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Deconstructing 5-D QED

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Abstract

We discuss periodic compactification and latticization of a 5-D $U(1)$ theory with a Dirac fermion, yielding a $1 + 3$ effective theory. We address subtleties in the lattice fermionic action, such as fermion doubling and the Wilson term. We compute the Coleman-Weinberg potential for the Wilson line which is finite for N -branes ≥ 3 , due to the Z_N symmetry, which replaces translations in the 5th dimension. This mode becomes a PNGB in the low energy $1 + 3$ theory. We derive its anomalous coupling to the “photon” and its KK-modes.

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1 Introduction

We consider a QED-like theory in $1 + 4$ dimensions, periodically compactified to $1 + 3$. The “electron” will be considered to be a heavy vectorlike fermion with an arbitrary mass, possibly larger than the compactification scale. We are particularly interested in the fate of the Wilson line in the low energy $1 + 4$ theory, or equivalently, the zero-mode of the fifth component of the vector potential, A_4 (our $1 + 4$ space-time indices run from 0 to 4). The Wilson line appears in the low energy $1 + 3$ effective theory as a dynamical degree of freedom which imitates a low mass pseudo-Nambu-Goldstone boson (PNGB), and we will refer to it as the WLPNGB.

In a nonabelian gauge theory WLPNGB’s acquire (mass)² of order \tilde{g}^2/R^2 where R is the size of the compact extra dimension and \tilde{g} the low energy effective coupling constant, arising from gauge interactions as well as matter interactions. However, in a $U(1)$ gauge theory the WLPNGB is neutral and receives no contribution to its mass from gauge interactions. It does however, acquire mass from its coupling to the vectorlike fermion. We derive in detail the Coleman-Weinberg effective potential [1] of the WLPNGB, after dealing with a number of technical issues.

We use a lattice approximation of the extra dimensional theory [2, 3] (see also [4]). When we latticize with N slices, or branes, the Coleman-Weinberg potential for the WLPNGB is finite for $N \geq 3$. The finite Coleman-Weinberg potential can, moreover, be reexpressed in terms of the low energy parameters and is therefore unambiguously determined in the lattice regulator scheme. The minimum of the Coleman-Weinberg potential determines the value of the Wilson line that wraps around the extra dimension. This Wilson line can be absorbed into the the fermion field and dictates its boundary conditions. Under this redefinition we find that the minimum of the potential corresponds to the fermion having *antiperiodic boundary conditions* in traversing the extra dimension.

The finiteness of the potentials for pseudo-Nambu-Goldstone bosons in models with Z_N symmetry was noticed long ago [5, 6]. The finiteness is a consequence of Z_N invariance of the full theory. The “schizon” models of Hill and Ross [6] exploited $Z_{2L} \times Z_{2R}$ to reduce the degree of divergence from quadratic to logarithmic and implement ultra-low-mass PNGB’s to provide natural “5th” forces in the Standard Model, and remedy certain limits on the axion. With Z_3 symmetry, finite neutrino-schizon models have been used to engineer the first “quintessence” models, late-time cosmological phase transitions, and place limits upon time dependent fundamental constants [7]. The finite temperature behavior of such

models is quite remarkable [8]. Remarkably, these models are structurally equivalent to the present extra-dimensional scheme with latticized fermions when written in the momentum space expansion in the fifth dimension!

In part, it is our interest in studying low energy PNGB's, such as the axion and ultra-low-mass cosmological PNGB's that has motivated the present study. The application of constructing models of ultra-low-mass PNGB's, such as the axion or quintessence, that are *immune from the effects of Planck-scale breaking of global symmetries*, will be described in a companion paper [9].

In contemplating extra dimensions of spacetime the lattice provides a useful tool for regulating the enhanced quantum loop divergences of the extra dimensional theory, and generates a gauge invariant low energy effective Lagrangian with a finite subset of Kaluza-Klein modes [2, 3]. A lattice description, or “deconstruction,” of a $1 + 4$ theory involves discretization of the fifth dimension, x^4 . It therefore becomes an equivalent effective $1 + 3$ theory with N copies of the gauge group and matter fields. This appears to be the only gauge invariant regulator for a fixed number N of KK-modes, where N plays the role of a cut-off. This procedure is powerful, and has suggested new directions and dynamics in building models of new physics beyond the Standard Model, e.g. [10, 11, 12].

Nevertheless, a faithful representation of a higher dimensional theory using a lattice involves subtleties which arise particularly when fermions are introduced. These issues can be side-stepped if one is only interested in a generalized concept of an extra dimension, e.g., “theory space.” However, we desire a *bona-fide* lattice description of a physical extra dimensional theory in which the lattice spectrum agrees precisely with the continuum spectrum at low energies, i.e., for n KK-modes, with $n \ll N$.

One of the key issues in latticizing fermions is the fermion flavor doubling problem. This is remedied by adding the Wilson term. There are also odd-even artifacts which one must reconcile. Finally, a redefinition of the parameters appearing in the lattice Lagrangian is required to match the continuum. As we will see, much of this in the $1 + 4 \rightarrow 1 + 3$ case can be understood diagrammatically. It is in the fermion structure of the theory that latticization of an extra dimension becomes most interesting [2, 3].

In addition to the Coleman-Weinberg potential, we also study the anomalous coupling of the WLPNGB to $F \star F$. Only the WLPNGB has anomalous couplings to the gauge fields. There occurs a universal anomalous coupling of the WLPNGB to each $F_n \star F_n$ for each KK-mode photon.

2 QED in 5-D Compactified to 4-D

2.1 Wilson Line and Gauge Invariance Under Compactification

Consider a field which lives in five spacetime dimensions, $\psi(x^\mu, x^4)$, and which transforms under a local $U(1)$ gauge transformation $e^{i\phi(x^4)}\psi(x^4)$. If we demand that $\psi(x^4)$ lives on a compact periodic manifold in the fifth dimension, e.g., we impose a periodic condition $\psi(x^\mu, x^4) = \psi(x^\mu, x^4 + R)$, then we must also require $e^{i\phi(x^\mu, x^4)}\psi(x^\mu, x^4) = e^{i\phi(x^\mu, x^4+R)}\psi(x^\mu, x^4+R)$. This requires a modular constraint on the gauge transformation:

$$\phi(x^\mu, x^4 + R) = \phi(x^\mu, x^4) + 2\pi n. \quad (2.1)$$

The vector potential A_A transforms under the gauge transformation as: $A'_A = A_A - \partial_A\phi$. Consider the Wilson line around the periodic manifold:

$$\chi(x^\mu) = \int_0^R dx^4 A_4(x^\mu, x^4). \quad (2.2)$$

The Wilson line, $\chi(x^\mu)$ behaves like a dynamical spin-0 field in the 1 + 3 theory. Indeed, χ , given a canonical normalization, is the Wilson-line pseudo-Nambu-Goldstone boson (WLPNGB). Under the gauge transformation the Wilson line transforms as:

$$\chi(x^\mu) \rightarrow \int_0^R dx^4 A'_4 = \chi(x^\mu) - 2\pi n. \quad (2.3)$$

The Wilson line is just the ($p^4 = 0$) zero-mode of A_4 . Under gauge transformations the zero-mode can only be shifted by a constant. Owing to the periodic compactification, gauge transformations of the zero mode are thus restricted:

$$A'_4(x^\mu) = A_4(x^\mu) - 2\pi n/R. \quad (2.4)$$

These results imply that the continuous $U(1)$ “chiral symmetry” of the χ field is explicitly broken to the center Z_N , and in general χ will acquire a mass. A lattice regularization of the quantum loops of the theory manifestly respects this constraint.

