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ViewMaps and Calibration in the SIunits Package

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And these shall be the measures thereof.
– Ezekiel 48:16

Contents

1 Introduction

The SIunits package allows for the use of world views other than the "standard" (std) view of dimensional quantities length, time, mass, current, temperature, and so forth. A commonly used world view says "we work in units where $c = 1$," for example. The package must do the work to support that new "relativistic" world view.

In this note, I want to provide concrete answers to several questions:

- What set of statements defines a world view?
- What does the SIunits Package need to know to be able to support each world view?
- How does this information appear in the files implementing SIunits?
- How can this information be derived from the defining statements, in the general case?
- What facility is placed into the SIunits package to derive the needed information?

An earier document, "World-Veiws in the SIunits Package," addressed these issues, but in the context of the needs of the earlier version of SIunits.

As a starting point, let me describe the *std* world view as having N_d dimensioned units labeled f_α. Thus f_1 is a meter, f_2 a kilogram, f_3 a second, and so forth. N_d is seven but that is moot; were we to eliminate Luminosity and/or Amount of Substance, for instance, the arguments in this note would remain unchanged. And in fact for the purposes of this document we will eliminate those two, to keep any matrices displayed to a reasonable size.

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2 Defining a World View

A world view works with dimensioned units δ_α. We shall see that in general some dimensions in the view are suppressed; thus the dimensions actually used can be said to be δ_μ where the range of μ is $\{1 \dots N_d\} - \{S\}$ where S is some (possibly empty) set of suppressed dimensions. When we take a product over mu, we will always mean the product over these non-suppressed values of mu.

The view is defined by statements of 3 sorts:

1. Denominating quantities. These are a set of dimensioned quantities which for now we consider to be Q_α, where α ranges from 1 to N_d. These must be chosen in such a way that every unit f_α can be expressed as a product of the denominating quantities:

$$f_\alpha = s_\alpha \prod_\beta Q_\beta^{R_{\alpha\beta}} \tag{1}$$

   We can write each dimensioned quantitity Q_β as

$$Q_\beta = q_\beta \prod_{\alpha=1}^{N_d} f_\alpha^{Q_{\beta\alpha}} \tag{2}$$

   The Q_β tells what is meant by each unit in the view. In the *std* view, each q_β will be 1, but for example, one might wish to choose $Q_1 = 1\text{cm} = .01 \cdot f_1^1$ to denominate the first dimension in centimeters rather than meters. In that case, q_1 would be .01, and Q_{11} would be 1.

   Notice that in this view, any physical quantity which happens to be one of the denominating quantities would have the value 1. Thus in the above example, one centimeter would have the numerical representation 1.

2. Physical constant constraints. These are each statements that in this view, some physical constant has the value (numerical representation) 1. That is, the physical constant

$$C_i \equiv c_i \prod_{\alpha=1}^{N_d} f_\alpha^{C_{i\alpha}} = 1 \cdot \prod_{\alpha=1}^{N_d} \delta_\alpha^{X_{i\alpha}} \tag{3}$$

   For example, in the relativistic model we say that $c = 1$; that is, that the δ_α are chosen such that some product of them, with coefficient 1, works out to $3 \cdot 10^8$ m/s.

   In general one can have N_c such constraints, labelled by the index i. $C_{i\alpha}$ is thus a matrix having N_c rows corresponding to the N_c constraints (the i index), and N_d columns corresponding to the fundamental dimensions (the α index). However, we will not require the values of i to run from 1 to N_c (the total number of constraints). Instead, we just say that the i are N_c numbers chosen from the set $1 \dots N_d$.

   Notice the similarity between a constraint and a denominating quantity: They both state that some combination of *std* view dimensions, multiplied by some numerical factor, is equal to a product of dimensions in this view, with a factor of 1. The distinction between these will come when suppression of dimensions is considered.

3. Suppression of dimensions: For each physical constant contraint, we choose one of the dimensions in this view, and say that that dimension δ_α is suppressed. (Normally one would choose to use the constraint to suppress a dimension which appears in that constraint to a non-zero power.) That is, because the constraint allows you to express one δ_α in terms of others, we are free to choose one δ_α and require that that dimension never be used when expressing quantities. We identify the suppressed dimension associated with C_i as δ_i.

For example, in the relativistic world view, the sole constraint is that $c = 1$, where c is a combination of length and time. This lets us state that the length-like dimension is not used in this world view. Instead, length is expressed in terms of the time-like dimension (and a unit of length would be interpreted in "light-seconds").

Note that the value supplied for either a constraint or a denominating quantity is expressed in the fundamental *std* units. I fact, what is being provided is a statement involving a physical constant or quantity havin some special place in the model; in principle, the physical constant carries with it information about its dimensionallity.

Of course, each suppressed dimension—thus each constraint—reduces by one the number of denominating quantities the view can have. Thus only $(N_d - N_c)$ denominating quantities are present. The view can be said to use $(N_d - N_c)$ dimensions δ_μ, where for each number α from 1 to N_d, either there is a constaint C_i with $i = \alpha$, or there is a dimension δ_μ with $\mu = \alpha$, but not both.

