi(t)$ and of its first two time derivatives $\dot\phi$ and $\ddot\phi$.

Similarly, the Hubble parameter and the scale factor are

$$\begin{aligned} H(t) &= H(t_0)\,(t/t_0)^{-1/3}, \\ \frac{a(t)}{a(t_0)} &= \exp\left\{\frac{15M_p^2 A_S(k_0)}{8\phi_0 t_0^{-1}}\left[\left(\frac{t}{t_0}\right)^{2/3}-1\right]\right\}. \end{aligned} \tag{44}$$

Note that the main contribution to the potential is given by the lowest-order term, whereas - as it will be shown in Sec. V - the next-order term contributes a correction which is actually of order ϕ^{-4}, important in the early stages of slow roll but negligible thereafter. Note also the order-of-magnitude relations

$$V(\phi) \sim H^2(\phi) \sim \epsilon(\phi) \sim \frac{A_S^2(k_0)}{\phi^2}, \tag{45}$$

so that a graph for the inflaton potential represents the potential, the square of the Hubble parameter and the slow-roll parameter.

B. Approximately Harrison–Zel'dovich: $n(k) = 1 - 2n_0^2,\ 2|n_0^2| \ll 1$

Having found a solution for the inflaton potential $V(\phi)$ that generates a Harrison–Zel'dovich density perturbation power spectrum, we now generalize the previous results for a k-independent spectral index different from unity. The motivation for this is mainly related to the experimental evidence that the measured spectral index differs from unity by at most a few percent. The following condition is therefore imposed:

$$n(k) = 1 - 2n_0^2. \tag{46}$$

In what follows, the case $n_0^2 > 0$ (and therefore spectral indices that are slightly smaller than unity) is considered in greater detail. The results for $n_0^2 < 0$ can be obtained by analytic continuation, with some care being taken over the number of solutions available in that case.

From Eq. (18), to lowest order we have

$$\eta(\epsilon) = 2\epsilon - n_0^2. \tag{47}$$

We want to express the inflaton potential as a function of the slow-roll parameter ϵ. Eq. (21) can still be used to express H^2 as a function of the rescaled tensor power spectrum A_T^2, but in this case Eq. (22) is no longer valid. Eq. (14) now becomes

$$\frac{d\ln A_S^2(k)}{d\ln k} = -2n_0^2, \tag{48}$$

hence

$$A_S^2(k) = A_S^2(k_0)\left(\frac{k}{k_0}\right)^{-2n_0^2}. \tag{49}$$

From Eq. (23), to lowest-order we have

$$A_T^2(k) \simeq \epsilon A_S^2(k) = A_S^2(k_0)\epsilon\left(\frac{k}{k_0}\right)^{-2n_0^2}. \tag{50}$$

To express the potential as a function of the slow-roll parameter we must express Eq. (50) as a function of ϵ only. From

$$\frac{d\epsilon}{d\ln k} = \frac{d\epsilon}{d\phi}\frac{d\phi}{d\ln k}, \tag{51}$$

and using Eq. (29), together with the exact relation [20]

$$\frac{d\phi}{d\ln k} = \frac{M_p}{2\sqrt{\pi}}\frac{\sqrt{\epsilon}}{(1-\epsilon)}, \tag{52}$$

yields

$$\frac{d\epsilon}{d\ln k} = \frac{2\epsilon(\epsilon-\eta)}{1-\epsilon} \simeq \frac{2\epsilon(n_0^2-\epsilon)}{1-\epsilon}. \tag{53}$$

This can be integrated to give

$$\begin{aligned} \int d\ln k &= \int d\epsilon\frac{\epsilon-1}{2\epsilon(\epsilon-n_0^2)} \\ &= \frac{1}{2n_0^2}\ln(\epsilon) + \frac{1}{2}\left(1-\frac{1}{n_0^2}\right)\ln|\epsilon-n_0^2|. \end{aligned} \tag{54}$$

Since $2n_0^2$ is at most a few percent, it is possible to approximate

$$-\frac{1}{n_0^2} + 1 \approx -\frac{1}{n_0^2}, \tag{55}$$

which then leads to the approximate result

$$\epsilon A_S^2(k) = \epsilon A_S^2(k_0)\left(\frac{k}{k_0}\right)^{-2n_0^2} \approx A_S^2(k_0)|\epsilon-n_0^2|. \tag{56}$$

Hence we obtain

$$H^2(\epsilon) \simeq \frac{25\pi M_p^2 A_S^2(k_0)}{4}|\epsilon-n_0^2|. \tag{57}$$

Using the above result in Eq. (9), the potential to lowest-order in the slow-roll parameter is obtained:

$$\begin{aligned} V(\epsilon) &= \frac{25 M_p^4 A_S^2(k_0)}{32} |\epsilon - n_0^2| (3 - \epsilon) \\ &\simeq \pm \frac{25 M_p^4 A_S^2(k_0)}{32} [\epsilon(3 + n_0^2) - 3n_0^2]. \end{aligned} \tag{58}$$

where the upper sign refers to the $\epsilon > n_0^2$ case and the lower sign to the $\epsilon < n_0^2$ case.