As we will see, with the lattice regulator, the Z_N symmetry, takes the place of the continuous S_1 translational symmetry of the continuum limit. The Wilson line effective potential is most naturally computed in a lattice approximation.

2.2 Gauge Field Lattice

Presently we consider a $U(1)$ gauge theory in 1 + 4 dimensions that is *periodically compactified* to 1 + 3. The latticization of a $U(1)$ gauge theory with a periodic fifth dimension

is straightforward. The effective Lagrangian becomes the gauged chiral Lagrangian in 1 + 3 dimensions for N copies of the $U(1)$ gauge group:

$$\mathcal{L} = \sum_{n=1}^N \left[-\frac{1}{4} F_{n\mu\nu} F_n^{\mu\nu} + D_\mu \Phi_n^\dagger D^\mu \Phi_n - V(\Phi_n) \right]. \quad (2.5)$$

Here we have N gauge groups $U(1)_n$ with a common (dimensionless) gauge coupling g , and N link-Higgs fields, Φ_n , having charges $(0, 0, \dots, -1_n, 1_{n+1}, \dots, 0)$ under $(U(1)_1, \dots, U(1)_N)$. The covariant derivative acts upon Φ_n as:

$$D_\mu \Phi_n = \partial_\mu \Phi_n + ig(\tilde{A}_{n\mu} - \tilde{A}_{n+1\mu})\Phi_n. \quad (2.6)$$

where all fields are functions of 1 + 3 spacetime x^μ . (Note: Henceforth we will denote the x^4 configuration space vector potentials $\tilde{A}_{n\mu}$ and Higgs-phase fields $\tilde{\chi}_n$ with a tilde; the corresponding fields without the tilde will be conjugate p^4 momentum space fields, $A_{p\mu}$ and χ_p ; $n = N + 1$ is identified with $n = 1$.)

As a boundary condition, we identify:

$$\Phi_n = \Phi_{mN+n} \quad (2.7)$$

for integer m . This implements the periodic compactification. Notice that eq. (2.5) with eq. (2.7) is Z_N invariant under $F_{n\mu\nu} \rightarrow F_{(n+m)\mu\nu}$ and $\Phi_n \rightarrow \Phi_{(n+m)}$. This Z_N invariance has replaced the continuous S_1 translational invariance of the compactified theory. It is an arbitrarily good approximation for large N in physical quantities that are insensitive to the short-distance (UV) structure of the theory. The Coleman-Weinberg potential is such a UV-safe quantity.

The potential $V(\Phi_n)$ causes each Φ_n to develop a common VEV. Φ_n is then interpreted as the Wilson link, linking brane n to brane $n + 1$:

$$\Phi_n = (v/\sqrt{2}g) \exp \left(ig \int_{x_n^4}^{x_{n+1}^4} dx^4 \tilde{A}_4 \right) \rightarrow (v/\sqrt{2}g) \exp(iga\tilde{A}_{4n}). \quad (2.8)$$

Thus, Φ_N links brane N to brane 1. Here a is the physical lattice constant, the distance between nearest neighboring branes in x^4 . In order for eq. (2.5) to reproduce the continuum limit kinetic terms, $-(1/4)(\partial_\mu \tilde{A}_4 - \partial_4 \tilde{A}_\mu)^2$, we must take $v = 1/a$, which is related to the compactification scale:

$$R = Na = N/v. \quad (2.9)$$

From the point of view of 1 + 3 dimensions, each Φ_n is effectively a nonlinear- σ model field:

$$\Phi_n \rightarrow (v/\sqrt{2}g) \exp(ig\tilde{\chi}_n/v). \quad (2.10)$$

The Φ_n kinetic terms then go over to a mass-matrix for the gauge fields:

$$\frac{1}{2}v^2 \sum_{n=1}^N \left((\tilde{A}_{n+1\mu} - \tilde{A}_{n\mu}) - \frac{1}{v} \partial_\mu \tilde{\chi}_n \right)^2. \quad (2.11)$$

To diagonalize eq. (2.11) it is useful to pass to a complex representation. Without loss of generality consider:

$$\tilde{A}_{n\mu} = \frac{1}{\sqrt{N}} \sum_{p=-J}^{J+\delta} A_{p\mu} \exp(2\pi i p n / N); \quad A_{p,\mu}^* = A_{-p,\mu}, \quad (2.12)$$

$$\tilde{\chi}_n = \frac{1}{\sqrt{N}} \sum_{p=-J}^{J+\delta} \chi_p \exp(2\pi i p n / N); \quad \chi_p^* = \chi_{-p}. \quad (2.13)$$

Here $J = (N - 1)/2$ and $\delta = 0$ for N odd [$J = (N - 2)/2$ and $\delta = 1$ for N even]. The p -representation preserves canonical normalizations, for example the $U(1)$ kinetic terms in the Lorentz gauge, $\partial_\mu A_n^\mu = 0$, become:

$$-\frac{1}{4} \sum_{n=1}^N F_{n\mu\nu} F_n^{\mu\nu} = -\frac{1}{2} (\partial_\mu A_{0\nu})^2 - \sum_{p=1}^J |\partial_\mu A_{p\nu}|^2 - \left\{ \frac{1}{2} \delta \right\} (\partial_\mu A_{(N/2)\nu})^2. \quad (2.14)$$

Note the last term is absent when N is odd ($\delta = 0$). Henceforth, for notational simplicity, we will refrain from writing the δ term, though it is implicitly present when N is even. Now, we define:

$$F_p = [\exp(2\pi i p / N) - 1], \quad F_p^* F_p = 4 \sin^2(\pi p / N), \quad (2.15)$$

and then see that eq. (2.11) becomes:

$$\frac{1}{2} (\partial_\mu \chi_0)^2 + \sum_{p=1}^J v^2 |A_{p\mu} F_p - \frac{1}{v} \partial_\mu \chi_p|^2. \quad (2.16)$$

We can perform a momentum space gauge transformation on each component, except $p = 0$:

$$A_p \rightarrow A_p + \frac{1}{v F_p} \partial_\mu \chi_p \quad p \neq 0; \quad A_0 \rightarrow A_0. \quad (2.17)$$

From this we obtain:

$$\frac{1}{2} (\partial_\mu \chi_0)^2 + 4v^2 \sum_{p=1}^J |A_{p\mu}|^2 \sin^2(\pi p / N). \quad (2.18)$$