Thus the denominating quantities are a set of $(N_d - N_c)$ dimensioned quantities Q_β which must be chosen in such a way that every unit f_α can be expressed as

$$f_\alpha = s_\alpha \prod_\mu Q_\mu^{R_{\alpha\mu}} \cdot \prod_i C_i^{P_{\alpha 1}} \tag{4}$$

That looks a lot more complicated than it is: The meaning is that there has to be some way to get to each f_α by multiplying some of the $\{Q_\beta\}$ and some of the physical constraint constants. For example, in the *rel* view model, we choose $C_1 = c(m/s)$ and for the $\{Q_\beta\}$ we use seconds, Amps, Kelvins, and eVs. Of these, eV is non-trivial and illustrates the fact that denominating quatities need not align with the dimensions used the the *std* view. Of course, for arbitrary choices there is the possibility of inconsistency, but here we can use eV because a mass (in kg) can be expressed in eV$/c^2$ and c is one of the physical constraint constants of the model.

We can write each denominating quantitity Q_β as

$$Q_\mu = q_\mu \prod_{\alpha=1}^{N_d} f_\alpha^{Q_{\mu\alpha}} \tag{5}$$

In general, as long as you have selected the proper number of denomination quantities, and the dimensions of the constraints and these quantities have no linear dependencies, this set of quantities can be used to denominate the units in the new model. The non-trivial matter of determining the s_α and P and R in (4) is discussed below.

In principle, we allow for the units in our new model to comprise arbitrary combinations of fundamental dimensions. That is, forming each of the f_α can require multiple Q_β. In practice, for the models supplied with the package, we mainly stick with one Q_β leading to each fundemental dimension.

2.1 Number and Nature of Statements

A complete world view definition based on N_c physical constant constraints will have N_c suppressed dimensions. Corresponding to each non-suppressed dimension we must have one Q_β.

These N_c numerical relations and $N_d - N_c$ choices of dimensioned quantities, plus the N_c discrete choices of which dimensions to suppress, fully define the world view.

2.2 (Near) Unification of Constraints and Denominating Quantities

Because of the similarity between constraints and statements specifying how quantities are denominated, they can both be considered as part of a single matrix of statements describing the view. Ultimately, the distinction comes down to this:

The set of N_d statements always defines how you would go from a measurement in the *std* view to one in this view. However, if one or more of the statements are constraints, then one or more dimensions are suppressed.

Consider as an illustration the constraint $c = 1$ (and length is suppressed), as opposed to the statement that length is denominated such that $c = 1$. In a view based on the constraint, one can freely add lengths and time; a Lorentz transformation is a matrix with homogeneous units throughout. In the other view, although c is still 1, units checking keeps track of length and time separately, and you would *not* be allowed to add a foot to a second.

2.3 Order of Statements

In principle, the indices which are assigned to each constraint or denominating quantity are moot, as long as all N_d indicies are ultimately covered by N_d statements. Thus, one could easily place all the constraints first (or last). The latter in fact might have desirable consequences in terms of allowing shorter "lists" of dimensions in models where some units are suppressed.

Here, however, we will persist in allowing an arbitrary sequence of indices, mixing constraints and denominating quantities. The reason is that most of the time, a unit in the model at hand will correspond in a natural way to one of the fundamental units in the *std* model, and it is attractive to retain this correspondence.

2.4 Consistency

The set of physical constant constraints and denominating quantities must be consistent, in the sense that the set of N_d-vectors comprising the powers of each dimension in the various of each constraints and denominating quantities must be linearly independant. If this were not the case, then either one or more constraints would be derivable from the others, or the set of constraints conuld not be satisfied, or it would be possible to construct some physical quantity which could not be expressed in the view being defined.

This is the obvious consistency condition but in principle there is another concern about the choice of constraints. This can be seen by imagining a constraint involving just one dimension. While it is allowed (and in fact typical) to have a denominating quantity based on just a single *std* view dimension, if you have a *constraint* based on just one dimension, then the dimension suppressed by that constraint would have no way to be converted to another dimension in the new view—it would of necessity become commeasurate with a pure number. This is probably not what one had in mind.

Similarly, normally every subset of N constraints deals in total with more than N dimensions, and this must span a vector space of at least N dimensions (or else the matrix of constraints and quanitites must be singular). But if the subset of N constraints spans a vectors spaces of dimension exactly N, then all those dimensions again are forced to be commeasurate with pure numbers. Again, this is probably not what the model was supposed to do.

The exception to this concern is when you mean (as in the *nat* view) for dimensions to become commeasurate with pure numbers. There, five constraints dealing with length, time, mass, current and temperature combine such that all the units in that sector are pure numbers.

3 Information Needed to Support Each World View

The previous section decribed how a view is defined in principle. Here, we describe the information the SIunits package uses to implement a view.