To compute $\epsilon(\phi)$ it is sufficient to insert Eq. (47) into Eq. (29), which yields

$$\frac{d\epsilon}{d\phi} = \frac{4\sqrt{\pi}}{M_p} \sqrt{\epsilon}(n_0^2 - \epsilon). \tag{59}$$

This equation is separable and gives

$$\begin{aligned} \phi(\epsilon) &= \frac{M_p}{2n_0\sqrt{\pi}} \coth^{-1}\left(\sqrt{\frac{\epsilon}{n_0^2}}\right) \quad \text{for } \epsilon > n_0^2, \\ \phi(\epsilon) &= \frac{M_p}{2n_0\sqrt{\pi}} \tanh^{-1}\left(\sqrt{\frac{\epsilon}{n_0^2}}\right) \quad \text{for } \epsilon < n_0^2. \end{aligned} \tag{60}$$

The integration has been carried out exactly in this case, because when ϵ approaches n_0^2 the geometric series expansion of the factor $(n_0^2 - 1)^{-1}$ becomes inaccurate.

The above results can then be inverted to obtain an explicit expression of the inflaton potential with respect to the field (again, the first expression is for $\epsilon > n_0^2$, and the second for $\epsilon < n_0^2$)

$$\begin{aligned} V(\phi) &= \frac{25 M_p^4 A_S^2(k_0)}{32} \\ &\times \left[n_0^2 \coth^2\left(\frac{2n_0\sqrt{\pi}}{M_p}\phi\right)(3 + n_0^2) - 3n_0^2 \right], \\ V(\phi) &= \frac{25 M_p^4 A_S^2(k_0)}{32} \\ &\times \left[3n_0^2 - n_0^2 \tanh^2\left(\frac{2n_0\sqrt{\pi}}{M_p}\phi\right)(n_0^2 + 3) \right]. \end{aligned} \tag{61}$$

Examples of such potentials for $\epsilon > n_0^2$ are illustrated in Fig. 2.

For $n_0^2 < 0$ the corresponding lowest-order results can be derived just retracing the steps taken in the previous subsections. To lowest-order the potential as a function of ϵ is now given by

$$V(\epsilon) \simeq \frac{25\pi M_p^4 A_S^2(k_0)}{32} \left[\epsilon(3 + n_0^2) - 3n_0^2\right], \tag{62}$$

where only one potential form is now available due to the absence of the absolute value $|\epsilon - n_0^2|$. A bit more care is needed to handle the calculation of the field as a function of the slow-roll parameter. Eq. (59) can be integrated in this case to yield

$$\phi(\epsilon) = \frac{M_p}{2\sqrt{\pi |n_0^2|}} \tan^{-1}\left(\sqrt{\frac{\epsilon}{|n_0^2|}}\right), \tag{63}$$

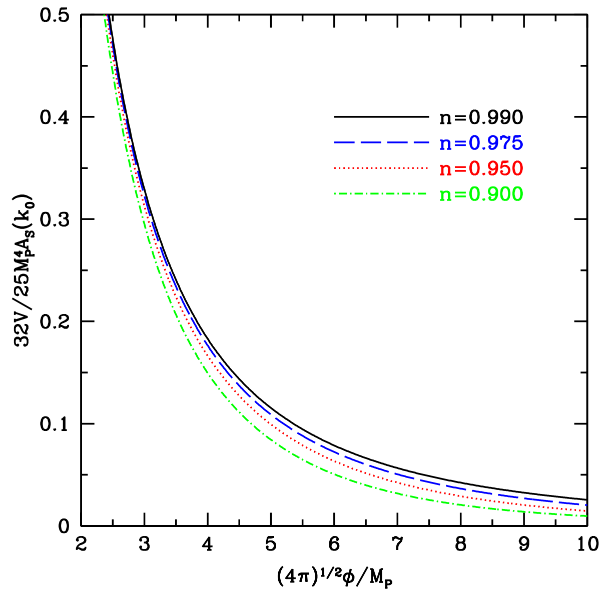

FIG. 2: Four potentials computed to lowest order, yielding density perturbation spectral indices of 0.9, 0.95, 0.975, 0.99.