The spectrum therefore contains the real zero-mode of A_μ , A_4 (which is χ_0), and a tower of doubled Kaluza-Klein-modes, appearing as massive photons, each pair labeled by p , of mass:

$$M_p^2 = 4v^2 \sin^2(\pi p / N). \quad (2.19)$$

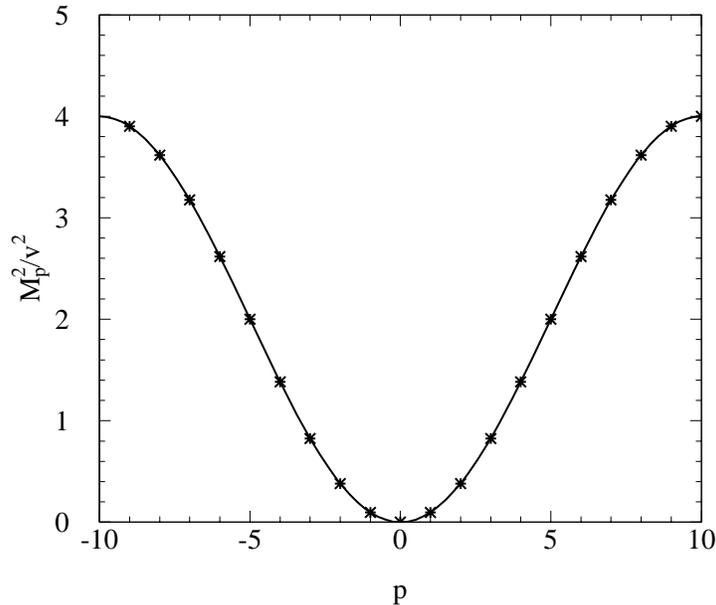


Figure 1: The vector boson mass spectrum is plotted for $N = 20$ and $N = 19$. For $N = 19$ all states are as indicated from $-9 \leq p \leq 9$, including the single $p = 0$ zero-mode photon. For $N = 20$ we include an extra state $p = 10$. The spectrum exhibits physical doubling, a consequence of periodic compactification, and encompasses a single Brillouin zone.

For N even ($\delta = 1$) the $p = N$ mode occurs as well, as a singlet with $(\text{mass})^2 4v^2$. Correspondingly, all but the zero-mode linear combination of the χ_0 are “eaten” to become longitudinal modes.

The spectrum of gauge fields for periodic compactification was discussed previously in [3]. For N odd [even] it admits a zero mode gauge field, and a tower of $(N - 1)/2$ [$(N - 2)/2$] doubled Kaluza-Klein modes [and a singlet highest mode]. This doubling is normal and a physical consequence of the periodic compactification; e.g., there will occur both left moving and right moving modes in the periodic manifold and these are degenerate (alternatively, the sine modes are degenerate with the corresponding cosine modes). This is shown in Figure 1 for $N = 20$ and $N = 19$. The first Kaluza-Klein gauge mode has a mass of $M_K \approx 2v\pi/N$ [3]. This is identified with the compactification scale $2\pi/R$, hence we again recover eq. (2.9), $vR = N$.

In summary, the master gauge group, $U(1)^N$, is broken to the diagonal subgroup $U(1)$ by the Φ_n . $N - 1$ of the link fields χ_n are eaten by the Higgs mechanism, giving $N - 1$ massive vector fields (the Kaluza-Klein states). We are left with one massless vector field and one massless scalar. The linear combination of the link fields that remains massless

in the classical limit is:

$$\chi_0 \equiv -\frac{iv}{g\sqrt{N}} \ln \left[\prod_{n=1}^N (\sqrt{2}g\Phi_n/v) \right] = \frac{1}{\sqrt{N}} \sum_{n=1}^N \tilde{\chi}_n. \quad (2.20)$$

This mode is the WLPNGB.

2.3 Latticizing Fermions

We now include a single Dirac fermion in the 1 + 4 theory with charge $q = -1$. The fermion-gauge boson system has the continuum action:

$$\mathcal{L} = \int d^5x \bar{\Psi} [(i\cancel{\partial} - g\cancel{A}) - (\partial_4 + igA_4)\gamma^5 - M]\Psi. \quad (2.21)$$

where the fifth γ -matrix is $i\gamma^5$. Consider $A_4 = 0$. For the fermion in the continuum with periodic compactification of the x^4 spatial manifold we can impose a periodic boundary conditions:

$$\Psi(x^4) = \psi(x^4 + R). \quad (2.22)$$

If, however, we view the manifold as the *boundary* of a 1 + 5 dimensional space then we must use the *antiperiodic* boundary condition:

$$\Psi(x^4) = -\psi(x^4 + R). \quad (2.23)$$

In the latter case the minus sign arises because the spinor is being rotated through 2π in the 1 + 5 space as we traverse the periodic compactification.

In the periodic case the x^4 momentum space basis functions are $\exp(ikx^4)$ where the momenta are $k = (2p)\pi/R$ where p is an integer that runs from $-\infty$ to $+\infty$. The $p = 0$ fermion is potentially a zero mode, but will always be vector-like (having both L - and R -handed components). In the antiperiodic case, the x^4 momenta are $k = (2p+1)\pi/R$ where p is an integer that runs from $-\infty$ to $+\infty$. There is no zero mode in the antiperiodic case.

In both periodic and antiperiodic boundary conditions all levels other than the zero mode are doubled as (p_1, p_2) : (periodic) $(-1, 1)$ $(-2, 2)$; (antiperiodic) $(-1, 0)$, $(-2, 1)$, etc. This doubling is physical, as we saw in the gauge field case, corresponding to the fact that a traveling wave packet moving to the left is degenerate with one moving to the right. The physical doubling stems from parity, i.e., the symmetry of $x^4 \rightarrow -x^4$.

Consider now a nontrivial Wilson line,

$$g \int_0^R dx^4 A_4 = \pi; \quad A_4 = \frac{\pi}{gR}, \quad (2.24)$$

which can be implemented with the indicated constant gauge field. If we begin with the antiperiodic boundary condition, we can nonetheless redefine the fermion field by a phase factor as:

$$\hat{\Psi}(x^4) = \exp\left(ig \int_0^{x^4} dx^4 A_4\right) \Psi(x^4). \quad (2.25)$$

With this redefinition we thus “gauge away” the Wilson line, but we have now modified the fermion boundary condition to be periodic. Indeed, we are free to use the Wilson line to achieve any desired boundary conditions on the fermion field. The presence of the $U(1)$ interaction thus influences the spectrum of fermions in a fundamental way. As we will see, however, the ground-state energy, when minimized, will determine the particular value of the Wilson line, hence the fermionic boundary conditions, dynamically. (Of course, if we view the $1 + 4$ manifold as a boundary of $1 + 5$, then there is additional field energy contributing through the magnetic field living in the $1 + 5$ bulk.) We will find that the *antiperiodic boundary conditions* are dynamically favored.