There are two pieces of information needed for the SIunits mechanisms to operate:

ViewMap

The ViewMap is a mapping between dimensions δ_α in the new view and products of powers of fundamental dimensions f_α in the *std* view. This is expressed as two sorts of statements. One set looks like

$$p(\delta_3) = p(f_1) + p(f_3)$$

meaning that if a quantity Q contains p_1 powers of f_1 and p_3 powers of f_3 then in the new view it will contain $p(\delta_3) = p_1 + p_3$ powers of δ_3. In general, this mapping looks like

$$p(\delta_\mu) = \sum_\alpha m_{\mu\alpha} p(f_\alpha) \tag{6}$$

where μ runs over the non-suppressed dimensions in the new model.

This sort of statement is realized, in the template code, by setting up a typedef which uses RatioAdd (and/or subtract and/or multiply) to form the combination needed for a dimension, and then using that typedef in the definition of `ViewMap::Ans` for the view.

The other set of statements in the ViewMap are of the form that some set of δ_α are suppressed, implying that these dimensions are unused in the view. In the language above, for suppressed dimensions $d_\alpha = 0$.

This sort of statement is realized, in the template code, by using `Unused` in the appropriate slot in the definition of `ViewMap::Ans` for the view.

Scale factors

For every fundamental dimension f_α, a scale factor (in the ViewMap, these are implemented via statements like `SI_calibrate ( meter, 3.335e-09, "?")`. Since this is associated with the fundamental dimensions in the *std* view, the new view needs to know all N_d of these even if it has several suppressed dimensions.

In terms of equation (4), we can identify this scale factor with s_α.

The meaning of these pieces of information is that a given *std* fundamental quantity (for example, a meter) is expressed in the new view as that scale factor, times a unit with dimensions given by the powers of each δ_α which come out when one power of that *std* quantity is plugged into equation (6). (Given that we can express any fundamental dimension in the new model, it is then trivial to express any general dimensioned quantity by multiplying the appropriate scales and dimensions.)

In terms of equation (4), we can identify s_α with the scale factor for f_α, and the $R_{\alpha\mu}$ can be identified with $m_{\mu\alpha}$ in equation (6). The latter can be shown as follows:

$$p(\delta_\mu) = \sum_\alpha m_{\mu\alpha} p(f_\alpha)$$

$$f_\alpha = s_\alpha \prod_\beta \delta_\beta^{R_{\alpha\beta}}$$

$$\prod_\alpha f_\alpha^{p(f_\alpha)} = \prod_\alpha s_\alpha^{p(f_\alpha)} \prod_\beta \delta_\beta^{R_{\alpha\beta}\cdot p(f_\alpha)}$$

So,

$$p(\delta_\mu) = \sum_\alpha R_{\alpha\mu} p(f_\alpha)$$

from which by inspection there follows

$$m_{\mu\alpha} = R_{\alpha\mu} \tag{7}$$

3.1 Simple Examples

First, consider a view in which the first dimension is denominated in units of a speed, such that $c = 1$. (This is different from saying that the constraint $c = 1$ is to be used to suppress the first dimension from the model—this view still has all seven active dimensions.) Here, considering only the first three dimensions,

$$Q = \left(\begin{array}{cccccl} & m & kg & s & \\ \delta_1 & 1 & 0 & -1 & c(\text{a speed}) \\ \delta_2 & 0 & 1 & 0 & kg \\ \delta_3 & 0 & 0 & 1 & s \end{array} \right)$$

$$R = \left(\begin{array}{cccc} & \delta_1 & \delta_2 & \delta_3 \\ f_1 & 1 & 0 & 1 \\ f_2 & 0 & 1 & 0 \\ f_3 & 0 & 0 & 1 \end{array} \right)$$

Reading off the first and last columns of R, we see that

$$p(\delta_{\text{speed}}) = p(f_{\text{len}})$$

$$p(\delta_{\text{time}}) = p(f_{\text{len}}) + p(f_{\text{time}})$$

so for example the quantity 5 meter-seconds would be expressed, in this view, as

$$\frac{5}{c \text{ (in m/s)}} \cdot c^1 \cdot \text{second}^2$$

Notice that the power of time does not match in the two views; the presence of powers of length translates to speed times time in the new view.

As a second example, in the *cgs* view, a denominating quantity is charge, rather than current. Looking only at the time and current (third and fourth) units, we have:

$$Q = \left(\begin{array}{cccl} & s & A & \\ \delta_3 & 1 & 0 & s \\ \delta_4 & 1 & 1 & esu \end{array} \right)$$

$$R = \left(\begin{array}{ccc} & \delta_3 & \delta_4 \\ f_3 & 1 & 0 \\ f_4 & -1 & 1 \end{array} \right)$$

so for example a Coulomb (an Amp-second) would have

$$p(\delta_{\text{time}}) = p(f_{\text{time}}) - p(f_{\text{current}}) = 0$$

$$p(\delta_{\text{charge}}) = p(f_{\text{current}}) = 1$$

while for an Ampere, we would have

$$p(\delta_{\text{time}}) = p(f_{\text{time}}) - p(f_{\text{current}}) = -1$$

$$p(\delta_{\text{charge}}) = p(f_{\text{current}}) = 1$$

At first sight, it may look strange that an Ampere, in the new view, suddenly picks up a power of time—but considering that it is a quantity of charge *per second* this is clearly right.