and for the potential to lowest order

$$\begin{aligned} V(\phi) &\simeq \frac{25\pi M_p^4 A_S^2(k_0)}{32} \\ &\times \left[|n_0^2| \tan^2\left(\frac{2\phi\sqrt{\pi|n_0^2|}}{M_p}\right)(3 + n_0^2) - 3n_0^2 \right]. \end{aligned} \tag{64}$$

At this point it seems rather puzzling that there are two different solutions for the potential arising in the $n_0^2 > 0$ case and only one in the $n_0^2 < 0$ case. To see the reason of this we need to consider the different behavior that Eq. (59) exhibits depending on the initial value of the slow-roll parameter, ϵ_0.

IV. THE FLOW OF ϵ

A. The $n_0^2 > 0$ case

Let's consider again the evolution of $\epsilon(\phi)$ in Eq. (59). Keeping in mind that we assume that the inflaton field is always increasing with respect to cosmic time, it is then interesting to consider the flow of the slow-roll parameter ϵ. It is straightforward to note from Fig. 3, which shows $d\epsilon/d\phi$ as function of ϵ, that $d\epsilon/d\phi$ is positive for $\epsilon < n_0^2$ and is negative for $\epsilon > n_0^2$. One can see that if ϵ_0, the initial value of ϵ, is smaller than n_0^2, then the slow-roll parameter ϵ will increase toward n_0^2, while if the initial value ϵ_0 is greater than n_0^2, then ϵ will decrease toward n_0^2. In the $n_0^2 > 0$ case, then, independent of its initial value ϵ_0, ϵ will tend toward the point $\epsilon = n_0^2$. Finally, if $\epsilon_0 = n_0^2$, the value of ϵ will remain constant as the field evolves with respect to time.

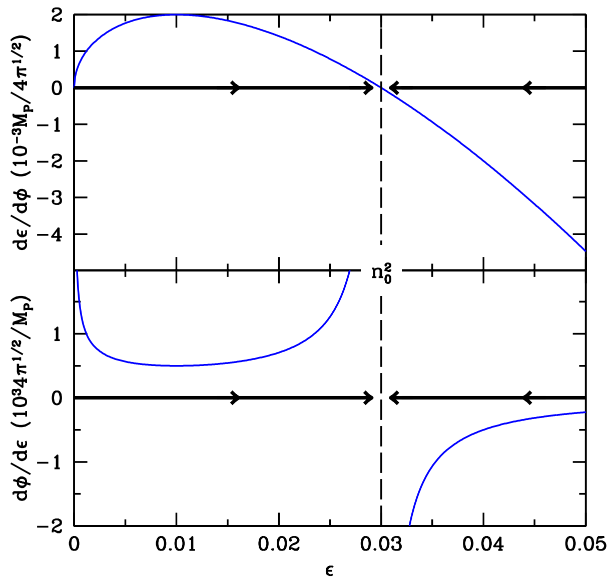

FIG. 3: The values of $d\epsilon/d\phi$ and $d\phi/d\epsilon$ for an assumed value of $n_0^2 = 0.03$. Notice that the sign of the derivatives implies that for $\epsilon \to n_0^2$ the value of the field tends toward infinity.

Let's first take a closer look at the $\epsilon = n_0^2$ point. It is not so difficult to note that it corresponds to power-law inflation. From the definition of ϵ, Eq. (5), the above condition leads to

$$\frac{M_p^2}{4\pi}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 = n_0^2. \tag{65}$$

Remembering that $\dot{\phi} > 0$ implies $dH/d\phi < 0$, the solution $H(\phi)$ is:

$$H(\phi) = A\exp\left[-\frac{2\sqrt{\pi}n_0\phi}{M_p}\right], \tag{66}$$

where A is the integration constant. The corresponding potential can be computed using Eq. (9) to be

$$V(\phi) \sim \exp\left(-\frac{4\sqrt{\pi}n_0\phi}{M_p}\right)(3 - n_0^2), \tag{67}$$

which in fact corresponds to the power-law potential, whose solution is known to be $a(t) \sim t^p$ with $p = n_0^{-2}$.