The fermionic $U(1)$ theory can be latticized as follows. We place independent fermions, ψ_n on brane n , i.e., having charge $(0, 0, \dots, -1_n, \dots, 0, 0..)$, hence the covariant derivative acts as:

$$D_\mu \tilde{\Psi}_n = (\partial_\mu + ig \tilde{A}_{n\mu}) \tilde{\Psi}_n. \quad (2.26)$$

Then a “naive” lattice in x^4 leads to the action:

$$\sum_{n=1}^N \int d^4x \left[\bar{\tilde{\Psi}}_n (i \not{D} - M) \tilde{\Psi}_n - \left(\frac{1}{\sqrt{2}} g \eta \bar{\tilde{\Psi}}_n \gamma_5 \Phi_n \tilde{\Psi}_{n+1} + h.c. \right) \right] \quad (2.27)$$

The nearest neighbor hopping term represents the kinetic term in the fifth dimension. We have taken the simplest lattice approximation to the derivative,

$$\partial_4 \Psi \rightarrow (\tilde{\Psi}_{n+1} - \tilde{\Psi}_n)/a. \quad (2.28)$$

Note that the normalization of the hopping term follows from the normalization of $\Phi_n = v/g\sqrt{2}$, when $A_4 = \chi_n = 0$, together with $v = 1/a$, and the fact that we latticized a Hermitian “backward-forward” derivative, $\partial_4 \equiv (-i/2)(\vec{\partial}_4 - \overleftarrow{\partial}_4)$, which together with the $i\gamma^5$ leads to a Hermitian kinetic term (or equivalently, use the “forward” derivative $(i/2) \vec{\partial}$ of eq. (2.28) and add $+h.c.$); this kills off the diagonal $\bar{\tilde{\Psi}}_n \gamma^5 \tilde{\Psi}_n$ terms. Here we have introduced a parameter η which will be needed for matching to the continuum theory spectrum.

Passing to a chiral projection basis on each brane, eq. (2.27) can be written as:

$$\sum_{n=1}^N \int d^4x \left[\bar{\tilde{\Psi}}_{nL} (i \not{D}) \tilde{\Psi}_{nL} + \bar{\tilde{\Psi}}_{nR} (i \not{D}) \tilde{\Psi}_{nR} - M (\bar{\tilde{\Psi}}_{nL} \tilde{\Psi}_{nR} + h.c.) \right]$$

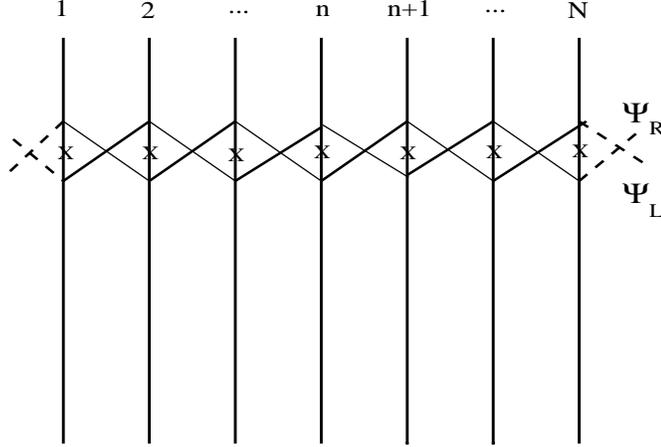


Figure 2: Dirac fermion has both chiral modes on each brane, where the upper (lower) vertices are R (L) modes. The \times 's denote the M terms on each brane which couple $\tilde{\Psi}_{nR}$ and $\tilde{\Psi}_{nL}$, and the cross-bars are the latticized fermion kinetic (hopping) terms, $\bar{\Psi}_n \Phi_n \tilde{\Psi}_{n+1}$ couplings. The spectrum has a singlet lowest massive mode of mass M , and doubled Kaluza-Klein modes; by adding a Wilson term one can remove one of the two cross-bars between adjacent branes, and eliminate fermion doubling in the spectrum [3].

$$-\frac{1}{\sqrt{2}} \sum_{n=1}^N \int d^4x \, g\eta \left[\bar{\tilde{\Psi}}_{nL} \Phi_n \tilde{\Psi}_{(n+1)R} - \bar{\tilde{\Psi}}_{nR} \Phi_n \tilde{\Psi}_{(n+1)L} + h.c. \right] \quad (2.29)$$

This is illustrated in Figure 2. We obtain the “zig-zag” pattern as each chirality hops to the opposite chirality on the neighboring brane.

Let us, for the sake of discussion, impose antiperiodic boundary conditions on the fermion field:

$$\tilde{\Psi}_n \equiv (-1)^m \tilde{\Psi}_{n+mN} \quad (2.30)$$

for integers n, m (Note that the action remains of Z_N invariant). The mass spectrum of eq. (2.29) is derived by diagonalizing the Lagrangian. Let:

$$\tilde{\Psi}_n = \frac{1}{\sqrt{N}} \sum_{p=-J}^{J+\delta} e^{(2p+1)i\pi n/N} \Psi_p. \quad (2.31)$$

We see that eq. (2.30) is therefore implemented. Note that there is no value of p for which $2p + 1$ vanishes, hence there is no fermionic zero mode with antiperiodic boundary

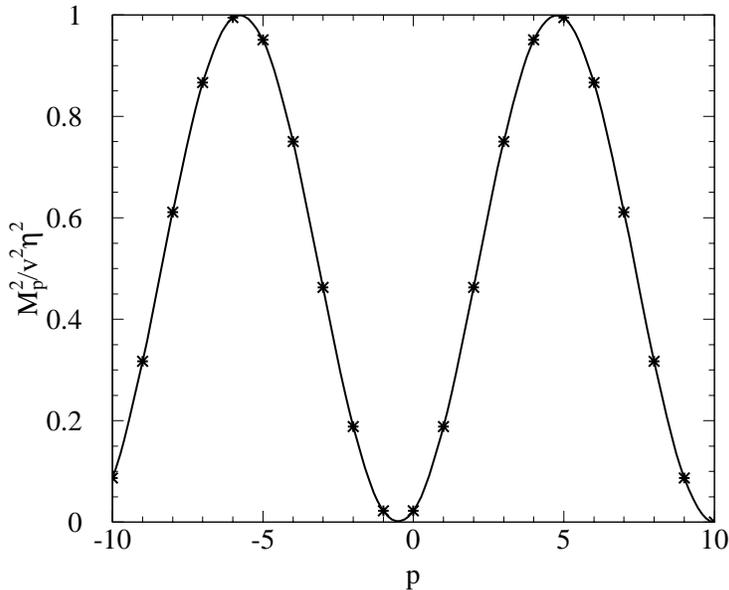


Figure 3: The fermion mass spectrum, with $M = 0$ and without the Wilson term (in units of $v^2 \eta^2$). Here we choose $N = 21$ odd. The spectrum exhibits physical doubling within each minimum, a consequence of periodic compactification. It encompasses two Brillouin zones. The state with $p = 10$ (which is equivalent to $p = -10$) has the lowest mass, but represents an unphysical second flavor, the usual fermion doubling problem for latticized fermions.

conditions, as in the continuum case. Eq. (2.27) becomes:

$$\sum_{p=-J}^{J+\delta} \int d^4x \left\{ \bar{\Psi}_p (i \not{\partial} - M) \Psi_p - i(\eta v) \sin [(2p+1)\pi/N] \bar{\Psi}_p \gamma_5 \Psi_p \right\}, \quad (2.32)$$

where we have substituted eq. (2.10) with $\chi_n = 0$ into this expression, and suppressed the $A_{q\mu}$ in the covariant derivatives. The mass of the p -th mode is therefore:

$$M_p^2 = M^2 + \eta^2 v^2 \sin^2 [(2p+1)\pi/N]. \quad (2.33)$$

Naively, we would argue that the low lying levels of this spectrum expanded about $p = 0$ match onto the continuum with the matching condition of eq. (2.9). Owing to the fermionic antiperiodic boundary condition the lowest modes in the continuum have masses $M_p^2 = M^2 + (2p+1)^2 \pi^2 / R^2$. We obtain presently, for the lattice theory with small p , $M_0^2 = M^2 + \eta^2 v^2 (2p+1)^2 \pi^2 / N^2$. Thus, using $v = 1/a = N/R$, the matching of the low energy modes to the continuum requires a choice of $\eta = 1$. However, we must examine the spectrum of the lattice theory in greater detail.