4 Calculating the Needed Information

4.1 Solving for R and P

We know the details of the denominating quatities and constraints: Matrices $Q_{\beta\alpha}$ and $C_{i\alpha}$, along with the associated scalar constants. But although we know (or at least take it on faith) that for our choice of constraints and denominating quantities there are matrices $R_{\alpha\mu}$ and $P_{\alpha i}$ that will make it possible for equation (4) to hold, we are not given the values of $R_{\alpha\mu}$ and $P_{\alpha i}$. These need to be derived. (The value of $R_{\alpha\mu}$ is needed directly, to read off the ViewMap. The value of $P_{\alpha i}$ is needed indirectly, to go from the values in the statements to the scale factors s_α.)

We will start from a horrendous-looking relation and demonstrate how it can be seen as an easily managed matrix equation.

When we insert the definitions of Q_μ (equation (5)) and C_i (equation (3)) into equation (4) we get that for each value of α,

$$f_\alpha = s_\alpha \prod_\mu \left(q_\mu \prod_{\alpha'} f_{\alpha'}^{Q_{\mu\alpha'}} \right)^{R_{\alpha\mu}} \prod_{i=1} \left(c_i \prod_{\alpha''} f_{\alpha''}^{C_{i\alpha''}} \right)^{P_{\alpha i}} \tag{8}$$

where as usual μ ranges over the denominating quantities.

In this horrendous equation, we first pull out all the scalar-valued terms:

$$f_\alpha = s_\alpha \prod_{\mu'} q_{\mu'}^{R_{\alpha\mu'}} \prod_{i'} c_{i'}^{P_{\alpha i'}} \prod_\mu \prod_i \prod_{\alpha'} \prod_{\alpha''} \left(f_{\alpha'}^{Q_{\mu\alpha}} \right)^{R_{\alpha\mu}} \left(f_{\alpha''}^{C_{i\alpha''}} \right)^{P_{\alpha i}} \tag{9}$$

Now we temporarily ignore the scalar out front, and do power counting to match powers of each $f_{\alpha\prime}$ in the equation for each f_α. The products become sums, and on the left hand side we require that only one power of f_α appear. So we get the equation

$$\delta_{\alpha\alpha\prime} = \sum_\mu R_{\alpha\mu} Q_{\mu\alpha'} + \sum_{i=1}^{N_c} P_{\alpha i} C_{i\alpha'} \tag{10}$$

or in a pseudo-matrix notation,

$$\mathbf{1} = RQ + PC \tag{11}$$

But because of the similarity between constraints and denominating quantity statements, we can combine these into one true matrix equation. If we form a N_d by N_d matrix U by using the rows of Q where there are non-suppressed dimensions and rows of C in the gaps, and a matrix K by putting the columns of R where there are non-suppressed dimensions and columns of P in the gaps, we find

$$UK = \mathbf{1} \tag{12}$$

So, solving for $R_{\alpha\beta}$ and $P_{\alpha i}$ is straightforward: We form U out of Q and P, invert to find K, and we then read R off the $(N_d - N_c)$ columns of K corresponding to non-suppressed dimensions, and we read P off the remaining columns.

4.2 The Dimension Mapping

To get the mapping needed (equation (6)) we identify $m_{\mu\alpha}$ with $R_{\alpha\mu}$. That is, we simply read off the coefficients of powers of $\{f_1 \ldots f_7\}$ for each non-suppressed δ_μ from a column of R.

4.3 The Scale Factors

To compute the scale factors s_α, we can start from equation (9). Once we use the proper P and R, the power-counting terms all work out correctly, and we are left to consider only the scalar term:

$$1 = s_\alpha \prod_\mu q_\mu^{R_{\alpha\mu}} \prod_{i=1} c_i^{P_{\alpha i}} \tag{13}$$

So each s_α can be read off by taking the q_β and c_i raised to powers read off by the corresponding elements of $-R_{\alpha\beta}$ and $-P_{\alpha i}$:

$$s_\alpha = \prod_\mu q_\mu^{-R_{\alpha\mu}} \prod_i c_i^{-P_{\alpha i}} \tag{14}$$

For example, in the relativity model, $R_{22} = 1$, q_2 is one eV (measured in Joules) $P_{21} = -2$, and c_1 is the speed of light c (measured in m/s). (The other relevant elements of R and P are zero.) So

$$s_2 = (eV/Joule)^{-1} c^2 \tag{15}$$

5 Examples from the Models

Here we present the full definitions and computations for each model. Since Amount-of-substance and Luminosity never come into non-trivial play, we ignore those dimensions. We are left with 5 dimensions; and we associate p1 (and d1) with length, p2 (and d2) with mass, p3 (and d3) with time, p4 (and d4) with current, and p5 (and d5) with temperature.

5.1 std model

This is trivial, and in the first version of this document this subsection was omitted. However, it serves to illustrate the pattern of information, so we include it here.

There are no physical constant constraints.