It is also interesting to compute the value of all the other slow-roll parameters λ_H^l in this case. The l-th derivative of the Hubble parameter is straightforward to compute, yielding

$$\frac{d^l H(\phi)}{d\phi^l} = A\left(-\frac{2\sqrt{\pi}n_0}{M_p}\right)^l \exp\left[-\frac{2\sqrt{\pi}n_0\phi}{M_p}\right]. \tag{68}$$

Inserting Eq. (68) into the definition of the slow-roll parameters, Eq. (8), their value is therefore given in this case by

$$\lambda_H^l = \left(\frac{M_p^2}{4\pi}\right)^l \left(-\frac{2\sqrt{\pi}n_0}{M_p}\right)^{2l} = n_0^{2l}. \tag{69}$$

Starting only from the general definition of ϵ, we have therefore derived a very interesting characteristic of power-law inflation: requiring the slow-roll parameter ϵ to be equal to a constant n_0^2 *automatically* implies that *all* the other slow-roll parameters will be equal to the *same constant*. It is then possible to take the condition $\epsilon = n_0^2$ as the general definition of power-law inflation.

The fixed point, corresponding to the condition $\epsilon = n_0^2$, is therefore associated with power-law inflation generating a k-independent density spectral index equal to $n(k) = 1 - 2n_0^2$. Furthermore, this result allows one to reconcile the apparent contradictory requirements for the generation of a Harrison–Zel'dovich power spectrum stemming from the lowest-order slow-roll approximation condition, $\eta = 2\epsilon$, and by power-law inflation definition $\epsilon = \eta = \xi = \cdots = n_0^2$. One can see once again that a Harrison–Zel'dovich power spectrum can be generated by power-law inflation in the limit $p \to \infty$, which corresponds to pure de Sitter expansion [20].

Turning our attention to the case $\epsilon_0 \neq n_0^2$ it is easier to consider the derivative of ϕ with respect to ϵ,

$$\frac{d\phi}{d\epsilon} = \left(\frac{d\epsilon}{d\phi}\right)^{-1} = \frac{M_p}{4\sqrt{\pi}}\frac{1}{\sqrt{\epsilon}(n_0^2 - \epsilon)}, \tag{70}$$

which is also shown in Fig. 3. First of all, recall that throughout this work it is assumed that the field ϕ is increasing as it rolls down the potential. Furthermore, it has been argued before that for $\epsilon_0 > n_0^2$ the value of ϵ tends to decrease, while for $\epsilon < n_0^2$ the value of ϵ tends to increase. It is then possible to note that in *all* regions of Fig. 3 the field tends to increase, which is consistent with the condition that has been imposed. The very interesting feature is that the point $\epsilon = n_0^2$ represents an asymptote of $d\phi/d\epsilon$: integrating it on either side with $\epsilon \to n_0^2$ then yields a logarithmically-diverging field. This necessarily implies that the value of the field, parametrized by ϵ, will tend to infinity while ϵ tends toward n_0^2. Remembering that Eq. (59) is integrated to yield $\phi(\epsilon)$, it is then possible to note that the three distinct regions $\epsilon < n_0^2$, $\epsilon = n_0^2$ and $\epsilon > n_0^2$ will give rise to three different dynamical behaviors for ϕ, which, once inserted in the expression for $V(\epsilon)$, are able to produce the same density perturbation spectral index. The apparent puzzle that arose at the end of Sec. III has therefore been solved: there are in fact two potentials, and both their domains are $\phi \in [0, \infty[$. It is now possible to understand that each one of them is able to generate the desired power spectrum, depending on the initial condition chosen for the slow-roll parameter.

B. The $n_0^2 \leq 0$ case

The cases for $n_0^2 = 0$ and $n_0^2 < 0$ are similar. From Eq. (59) it is in fact possible to note that independent of ϵ_0, the value of ϵ will tend towards zero as inflation proceeds. In the $n_0^2 < 0$ case the solution derived Sec.

III is the only one available, while in the special case $n_0^2 = 0$ (Harrison–Zel'dovich) it is possible to claim that two different inflationary potentials will be able to generate such a power spectrum: the flat one giving rise to the classical de Sitter expansion, and the one derived in Sec. III A, whose first term is proportional to ϕ^{-2}. Given the importance of the Harrison–Zel'dovich case, it is interesting to investigate the next-order contributions to its inflationary potential.

V. THE NEXT-ORDER ANALYSIS OF THE HARRISON–ZEL'DOVICH CASE

In this case the approximation order is $l_0 = 2$ and the set of slow-roll parameters consistent with it is given by $\{\epsilon, \eta, \xi^2\}$. Also, order consistency will require to expand exact expressions in power series and to retain only terms up to second degree in the slow-roll parameters. In what follows two results will be presented for A_T^2, H^2 and V, the first one being the "exact" result and the second one being the power series expansion of the former truncated at terms of second degree.