First, consider N odd (see Fig. (3)), and for simplicity we presently take $M = 0$ and $\eta = 1$. Then we see that the lowest mass fermion is the $p = (N-1)/2$ mode, which is the

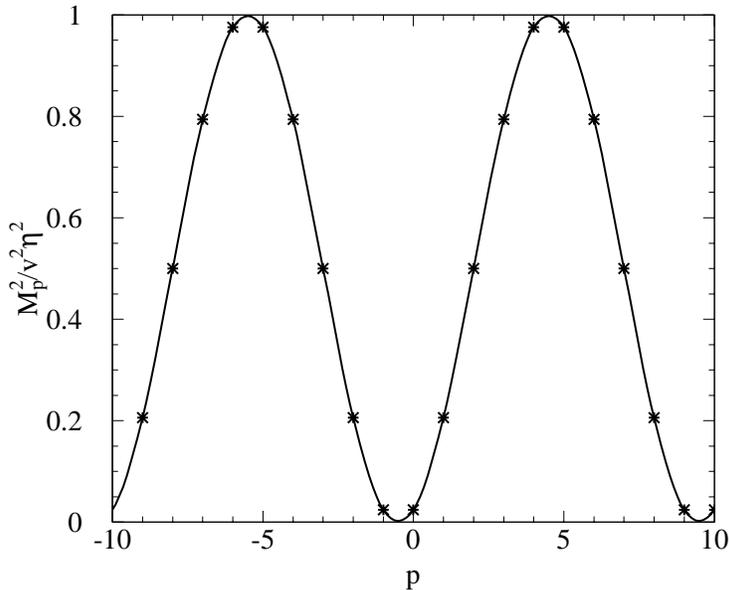


Figure 4: The fermion mass spectrum, with $M = 0$ and without the Wilson term (in units of $v^2 \eta^2$). Here we choose $N = 20$ even. Notice the quadrupling of levels and the absence of a fermionic zero mode due to the antiperiodic boundary conditions. The spectrum exhibits physical doubling within each minimum, a consequence of periodic compactification. It encompasses two Brillouin zones, however, leading to an overall quadrupling of the spectrum. The states in the vicinity of $p = 10$ are an unphysical second flavor.

lowest mass state of zero mass, and is an undoubled singlet. This state was absent in the continuum case and must be a lattice artifact. All other modes are doubled. The $p = 0$ mode is the next lightest state and is degenerate with $p = -1$ with mass $v^2 \sin^2(\pi/N)$; the $p = -(N-1)/2$ with $p = (N-3)/2$ with mass $v^2 \sin^2(2\pi/N)$, etc. This doubling is just the conventional $x^4 \rightarrow -x^4$ invariance. The mass as a function of p , therefore, has two basins about the two distinct minima of $\sin^2((2p+1)\pi/N)$ corresponding to the towers of states: $p = [(N-1)/2][-(N-1)/2, (N-3)/2] \dots$ and $p = [0, -1], [1, -2] \dots$. These two basins of states correspond to two distinct Brillouin zones. This is the familiar lattice fermion flavor doubling problem.

We *cannot* interpret the $p = (N-1)/2$ mode as the ground state of a tower with the $p = [0, -1]$ modes as next in the sequence. The transition from $p = 0$ state of mass $\approx v\pi/N$ to the $p = (N-1)/2$ of mass = 0 is allowed virtually with the emission of a $p = [(N-1)/2]$ heavy photon. However the mass of this photon is $2v \sin(\pi(N-1)/2N) \approx 2v$, and the transition can never match energetically, even approximately. On the other hand, the transition from the $p = 1$ state of mass $\approx 3v\pi/N$ to $p = 0$ of mass $\approx v\pi/N$ does match the $p = 1$ photon of mass $2v \sin(\pi/N) \approx 2v\pi/N$. Of course, these energetics only make

sense in the $N \rightarrow \infty$ limit. Hence, we must conclude that the second basin of states (Brillouin zone) represents a second spurious fermionic flavor.

For N even (see Fig. (4)), we see that the ground-state fermion mode $p = 0$ is a non-zero mode, and there is a 4-fold degeneracy with the $p = -1, p = N/2, p = N/2 - 1$ modes; the $p = 1$ is degenerate with $p = -2$ and the $p = N/2 + 1, p = N/2 - 2$ modes, etc, spanning the tower of states sequentially. Again, these form two distinct basins, $p = [0, -1], [1, -2], \dots$ and $p = [N/2, N/2 - 1], [-N/2 + 1, N/2 - 2], \dots$ with accidental degeneracy, and represent two distinct flavors.

The fermion flavor doubling behavior can be understood graphically from Figure 2. Consider the zig-zag hopping line emanating from Ψ_{R1} , coursing through $\Psi_{L2}, \Psi_{R3}, \dots, \Psi_{LN}$. The accidental degeneracy for N even arises because the line which started on Ψ_{1R} ends on $\tilde{\Psi}_{nL}$. When we make the periodic hop we have Ψ_{LN} rejoining to the starting point, Ψ_{R1} . Similarly, Ψ_{L1} is rejoined by its starting point Ψ_{RN} . Hence there are two independent zig-zag hopping paths that course through the lattice. For $M = 0$ these are two independent flavors, coupled only by chirality conserving gauge field interactions. These are equivalent under the Z_2 transformation (parity) which swaps L and R , hence there is quadrupling. With N odd, there is only one zig-zag line since, starting from Ψ_{R1} , one reaches Ψ_{RN} and rejoins Ψ_{L1} , etc. Thus, the quadrupled degeneracy is lifted, and the ground state is a singlet, and all other modes are doubled. This lifting of degeneracy is a small effect in the large N limit. In this limit we just ignore the connection from Ψ_N to Ψ_1 then the two zig-zag paths are degenerate. This then leads to the fermion flavor doubling in the spectrum.

As mentioned above, the usual lattice fermion flavor doubling is always present with the simplest hopping, or first order approximation to the derivative, and the second Brillouin zone then appears in the spectrum. The Brillouin zones are defined by the basins in momentum space localized around minima of the energy (mass), and each tower of states built around the minima.