The denominating quantities are

1. one meter
2. one kilogram
3. one second
4. one Amp
5. one Kelvin

Then

$$Q = \begin{array}{cc} & \begin{array}{ccccccc} m & kg & s & A & K & & \end{array} \\ \begin{array}{c} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{array} & \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & m & q_1 = 1 \\ 0 & 1 & 0 & 0 & 0 & kg & q_2 = 1 \\ 0 & 0 & 1 & 0 & 0 & s & q_3 = 1 \\ 0 & 0 & 0 & 1 & 0 & A & q_4 = 1 \\ 0 & 0 & 0 & 0 & 1 & K & q_5 = 1 \end{array} \right) \end{array}$$

$$U = \begin{array}{cc} & \begin{array}{ccccc} m & kg & s & A & K \end{array} \\ \begin{array}{c} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{array} & \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}$$

$$K = U^{-1} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{array}{ccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \end{array} \atop \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R = \begin{array}{c|ccccc} & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \\ \hline f_1 & 1 & 0 & 0 & 0 & 0 \\ f_2 & 0 & 1 & 0 & 0 & 0 \\ f_3 & 0 & 0 & 1 & 0 & 0 \\ f_4 & 0 & 0 & 0 & 1 & 0 \\ f_5 & 0 & 0 & 0 & 0 & 1 \end{array}$$

So each δ_μ has the same power as the corresponding f_μ (in fact, it is the same as f_μ). Thus ViewMap::Ans is the trivial identity map of the template arguments.

And reading exponents of $-R$ to use as exponents of q_μ, we get:

- SI_calibrate (meter, 1.0)
- SI_calibrate (kilogram, 1.0)
- SI_calibrate (second, 1.0)
- SI_calibrate (ampere, 1.0)
- SI_calibrate (kelvin, 1.0)

5.2 cgs model

This model was not present in the first version of SIunits, but is an obvious view to have available. There are still seven non-suppressed dimensions. The only minor subtlety is a question of whether current or charge is to be considered a fundamental quantity. In the *std* model, that question is resolved unambiguously: The Ampere is one of the seven fundamental units. When working in *cgs* units, the statement is not made as clearly; but it seems more common to take the esu (or statcoulomb) to be fundamental—see for example in table 1.1 of Review of Particle Physics, where it is stated that 2.997×10^9 esu = 1C, while nothing is said about statamperes.

Thus, we take the unit of charge to be one esu that is, we state that the esu is a denominating quantity.

There are no physical constant constraints.

The denominating quantities are

1. one centimeter
2. one gram
3. one second
4. one esu
5. one Kelvin

Then

$$Q = \begin{array}{ccc} & \begin{array}{ccccc} m & kg & s & A & K \end{array} & \\ \begin{array}{c} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{ll} cm & q_1 = cm/m \\ g & q_2 = g/kg \\ s & q_3 = 1 \\ esu & q_4 = \text{esu}/(\text{Amp-sec}) \\ K & q_5 = 1 \end{array} \end{array}$$

$$U = \begin{array}{cc} & \begin{array}{ccccc} m & kg & s & A & K \end{array} \\ \begin{array}{c} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$K = U^{-1} = \begin{array}{cc} & \begin{array}{ccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$R = \begin{array}{cc} & \begin{array}{ccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

So $p(\delta_3) = p(f_4) - p(f_3)$, and the remaining δ_μ have the same power as the corresponding f_μ.

Thus we need to define TimCur as RatioAdd<Tim,Cur> and use TimCur as the 3^{rd} item in ViewMap::Ans.

And reading exponents of $-R$ to use as exponents of q_μ, we get:

- SI_calibrate (meter, $q_1^{-R_{11}} = \text{cm/m}^{-1} = 100.0$)
- SI_calibrate (kilogram, $q_2^{-R_{22}} = \text{g/kg}^{-1} = 1000.0$)
- SI_calibrate (second, 1.0)
- SI_calibrate (ampere, 1.0, $q_3^{-R_{43}} \cdot q_4^{-R_{44}} = \text{s/s}^1 \cdot \text{esu/Amp-sec}^{-1} = 1$ C/ 1 esu)
- SI_calibrate (kelvin, 1.0)

(Here, terms where $q_\mu = 1$ are not shown since they do not affect the scale factors regardless of exponent.)

5.3 rel model

This is the first model in which we suppress a dimension, and can be examined carefully to understand how the mechanism arrives at the ViewMap and scaling factors.

The constant constraints are

1. $c = 1$

The denominating quantities are

1. (dimension corresponding to length is suppressed)
2. one eV
3. one second
4. one Amp
5. one Kelvin

Then

$$C = \left(\begin{array}{cccccc} m & kg & s & A & K & \\ 1 & 0 & -1 & 0 & 0 & c_1 = c \end{array} \right)$$

$$Q = \left(\begin{array}{ccccccccc} & m & kg & s & A & K & & \\ \delta_2 & 2 & 1 & -2 & 0 & 0 & eV & q_2 = eV/J \\ \delta_3 & 0 & 0 & 1 & 0 & 0 & s & q_3 = 1 \\ \delta_4 & 0 & 0 & 0 & 1 & 0 & A & q_4 = 1 \\ \delta_5 & 0 & 0 & 0 & 0 & 1 & K & q_5 = 1 \end{array} \right)$$