We start by imposing the condition $n(k)-1 = 0$ in Eq. (18),

$$4\epsilon - 2\eta + 8(C+1)\epsilon^2 - (6+10C)\epsilon\eta + 2C\xi^2 = 0. \quad (71)$$

The k-independence of n also requires

$$\frac{dn(k)}{d\ln k} = -2\xi^2 - 8\epsilon^2 + 10\epsilon\eta = 0. \quad (72)$$

This formula can then be exploited to express ξ^2 as a function of the other two parameters, ϵ and η. Eq. (71) then becomes

$$4\epsilon - 2\eta + 8\epsilon^2 - 6\eta\epsilon = 0, \quad (73)$$

hence

$$\eta(\epsilon) = \frac{2\epsilon + 4\epsilon^2}{3\epsilon + 1} \simeq 2\epsilon - 2\epsilon^2. \quad (74)$$

Inserting Eq. (74) into Eq. (72) yields

$$\xi^2(\epsilon) = \frac{6\epsilon^2 + 8\epsilon^3}{3\epsilon + 1} \simeq 6\epsilon^2. \quad (75)$$

The requirements $n(k) = 1$ and $dn(k)/d\ln k = 0$ have therefore allowed the expression of all slow-roll parameters as functions of ϵ.

It is now necessary to find an appropriate expression for $H^2(\epsilon)$ that can then be inserted into Eq. (9) to yield the potential as a function of the slow-roll parameter. Retracing the steps taken in Sec. III A, it is important to note that to *next-order* the appropriate equation for the rescaled gravitational perturbation power spectrum is Eq. (17) with the $\{\cdots\}$ included. Inverting it produces

$$H^2 \simeq \frac{25\pi M_p^2}{4\left[1 - \epsilon(C+1)\right]^2} A_T^2. \quad (76)$$

But to *next-order* approximation we also have [20]

$$\epsilon \simeq \frac{A_T^2}{A_S^2}\left[1 - 2C(\epsilon - \eta)\right]. \quad (77)$$

This relation can then be used together with Eq. (22) to express $A_T^2(k)$ as a function of ϵ,

$$A_T^2(k) = A_S^2(k_0)\frac{\epsilon}{1 - 2C(\epsilon - \eta)}. \quad (78)$$

It is then important to compute the $(\epsilon - \eta)$ factor

$$\epsilon - \eta = -\frac{\epsilon + \epsilon^2}{3\epsilon + 1}, \quad (79)$$

hence

$$A_T^2 = A_S^2(k_0)\frac{\epsilon(3\epsilon + 1)}{2C\epsilon^2 + (3 + 2C)\epsilon + 1}, \quad (80)$$

and

$$H^2(\epsilon) = \frac{25\pi M_p^2}{4\left[1 - \epsilon(C+1)\right]^2}\frac{A_S^2(k_0)\epsilon(3\epsilon + 1)}{\left[2C\epsilon^2 + (3 + 2C)\epsilon + 1\right]}. \quad (81)$$

This finally leads to

$$V(\epsilon) = \frac{25}{32}\frac{M_p^4 A_S^2(k_0)}{\left[1 - \epsilon(C+1)\right]^2}\frac{\epsilon(3\epsilon + 1)(3 - \epsilon)}{\left[2C\epsilon^2 + (3 + 2C)\epsilon + 1\right]}. \quad (82)$$

The appropriate expressions valid to next-order for Eqs. (80-82) are obtained by power series expansions where terms of degree ϵ^3 and higher are neglected:

$$A_T^2 \simeq A_S^2(k_0)(\epsilon - 2C\epsilon^2), \quad (83)$$

$$H^2(\epsilon) \simeq \frac{25\pi M_p^2 A_S^2(k_0)}{4}(\epsilon + 2\epsilon^2), \quad (84)$$

$$V(\epsilon) \simeq \frac{25 M_p^4 A_S^2(k_0)}{32}(3\epsilon + 5\epsilon^2). \quad (85)$$

Having succeeded in writing the potential as a function of ϵ, it is necessary to find an expression for ϵ as a function of the inflaton field ϕ. To do so, let's proceed as in Sec. III using Eq. (79)

$$\frac{d\epsilon}{d\phi} = -\frac{4\sqrt{\pi}}{M_p}\sqrt{\epsilon}\,\frac{\epsilon(\epsilon + 1)}{3\epsilon + 1}. \quad (86)$$