Notice that, with the Dirac mass $M \neq 0$ and large, these transition energetics can never match, even in the continuum theory! For example, expanding in v/M for large M , we can consider the $p = 1$ state of mass $\approx M + 9v^2\pi^2/2N^2M$ decaying to the $p = 0$ state of mass $\approx M + v^2\pi^2/2N^2M$. The transition energy is thus $\approx 4v^2\pi^2/N^2M$ and cannot match the $p = 1$ photon of mass $2v \sin(\pi/N) \approx 2v\pi/N$. We emphasize that this *is not a lattice artifact!* A heavy fermion will have slow, virtually mediated, decays of its KK-modes, i.e., all KK-modes become quasistable in the large M limit.

This discussion is *not to imply* that we cannot build a model in which the lattice is real, and the second Brillouin zone is therefore physical, e.g., perhaps it is possible to interpret sequential generations of flavors in this way (!) For the present discussion, however, we are interested in a faithful lattice representation of the continuum, hence we view the flavor doubling as an unwanted artifact.

2.4 Incorporating the Wilson Term

The lattice artifact problems described above have a well-known remedy, the addition of the “Wilson term” (see, e.g., the lectures of A. Kronfeld in [14]). The Wilson term is a higher dimension operator that we add to the continuum theory of eq. (2.21), which acts like a bosonic kinetic term:

$$\int d^5x \bar{\Psi}[(i\cancel{D} - g\cancel{A}) - (\partial_4 + igA_4)\gamma^5 - M - \frac{1}{M_X}(\partial_4 + igA_4)^2]\Psi. \quad (2.34)$$

The sign of the Wilson term is fixed by positivity of the energy, while the relative sign of the γ^5 term is arbitrary. In the lattice theory this amounts to adding a term to eq. (2.29) of the form:

$$-\sum_{n=1}^N \int d^4x \frac{\eta'}{2a^2vM_X} \left(\sqrt{2}g\bar{\Psi}_n\Phi_n\tilde{\Psi}_{n+1} + \sqrt{2}g\bar{\Psi}_n\Phi_{n-1}^\dagger\tilde{\Psi}_{n-1} - 2v\bar{\Psi}_n\tilde{\Psi}_n \right). \quad (2.35)$$

We have installed a coefficient η' which will be determined momentarily. For convenience we define $M_X = 1/a \equiv v$. Eq. (2.35) then takes the form in the chiral basis:

$$-\frac{1}{2}\eta' \sum_{n=1}^N \int d^4x \left(\sqrt{2}g\bar{\Psi}_{nL}\Phi_n\tilde{\Psi}_{(n+1)R} + \sqrt{2}g\bar{\Psi}_{nR}\Phi_n\tilde{\Psi}_{(n+1)L} - 2v\bar{\Psi}_{nL}\tilde{\Psi}_{n,R} + h.c. \right) \quad (2.36)$$

Note that we conjugated and used the shift symmetry in the sum $n \rightarrow n+1$ for the second term above. Adding the Wilson term then modifies eq. (2.29):

$$\begin{aligned} & \sum_{n=1}^N \int d^4x \left[\bar{\Psi}_{nL}(i\cancel{D})\tilde{\Psi}_{nL} + \bar{\Psi}_{nR}(i\cancel{D})\tilde{\Psi}_{nR} - \tilde{M}(\bar{\Psi}_{nL}\tilde{\Psi}_{nR} + h.c.) \right] \\ & - \frac{1}{\sqrt{2}} \sum_{n=1}^N \int d^4x \left[g(\eta' + \eta)\bar{\Psi}_{nL}\Phi_n\tilde{\Psi}_{(n+1)R} + g(\eta' - \eta)\bar{\Psi}_{nR}\Phi_n\tilde{\Psi}_{(n+1)L} + h.c. \right] \end{aligned} \quad (2.37)$$

where:

$$\tilde{M} = M - v\eta'. \quad (2.38)$$

We can thus choose $\eta' = \eta$ to cancel the $nR \rightarrow (n+1)L$ hopping terms, or $\eta' = -\eta$ to cancel the $nL \rightarrow (n+1)R$ hopping terms. Note that our freedom to choose the sign of the

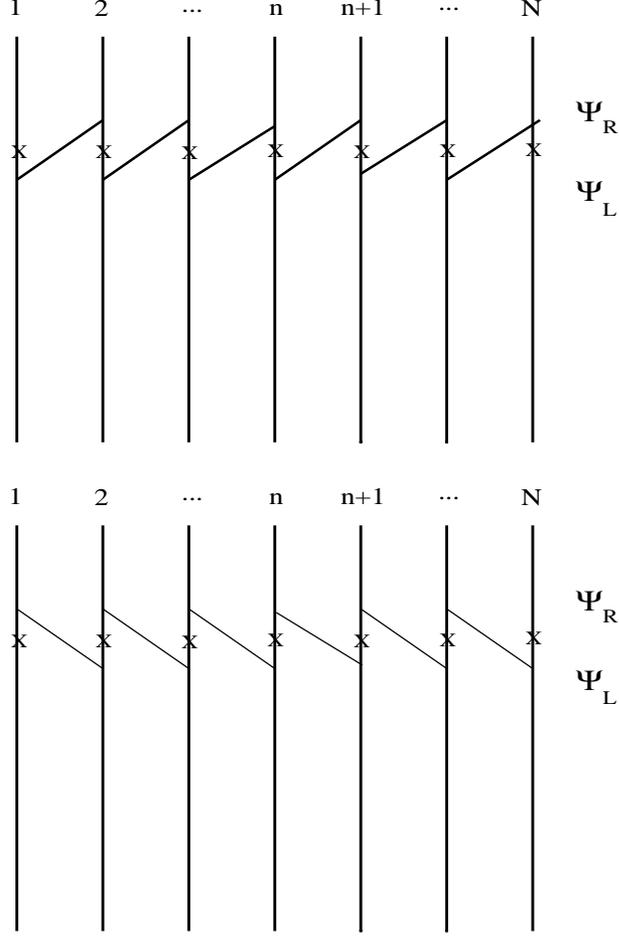


Figure 5: Adding a Wilson term annihilates half of the links. The choice of lattice is controlled by the relative sign, $\eta' = \eta$ ($\eta' = -\eta$) yields the top (bottom) figure. The doubling problem is now solved. Both lattices are periodic [3].

$(\partial_4 + igA_4)\gamma^5$ term relative to the Wilson term by the Z_2 parity inversion of γ_5 , amounts to a freedom of choice in the sign of η . This flips L and R in eq. (2.37) and allows Figure 2 to be replaced by either the upper or lower of Fig. (5).