$$U = \left(\begin{array}{cccccc} & m & kg & s & A & K \\ & 1 & 0 & -1 & 0 & 0 \\ \delta_2 & 2 & 1 & -2 & 0 & 0 \\ \delta_3 & 0 & 0 & 1 & 0 & 0 \\ \delta_4 & 0 & 0 & 0 & 1 & 0 \\ \delta_5 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$K = U^{-1} = \left(\begin{array}{cccccc} & c_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \\ f_1 & 1 & 0 & 1 & 0 & 0 \\ f_2 & -2 & 1 & 0 & 0 & 0 \\ f_3 & 0 & 0 & 1 & 0 & 0 \\ f_4 & 0 & 0 & 0 & 1 & 0 \\ f_5 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$R = \left(\begin{array}{ccccc} & \delta_2 & \delta_3 & \delta_4 & \delta_5 \\ f_1 & 0 & 1 & 0 & 0 \\ f_2 & 1 & 0 & 0 & 0 \\ f_3 & 0 & 1 & 0 & 0 \\ f_4 & 0 & 0 & 1 & 0 \\ f_5 & 0 & 0 & 0 & 1 \end{array} \right)$$

So δ_1 is suppressed, $p(\delta_3) = p(f_1) + p(f_3)$, and the remaining δ_μ have the same power as the corresponding f_μ.

Thus we need to define LenTim as RatioAdd<Len,Tim> and use LenTim as the 3^{rd} item in ViewMap::Ans.

$$P = \begin{array}{cc} & c_1 \\ f_1 & \\ f_2 & \\ f_3 & \\ f_4 & \\ f_5 & \end{array}\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And reading exponents of $-R$ and $-P$ to use as exponents of q_μ and c_i, we get:

- SI_calibrate (meter, $c_1^{-P_{11}} = C^{-1}$)
- SI_calibrate (kilogram, $c_1^{-P_{21}} q_2^{-R_{22}} = c^2 * (eV/J)^{-1}$)
- SI_calibrate (second, 1.0)
- SI_calibrate (ampere, 1.0)
- SI_calibrate (kelvin, 1.0)

(Here, terms where $q_\mu = 1$ are not shown since they do not affect the scale factors regardless of exponent.)

When we show physical constants in this way, we mean measured in std units, for example c is measured in m/s.

5.4 hep model

Now we add the constraint that k (which in std is measured in J/K) is one, change over to GeV and ns, and also demoninate charge in units of e^+.

The physical constant constraints are

1. $c = 1$
5. $k = 1$

The denominating quantities are

1. (dimension corresponding to length is suppressed)
2. one GeV
3. one nanosecond
4. one electron-charge (= eplus/Coulomb Amp-Seconds)
5. (dimension corresponding to temperature is suppressed)

(In this and subsequent models, we also denominate Amount-of-substance in molecules; this was intended from the start but was not done in the original package. For the purpose of brevity in this note we are ignoring both the Amount-of-substance and Lumunous Intensity units.)

$$C = \begin{pmatrix} m & kg & s & A & K & \\ 1 & 0 & -1 & 0 & 0 & c_1 = c \\ 2 & 1 & -2 & 0 & -1 & c_5 = k \end{pmatrix}$$

(We call the second contraint c_5 so that there will be a nice coverage of indices: For any i there will be either a non-suppressed d_i or a c_i.)

$$Q = \begin{pmatrix} & m & kg & s & A & K & & \\ \delta_2 & 2 & 1 & -2 & 0 & 0 & GeV & q_2 = GeV/J \\ \delta_3 & 0 & 0 & 1 & 0 & 0 & ns & q_3 = ns/s = 10^{-9} \\ \delta_4 & 0 & 0 & 1 & 1 & 0 & e^+ & q_4 = e^+/\text{Coulomb} \end{pmatrix}$$

$$U = \begin{pmatrix} & m & kg & s & A & K \\ & 1 & 0 & -1 & 0 & 0 \\ \delta_2 & 2 & 1 & -2 & 0 & 0 \\ \delta_3 & 0 & 0 & 1 & 0 & 0 \\ \delta_4 & 0 & 0 & 1 & 1 & 0 \\ & 2 & 1 & -2 & 0 & -1 \end{pmatrix}$$

$$K = U^{-1} = \begin{pmatrix} & c_1 & \delta_2 & \delta_3 & \delta_4 & c_5 \\ f_1 & 1 & 0 & 1 & 0 & 0 \\ f_2 & -2 & 1 & 0 & 0 & 0 \\ f_3 & 0 & 0 & 1 & 0 & 0 \\ f_4 & 0 & 0 & -1 & 1 & 0 \\ f_5 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

$$R = \begin{pmatrix} & \delta_2 & \delta_3 & \delta_4 \\ f_1 & 0 & 1 & 0 \\ f_2 & 1 & 0 & 0 \\ f_3 & 0 & 1 & 0 \\ f_4 & 0 & -1 & 1 \\ f_5 & 1 & 0 & 0 \end{pmatrix}$$

So δ_1 and δ_5 are suppressed, $p(\delta_3) = p(f_1) + p(f_3) - p(f_4)$, and $p(\delta_2) = p(f_2) + p(f_5)$, and the remaining δ_μ have the same power as the corresponding f_μ.