It is then possible to integrate exactly the above differential equation,

$$\begin{aligned}\phi(\epsilon) &= \frac{M_p}{2\sqrt{\pi}}\left[\frac{1}{\sqrt{\epsilon}} - 2\tan^{-1}(\sqrt{\epsilon})\right] \\ &\simeq \frac{M_p}{2\sqrt{\pi}}\left[\frac{1}{\sqrt{\epsilon}} - 2\sqrt{\epsilon}\right]. \end{aligned} \quad (87)$$

In this case it is neither straightforward nor very enlightening to obtain an explicit expression for the potential as a function of the field. However, inverting Eq. (87) is not really necessary, since both the inflaton field and its potential have been successfully parametrized by the slow-roll parameter ϵ: Eqs. (85) and (87) are sufficient to fully specify the potential as a function of the field.

The potentials that are able to generate a Harrison-Zel'dovich density power spectrum calculated at lowest-order and next-order are thus illustrated in Fig. 4.

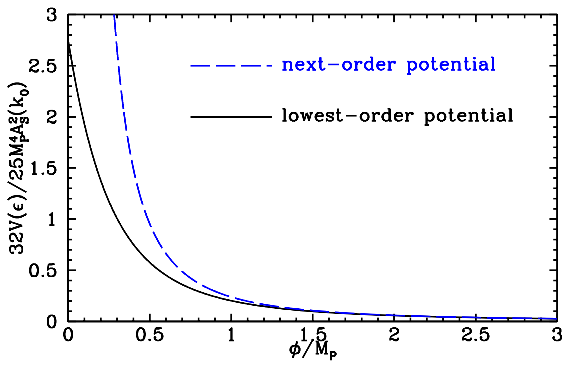

FIG. 4: Potentials giving the Harrison–Zel'dovich density spectral index, computed to lowest-order approximation and to next order approximation.

VI. DISCUSSION

The analysis that has been carried out shows that inflaton potentials yielding the Harrison–Zel'dovich flat spectrum can be determined to *lowest-order* and *next-order* approximation in the slow-roll parameters. Similarly, potentials producing a k-independent spectral index slightly different from unity have been derived to lowest-order.

We expect that the same procedure can be carried out to any order of expansion in the slow-roll parameters. This is because the implications of the spectral index k-independence are not as trivial as they may seem at first glance. Every time a higher approximation order is assumed, new slow-roll parameters will appear in the expression for the spectral index[3]: going from lowest-order to next-order, for example, ξ^2 was introduced. However, the requirement of the spectral index to be *k-independent* implies not only a particular value for $n(k)$ but also that all its derivatives are equal to zero:

$$\frac{d^i n(k)}{d(\ln k)^i} = 0. \tag{88}$$

Furthermore, it is possible to note that the expression for the $(l_0 - 1)^{th}$ derivative of the spectral index contains slow-roll parameters up to the l_0^{th} one. So once the approximation order l_0 is chosen, the problem is characterized by l_0+1 parameters and l_0 equations of constraint relating them. This allows the expression of all the slow-roll parameters as functions of a single one, which is chosen to be ϵ. The choice of ϵ is not so arbitrary, because then the *exact* expression for $d\epsilon/d\phi$, Eq. (29), can be used to compute ϵ as a function of ϕ.[4]

On the other side of the treatment, the expression for the potential as a function of the slow-roll parameter $V = V(\epsilon)$ appropriate for a definite order of the approximation depends on two crucial points. The first one is the expression of the rescaled tensor perturbation power spectrum A_T^2, which at every order acquires new contributions from slow-roll parameters [see Eq. (17)]. The second one is the equation expressing ϵ as a function of the ratio between the rescaled power spectra, which also gets new contributions from slow-roll parameters at every order [compare Eqs. (23) and (77)]. Furthermore, in the $n_0^2 \neq 0$ case an expression for the $\epsilon A_S^2(k)$ factor appropriate to the level of approximation assumed needs to be derived along the lines outlined in Sec. III B.