Let us choose $\eta = \eta'$ and replace $\Phi_n = v/\sqrt{2}g$, whence the Lagrangian eq. (2.37) becomes:

$$\begin{aligned}
& \sum_{n=1}^N \int d^4x \left[\bar{\tilde{\Psi}}_{nL}(iD)\tilde{\Psi}_{nL} + \bar{\tilde{\Psi}}_{nR}(iD)\tilde{\Psi}_{nR} - \tilde{M}(\bar{\tilde{\Psi}}_{nL}\tilde{\Psi}_{nR} + h.c.) \right] \\
& - \sum_{n=1}^N \int d^4x \left[\eta v \bar{\tilde{\Psi}}_{nL}\tilde{\Psi}_{(n+1)R} + h.c. \right]
\end{aligned} \tag{2.39}$$

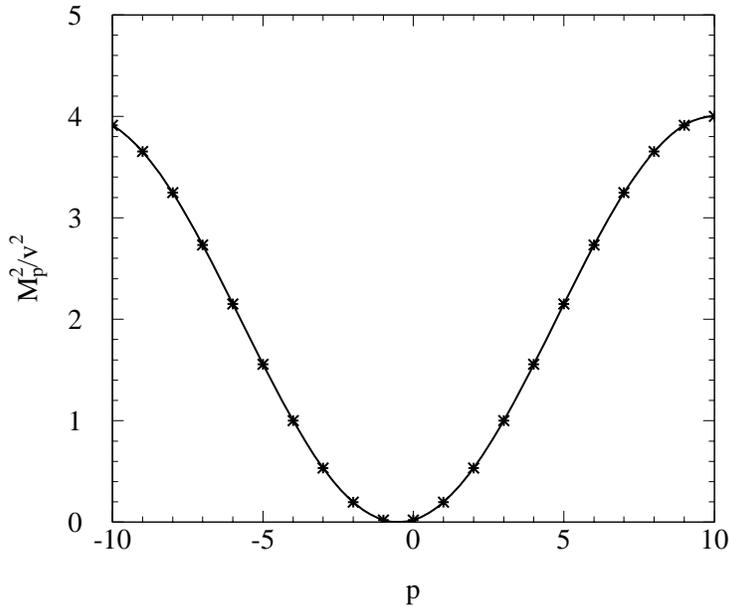


Figure 6: The fermion mass spectrum, with $M = 0$ and with the Wilson term (in units of $v^2\eta^2$). Here we choose $N = 21$ odd. The spectrum exhibits physical doubling a consequence of periodic compactification. It encompasses now only one Brillouin zones, eliminating the unphysical second flavor.

In the momentum eigenbasis, eq. (2.31), with $\Phi_n = v/g\sqrt{2}$, the mass term becomes:

$$- \sum_{p=-J}^{J+\delta} \int d^4x \left[\bar{\Psi}_p (\widetilde{M} + \eta v \cos [(2p+1)\pi/N]) \Psi_p + i\eta v \sin [(2p+1)\pi/N] \bar{\Psi}_p \gamma_5 \Psi_p \right], \quad (2.40)$$

The spectrum with the Wilson term becomes:

$$M_p^2 = M^2 + 4(v\eta - M)v\eta \sin^2 [(2p+1)\pi/2N]. \quad (2.41)$$

With the antiperiodic fermionic boundary condition the lowest fermionic mode in the continuum has a mass $M_0^2 = M^2 + \pi^2/R^2$. In the small $M \ll \eta v$ limit we obtain $M_0^2 = M^2 + \eta^2 v^2 \pi^2 / N^2$ and, using $vR = N$ from the bosonic case the matching of the low energy modes to the continuum requires a choice of $\eta = 1$. More generally, for any M we require for matching to the continuum

$$- \widetilde{M}\eta/v = (\eta v - M)\eta/v = 1. \quad (2.42)$$

For $M > \eta v$ the sign of η flips.

The spectrum of eq. (2.41) is now faithful to the continuum limit (see Fig. 6). Consider N -odd: We have the levels $p = [0, -1]$, $p = [1, -2]$, ... and the previous zero-mode

becomes now the high-mass singlet $p = [(N - 1)/2]$. For N -even we have the same tower of low mass states terminating at the doubled highest mass levels $p = [N/2 - 1, N/2]$. We do not really care about the highest energy states since we cut the theory off for some level $p \ll N/2$. There is now only one basin of levels in both cases.

The fermion doubling problem underlies difficulties for the faithful lattice implementation of SUSY. With the naive latticization, fermions are doubled, while bosons are not, thus breaking SUSY. Of course, this traces to the fact that the Lorentz group, $O(4, 1)$ has been replaced by the group $O(3, 1) \times Z_N$. Thus, at the most fundamental level, the supersymmetric grading of the Lorentz group $O(4, 1)$ is not expected to be implemented in the lattice approximation. Deconstructions of SUSY theories to date treat the SUSY hopping terms as parts of a superpotential [12]. However, they should properly emerge from Kahler potentials in the higher dimensional theory, and it is likely that other terms or constraints will arise that are required for anomaly matching, etc. There is also the issue of chirality, such as the localization of domain wall fermions [15], which is an interesting story in itself. This has a nice realization in terms of the lattice construction: the chiral fermion shows up as a dislocation in the lattice hopping terms [3] (see also Hill, He and Tait in ref.([11])). We will return to these and other issues elsewhere [16].

3 The Effective Potential

We now calculate the Coleman-Weinberg potential for the WLPNGB field χ by integrating out the fermions. This amounts to computing the fermionic determinant for the Lagrangian eq. (2.27) in the classical background ϕ :

$$V = +i\hbar \ln [\det(i\mathcal{D} - M(\chi))]. \quad (3.43)$$

To leading order we neglect the 1 + 3 vector potentials, and replace each Φ_n by their common dependence upon the zero mode, χ_0 :

$$\Phi_n \rightarrow \Phi \equiv (v/\sqrt{2}g) \exp(ig\chi_0/\sqrt{N}v), \quad (3.44)$$

where we have used the normalization conventions of eq. (2.10). The Dirac action with the Wilson term is now:

$$\sum_{n=1}^N \int d^4x \left[\bar{\tilde{\Psi}}_n (i\mathcal{D} - \tilde{M}) \tilde{\Psi}_n - \eta v \bar{\tilde{\Psi}}_{nL} \exp(ig\chi_0/\sqrt{N}v) \tilde{\Psi}_{(n+1)R} + h.c. \right] \quad (3.45)$$

Going to the p -basis we obtain:

$$\sum_{p=-J}^{J+\delta} \int d^4x \left[\bar{\Psi}_p(i\cancel{\partial} - \tilde{M} - \eta v \cos[\Omega_p])\Psi_p - i\eta v \sin[\Omega_p] \bar{\Psi}_p \gamma_5 \Psi_p \right] \quad (3.46)$$

where:

$$\Omega_p \equiv \frac{(2p+1)\pi v + g\sqrt{N}\chi_0}{Nv}. \quad (3.47)$$

This shifts the mass of each mode:

$$M_p^2 = M^2 + 4(v\eta - M)v\eta \sin^2\left[\frac{1}{2}\Omega_p\right] = \tilde{M}^2 + \eta^2 v^2 + 2\tilde{M}v\eta \cos(\Omega_p). \quad (3.48)$$

The mass term is periodic under:

$$\chi_0 \rightarrow \chi_0 + 2\pi\sqrt{N}v/g \quad (3.49)$$

which corresponds, with our normalization, to the modular invariance of the Wilson line under gauge transformations.