Thus we need to define LenTimCur as RatioSub<LenTim,Cur> and use LenTimCur as the 3^{rd} item in ViewMap::Ans. And we need to define MasTmp as RatioAdd<Mas,Tmp> and use MasTmp as the 2^{nd} item in ViewMap::Ans.

$$P = \begin{pmatrix} & c_1 & c_5 \\ f_1 & 1 & 0 \\ f_2 & -2 & 0 \\ f_3 & 0 & 0 \\ f_4 & 0 & 0 \\ f_5 & 0 & -1 \end{pmatrix}$$

And reading exponents of $-R$ and $-P$ to use as exponents of q_β and c_i, we get:

- SI_calibrate (meter, $c_1^{-P_{11}} = c^{-1}$)
- SI_calibrate (kilogram, $c_1^{-P_{21}} q_2^{-R_{22}} = c^2 * (GeV/J)^{-1}$)
- SI_calibrate (second, $q_3^{-R_{33}} = (ns/s)^{-1} = 10^9$)
- SI_calibrate (ampere, $q_3^{-R_{43}} q_4^{-R_{44}} = (ns/s)^{+1} (e^+/\text{Coulomb})^{-1} = 10^{-9} * (e^+/\text{Coulomb})^{-1}$)
- SI_calibrate (kelvin, $c_5^{-P_{55}} q_2^{-R_{52}} = k^{+1} * (GeV/J)^{-1}$)

Note that in this model, even though we fix the charge on the positron to be +1, we do that by choosing how we denominate charge, not by using a phsyical constant constraint to suppress the current dimension. Thus dimension checking still goes on for quantities having a net power of current (or charge), and current does not appear in any non-trivial way in the ViewMap.

5.5 qtm model

Now we add the constraints that $\hbar = 1$, and $\varepsilon_0 = 1$. We continue to denominate in GeV, but the because of the added constraints we no longer use ns to denominate time, nor can we use e^+ to denominate charge. (With ε_0 set to unity, the charge on the electron is fixed by the square root of 2π times the fine structure constant.)

The physical constant constraints are

1. $c = 1$
3. $\hbar = 1$
4. $\varepsilon_0 = 1$ (This is measured in Farads/meter)
5. $k = 1$

The denominating quantities are

1. (dimension corresponding to length is suppressed)
2. one GeV
3. (dimension corresponding to time is suppressed)
4. (dimension corresponding to current is suppressed)
5. (dimension corresponding to temperature is suppressed)

$$C = \begin{array}{c} \begin{matrix} m & kg & s & A & K \end{matrix} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 \\ -3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -2 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{l} \\ c_1 = c \\ c_3 = \hbar \\ c_4 = \varepsilon_0 \\ c_5 = k \end{array}$$

$$Q = \begin{array}{c} \begin{matrix} m & kg & s & A & K \end{matrix} \\ \delta_2 \begin{pmatrix} 2 & 1 & -2 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ GeV \quad q_2 = GeV/J \end{array}$$

$$U = \begin{array}{c} \begin{matrix} m & kg & s & A & K \end{matrix} \\ \begin{array}{c} \\ \delta_2 \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 \\ -3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -2 & 0 & -1 \end{pmatrix} \end{array}$$

$$K = U^{-1} = \begin{array}{c} \begin{matrix} c_1 & \delta_2 & c_3 & c_4 & c_5 \end{matrix} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ .5 & 1 & -.5 & .5 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Notice that tor the first time, fractional powers are creeping in. This is nothing to worry about; but absent a solid formalism it would make the computation of scale factors quite tricky.

$$R = \begin{array}{cc} & \delta_2 \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} & \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \end{array}$$

So δ_1, δ_3, δ_4, and δ_5 are suppressed, and and $p(\delta_2) = p(f_2)+p(f_4)+p(f_5)-p(f_1)-p(f_3)$.

We implement this by defining MasCur as RatioAdd<Mas,Cur>, then MasCurTmp as RatioAdd<MasCur,Tmp>, then All5 as RatioSub<MasCurTmp,LinTim> and using All5 as the 2^{nd} item in ViewMap::Ans.

$$P = \begin{array}{cc} & \begin{array}{cccc} c_1 & c_3 & c_4 & c_5 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .5 & -.5 & .5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

And reading exponents of $-R$ and $-P$ to use as exponents of q_β and c_i, we get:

- SI_calibrate (meter, $c_1^{-P_{11}} c_3^{-P_{13}} q_2^{-R_{12}} = c^{-1}\hbar^{-1} \cdot (GeV/J)^{+1}$)
- SI_calibrate (kilogram, $c_1^{-P_{21}} q_2^{-R_{22}} = c^2 \cdot (GeV/J)^{-1}$)
- SI_calibrate (second, $c_3^{-P_{33}} q_2^{-R_{32}} = \hbar^{-1} \cdot (GeV/J)^{+1}$)
- SI_calibrate (ampere, $c_1^{-P_{41}} c_3^{-P_{43}} c_4^{-P_{44}} q_2^{-R_{42}} = \frac{1}{\sqrt{c}} \cdot \sqrt{\hbar} \cdot \frac{1}{\sqrt{\varepsilon_0}} \cdot (GeV/J)^{-1}$)
- SI_calibrate (kelvin, $c_5^{-P_{55}} q_2^{-R_{52}} = k^{+1} \cdot (GeV/J)^{-1}$)

5.6 nat model

Finally we add one more constraint: The gravitational constant G = 1. Now the last of the dimensinal quantities goes away, and we have no freedom concerning how to denominate quantities. We work in units of the Planck mass, Planck length, Planck time, and so forth.