To summarize, the inflaton potential yielding a density perturbation spectral index $n(k) = 1 - 2n_0^2$ can be derived with the aid of the parametrization approach by the following procedure. First, derive the expression for $n(k)$ to whatever order is desired (say l_0). Second, impose the constraints $n(k) = 1 - 2n_0^2$ and $d^j n/d\ln k^j = 0$ with $j = 1, \ldots l_0 - 1$, yielding a system of l_0 equations in $l_0 + 1$ unknowns. Third, solve the above system expressing all the slow-roll parameters as power series expansion with respect to ϵ. Fourth, derive the expressions for the rescaled gravitational power spectrum and for ϵ as a function of the ratio of the rescaled power spectra appropriate to the same order. This, together with the approximate formula for the $\epsilon A_S^2(k)$ factor, will allow the expression of the Hubble parameter $H = H(\epsilon)$, which can then be plugged into the exact form of the potential, yielding the potential as a function of the only slow-roll parameter available ϵ. Once this is done, take the expression for $\eta = \eta(\epsilon)$ from the solution of the above system of equations and plug it into the exact differential equation for $d\epsilon/d\phi$. Integration of this equation gives $\phi = \phi(\epsilon)$. Finally recognize that both the inflaton field and the potential are now parametrized with respect to ϵ and therefore it is straightforward to compute the inflaton potential as a function of the value assumed by the field $V = V(\phi)$. An outline of the application of the above procedure to the next-order analysis of the approximately Harrison–Zel'dovich case is given in Appendix A.

Acknowledgments

This work was supported in part by NASA grant NAG5-10842. A.V. would like to thank the David and Lucile Packard Foundation and Hotel Victoria, Torino, for financial support. E.J.C. thanks the Kavli Institute

[3] The above fact is hardly surprising, though, because these new parameters just correspond to higher derivatives of $V(\phi)$ or $H(\phi)$ (whatever is the degree of freedom chosen to express the slow-roll parameters) and a higher-order treatment necessarily needs to take into account more derivative terms of the potential.

[4] Notice that Eq. (29) depends on the difference $(\epsilon - \eta)$, but since it contains the expression for $\eta = \eta(\epsilon)$, it needs to be reevaluated at every order of approximation.

for Theoretical Physics, Santa Barbara for their support during the completion of part of this work. This work was initiated during a visit by E.W.K. to Sussex supported by PPARC. A.R.L. was supported in part by the Leverhulme Trust.

APPENDIX A: NEXT-ORDER ANALYSIS OF THE APPROXIMATELY HARRISON–ZEL'DOVICH CASE

In principle it is possible to retrace the steps taken in Sec. V to compute the potential that originates a k-independent spectral index slightly different from unity to next-order approximation. Given the increased algebraic intricacy and the fact that the results are not so enlightening from the physics point of view, only an outline of the process required will be given.

First it is necessary to express the slow-roll parameters η and ξ^2 as functions of ϵ and n_0^2. The $n(k) = 1 - 2n_0^2$ condition at next-order takes the form

$$4\epsilon - 2\eta + 8(C+1)\epsilon^2 - (6+10C)\epsilon\eta + 2C\xi^2 = 2n_0^2. \quad \text{(A1)}$$

Imposing the auxiliary condition $dn(k)/d\ln k = 0$ allows again the expression of the slow-roll parameters as functions of ϵ:

$$\begin{aligned} \eta(\epsilon; n_0^2) &= \frac{2\epsilon + 4\epsilon^2 - n_0^2}{3\epsilon + 1} && \text{(A2)} \\ &\simeq -n_0^2 + (2 + 3n_0^2)\epsilon - (2 + 9n_0^2)\epsilon^2 + O(\epsilon^3). \\ \xi^2(\epsilon; n_0^2) &= \frac{6\epsilon^2 + 8\epsilon^3 - 5n_0^2\epsilon}{3\epsilon + 1} && \text{(A3)} \\ &\simeq -5n_0^2\epsilon + (6 + 15n_0^2)\epsilon^2 + O(\epsilon^3). \end{aligned}$$

Next it is necessary to invert Eq. (17) to obtain an expression for $H^2 = H^2(\epsilon; n_0^2)$. Remembering also that to *next-order* approximation [20]

$$\epsilon \simeq \frac{A_T^2}{A_S^2}\left[1 - 2C(\epsilon - \eta)\right], \quad \text{(A4)}$$

it is possible to derive the following expression:

$$H^2(\epsilon; n_0^2) \simeq \frac{25\pi M_p^2 \epsilon A_S^2(k)}{4\left[1 - \epsilon(C+1)\right]^2 \left[1 - 2C(\epsilon - \eta)\right]}. \quad \text{(A5)}$$

Also, given Eq. (A2) for $\eta(\epsilon; n_0^2)$, it is then possible to derive the next-order expression for the $(\epsilon - \eta)$ factor

$$\epsilon - \eta = \frac{n_0^2 - \epsilon - \epsilon^2}{3\epsilon + 1}. \quad \text{(A6)}$$