The constant constraints are

1. $c = 1$
2. $G = 1$ (This is measured in cubic meters/kg per second squared)
3. $\hbar = 1$
4. $\varepsilon_0 = 1$ (This is measured in Farads/meter)
5. $k = 1$

There are no denominating quantities.

1. (dimension corresponding to length is suppressed)
2. (dimension corresponding to mass is suppressed)

3. (dimension corresponding to time is suppressed)
4. (dimension corresponding to current is suppressed)
5. (dimension corresponding to temperature is suppressed)

$$C = U = \begin{array}{cc} \begin{matrix} m & kg & s & A & K \end{matrix} & \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 \\ -3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -2 & 0 & -1 \end{pmatrix} & \begin{matrix} c_1 = c \\ c_2 = G \\ c_3 = \hbar \\ c_4 = \varepsilon_0 \\ c_5 = k \end{matrix} \end{array}$$

$$P = K = U^{-1} = \begin{array}{cc} & \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \\ \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{matrix} & \begin{pmatrix} -1.5 & 0.5 & 0.5 & 0.0 & 0.0 \\ 0.5 & -0.5 & 0.5 & 0.0 & 0.0 \\ -2.5 & 0.5 & 0.5 & 0.0 & 0.0 \\ 3.0 & -0.5 & 0.0 & 0.5 & 0.0 \\ 2.5 & -0.5 & 0.5 & 0.0 & -1.0 \end{pmatrix} \end{array}$$

δ_1, δ_2, δ_3, δ_4 and δ_5 are all suppressed.
And reading exponents of $-P$ to use as exponents of c_i, we get:

- SI_calibrate (meter, $c_1^{-P_{11}} c_2^{-P_{12}} c_3^{-P_{13}} = \sqrt{c^3 G^{-1} \hbar^{-1}}$)
- SI_calibrate (kilogram, $c_1^{-P_{21}} c_2^{-P_{22}} c_3^{-P_{23}} = \sqrt{c^{-1} G^{+1} \hbar^{-1}}$)
- SI_calibrate (second, $c_1^{-P_{31}} c_2^{-P_{32}} c_3^{-P_{33}} = \sqrt{c^5 G^{-1} \hbar^{-1}}$)
- SI_calibrate (ampere, $c_1^{-P_{41}} c_2^{-P_{42}} c_4^{-P_{44}} = c^{-3} \sqrt{G^{+1} \varepsilon_0^{-1}}$)
- SI_calibrate (kelvin, $c_1^{-P_{51}} c_2^{-P_{52}} c_3^{-P_{53}} c_5^{-P_{55}} = k^{+1} \sqrt{c^{-5} G^{+1} \hbar^{-1}}$)

6 Where is this Computed and Embedded in SIunits

The information needed to define and implement a view is embedded in View.h, in the form of two types of information inside the *Xxx*View structure (for instance, StdView or RelView):

1. The nested `ViewMap` structure, which has a typedef `Ans` which tells how the units in the *std* view translate to units in this view. The way this is coded is guided by the components of $R_{\alpha\beta}$.

   `ViewMap::Ans` is used to decide whether two quantities are commensurate.

2. A set of seven `SI_calibrate` macros. Each defines (among other things) a function like meter(), and uses the scale factor (which comes from our s_α) to define the result of that function.

   The templated `Conversion` structure makes use of appropriate powers of those functions to provide for conversion of dimensioned quantities to this view.

6.1 The viewMaker Method

The information needed to intelligently create the `ViewMap` structure, and the seven calibration constants used in `SI_calibrate` macros, is generated by an auxilliary free function in viewMaker.cc. This takes as input seven "Statements," where each Statement comprises a value and a seven-tuple of powers of dimensions, plus a bool to select whether this represents a physical constant constraint or a denominating quantity.

The order of these Statements is relevant in the following sense: The n-th Statement describes how to treat dimension n. If that Statement is a constraint, for example, then dimension n will be suppressed.

(It may be noted, however, that the behavior of the SIunits package as seen by the user is invariant under premutations of which dimension is internally labelled 1, 2, ... N_d. That is, when using the output of `viewMaker()` to code the `ViewMap` structure, there is freedom to re-arrange the dimension labels such that, for example, all the suppressed dimensions lie at the higher numbers. This may be beneficial in permitting the use of shorter lists in the more heavily constrained models.)

Assuming the set of Statements is not inconsistent, and that each dimension which was directed to be suppressed can indeed be eliminated based on the constraints, viewmaker() will set up and invert the U matrix, and return the calibrations and values which can be read off to from the ViewMap.