It is important to note, however, that also Eq. (53) used in Sec. III B to derive the approximate expression for the $\epsilon A_S^2(k)$ term depends on the factor $(\epsilon - \eta)$. It is therefore necessary to insert Eq. (A6) into Eq. (53) and to integrate it again in order to derive an approximate expression for $\epsilon A_S^2(k)$ appropriate to next-order, which can subsequently be used in Eq. (A5).[5] The expression for $H^2 = H^2(\epsilon; n_0^2)$ so obtained can then be plugged into Eq. (9) to obtain an expression for the potential $V = V(\epsilon; n_0^2)$; both expressions finally needs to be expanded in power series with respect to ϵ, and terms up to ϵ^2 retained.

Lastly, it is necessary to insert the next-order expression for $(\epsilon - \eta)$, Eq. (A6), into Eq. (29) which then produces

$$\frac{d\epsilon}{d\phi} = -\frac{4\sqrt{\pi}}{M_p}\sqrt{\epsilon}\,\frac{\epsilon(\epsilon+1) - n_0^2}{3\epsilon + 1}. \quad \text{(A7)}$$

Integration of Eq. (A7) then yields an expression for $\phi = \phi(\epsilon; n_0^2)$. Having carried out this step, it is finally possible to note that both the potential $V = V(\epsilon; n_0^2)$ and the field $\phi = \phi(\epsilon; n_0^2)$ are parametrized with respect to ϵ and n_0^2 and therefore the value of the potential as a function of the field can be obtained.

[5] It is possible to speculate that to next-order a good approximation is given by $\epsilon A_S^2(k) \approx |\epsilon^2 + \epsilon - n_0^2|$.

[1] A. Guth, Phys. Rev. D **23**, 347 (1981)
[2] D. H. Lyth and A. Riotto, Phys. Rep. **314** 1 (1999); A. Riotto, hep-ph/0210162; W. H. Kinney, astro-ph/0301448.
[3] V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33**, 532 (1981).
[4] S. W. Hawking, Phys. Lett. **115B**, 295 (1982).
[5] A. Starobinsky, Phys. Lett. **117B**, 175 (1982).
[6] A. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).
[7] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).
[8] S. Dodelson, W. H. Kinney, and E. W. Kolb, Phys. Rev. D **56**, 3207 (1997).
[9] W. H. Kinney, Phys. Rev. D **58**, 123506 (1998).
[10] W. H. Kinney, A. Melchiorri, A. Riotto, Phys. Rev. D **63**, 023505 (2001); S. Hannestad, S. H. Hansen, and F. L. Villante, Astropart. Phys. **17** 375 (2002); S. Hannestad, S. H. Hansen and F. L. Villante, Astropart. Phys. **16**, 137 (2001); D. J. Schwarz, C. A. Terrero-Escalante, and A. A. Garcia, Phys. Lett. B **517**, 243 (2001); X. Wang, M. Tegmark, B. Jain and M. Zaldarriaga, astro-ph/0212417.
[11] H. V. Peiris *et al.*, astro-ph/0302225.
[12] V. Barger, H. S. Lee and D. Marfatia, hep-ph/0302150.
[13] W. H. Kinney, E. W. Kolb, A. Melchiorri, and A. Riotto, hep-ph/0305130.
[14] S. M. Leach and A. R. Liddle, astro-ph/0306305.

[15] P. J. Steinhardt and M. S. Turner, Phys. Rev. D **29**, 2162 (1984); E. W. Kolb and M. S. Turner, *The Early Universe*, Addison–Wesley, Redwood City (1990).

[16] F. Lucchin and S. Matarrese, Phys. Rev. D 32, 1316 (1985); Phys. Lett. B 164, 282 (1895).

[17] D. S. Salopek and J. R. Bond, Phys. Rev D**42**, 3936 (1990).

[18] A. R. Liddle, P. Parsons, and J. D. Barrow, Phys. Rev. D **50**, 7222 (1994).

[19] E. D. Stewart and D. H. Lyth, Phys. Lett. B **302**, 171 (1993).

[20] J. E. Lidsey, A. R. Liddle, E. W. Kolb, E. J. Copeland, T. Barreiro, and M. Abney, Rev. Mod. Phys. **69**, 373 (1997).

[21] J. D. Barrow and A. R. Liddle, Phys. Rev. D **47**, 5219 (1993).

[22] D. Wands, Phys. Rev. D **60**, 023507 (1999).