The functional integral over fermions yields the Dirac determinant:

$$Z = \det(i\cancel{D} - M(\chi_0)) = \prod_{k,p} \det(\cancel{k} - M_p(\chi_0)), \quad (3.50)$$

where the second expression contains a product over all 4-momenta and the discretized 5th momentum, and the determinant runs of the all four Dirac spin states for a given (k, p) . This latter expression applies when the mass matrix is diagonal. Z is symmetric under an overall change in the signs of the γ_μ , hence we can write:

$$Z^2 = \prod_{k,p} [-k^2 + M_p^2(\chi_0)]^4 \quad (3.51)$$

and the effective potential is:

$$V = i \ln Z = 2i \int \frac{d^4k}{(2\pi)^4} \sum_p \ln(-k^2 + M_p^2(\chi_0)). \quad (3.52)$$

Here, and subsequently, we discard additive (non- χ_0 dependent) constants. The k -integrals can be performed with a Wick-rotation and we use a Euclidean cut-off, Λ , obtaining to order $1/\Lambda^2$:

$$V = -\frac{1}{16\pi^2} \sum_p \left[\Lambda^4 \left(\ln \Lambda^4 - \frac{1}{2} \right) + 2M_p^2 \Lambda^2 - M_p^4 \ln \left(\frac{\Lambda^2}{M_p^2} \right) - \frac{1}{2} M_p^4 \right]. \quad (3.53)$$

Under Z_N transformations, $\tilde{\Psi}_n \rightarrow \tilde{\Psi}_{n+m}$, χ_0 is invariant, hence the induced potential for χ_0 is invariant under Z_N . For $N \geq 3$ the explicit dependence upon χ_0 in V is finite.

Therefore, the only Z_N invariants that can arise for $N \geq 3$ in the sum involve the term with logarithm. Hence, V can be written in a form, summed on p for $N \geq 3$:

$$V = -\frac{v^4}{16\pi^2} \sum_p \left[\left(\eta^{-2} + \eta^2 - 2 \cos(\Omega_p) \right)^2 \ln \left(1 - \frac{2 \cos(\Omega_p)}{\eta^{-2} + \eta^2} \right) \right], \quad (3.54)$$

where we have used the matching condition, $-\widetilde{M}\eta = v$. This does not yet display the full Z_N suppression, and we must therefore expand the logarithm (the $n < 3$ terms average to zero upon summing over p):

$$V = \frac{v^4}{8\pi^2} \sum_p \sum_{n=3}^{\infty} \left[\frac{1}{n(n-1)(n-2)} \left(\frac{(\exp[i\Omega_p] + h.c.)^n}{(\eta^{-2} + \eta^2)^{n-2}} \right) \right], \quad (3.55)$$

where additive constants that have no χ_0 dependence and terms unimportant for $N \geq 3$ are dropped.

Let us consider the natural limit in which M is small compared to v (but M can be arbitrary compared to $1/R$ since $v = N/R$). We have $\widetilde{M} = M - \eta v$ and the matching condition, $-\widetilde{M}\eta = v$. Hence, in the limit $M/v \rightarrow 0$ we have $\eta \rightarrow \pm 1$ and $\widetilde{M} \rightarrow \mp v$:

$$V = \frac{v^4}{2\pi^2} \sum_p \sum_{n=3}^{\infty} \frac{1}{n(n-1)(n-2)} \cos^n[\Omega_p]. \quad (3.56)$$

Expanding $\cos^n(\Omega) = (e^{i\Omega}/2 + e^{-i\Omega}/2)^n$ gives:

$$V = \frac{v^4}{\pi^2} \sum_p \sum_{n=3}^{\infty} \frac{2^{-n}}{n(n-1)(n-2)} \sum_{m=0}^{[n/2]} \binom{n}{m} \cos[(n-2m)\Omega_p]. \quad (3.57)$$

The only χ_0 dependent terms that can survive the sum over p in the expansion involve

$$(\exp[i\Omega_p])^{qN} \quad (3.58)$$

for integer q , a consequence of the Z_N invariance. Such terms reduce the sum on p to an overall factor of $(-1)^q N$. (Note: with periodic fermion boundary conditions the sum on p gives N and the overall sign of the potential is flipped; this is consistent with the Wilson line redefinition of the fermionic boundary condition.) Therefore, the sum over p will only give contributions for $n - 2m = qN$ for integer q :

$$V = \frac{v^4 N}{\pi^2} \sum_{q=1}^{\infty} \sum_{n=qN}^{\infty} \frac{(-1)^q 2^{-n}}{n(n-1)(n-2)} \binom{n}{(n-qN)/2} \cos[q\sqrt{N}g\chi_0/v], \quad (3.59)$$

$$= \frac{v^4 N}{\pi^2} \sum_{q=1}^{\infty} (-1)^q \cos[q\sqrt{N}g\chi_0/v] \sum_{k=0}^{\infty} 2^{-qN-2k} \frac{(qN+2k-3)!}{k!(qN+k)!}, \quad (3.60)$$

where $2k = n - qN$. Using an exact result:

$$\sum_{k=0}^{\infty} 2^{-a-2k} \frac{(a+2k-b)!}{k!(a+k)!} = \frac{2^{b-1} \Gamma[1+a-b] \Gamma[b-1/2]}{\sqrt{\pi} \Gamma[a+b]} \quad (3.61)$$

we get:

$$V = \frac{3v^4}{\pi^2} \sum_{q=1}^{\infty} \frac{(-1)^q \cos(q\sqrt{N}g\chi_0/v)}{q(q^2N^2-1)(q^2N^2-4)}. \quad (3.62)$$

Hence, for large N the leading term in the above series is:

$$V \approx -\frac{3v^4}{N^4\pi^2} \cos(g\sqrt{N}\chi_0/v) \quad (3.63)$$

$$= -\frac{3}{\pi^2 R^4} \cos(\chi_0/f_\chi), \quad f_\chi = 1/\tilde{g}R, \quad (3.64)$$

where $\tilde{g} = g/\sqrt{N}$ is the low energy value of the coupling constant.

From this analysis we obtain the *decay constant* of the χ_0 field, $f_\chi = 1/\tilde{g}R$. We have verified that this result is precise for large- N by performing the sums numerically. Remarkably, the effective potential is finite for $N \geq 3$. Moreover, it has no cut-off dependence upon N , once we reexpress the parameters in terms of the low energy variables, \tilde{g} and R .

The minima of the potential occur for $\chi_0/f_\chi = 2n\pi$. This corresponds to the fermion acquiring *antiperiodic boundary conditions*. (had we computed the potential with fermions having periodic boundary conditions, the overall sign of V would have flipped, and the minima would occur at $\chi_0/f_\chi = (2n+1)\pi$, hence the two computations are consistent.) The mass of the χ_0 field is obtained by expanding about a minimum:

$$m_\chi^2 = \frac{3\tilde{g}^2}{\pi^2 R^2} = \frac{12}{\pi R^2} \tilde{\alpha}. \quad (3.65)$$

This is similar to the result for a nonabelian gauge theory WLPNGB, $\propto \tilde{\alpha}/R^2$ [4].

We also consider the limit, $\eta \ll 1$, ($\eta \gg 1$ can be gotten by flipping the sign of the η exponent in eq. (3.67)) which is a fermion with the hopping links suppressed (strongly coupled). Eq. (3.55) yields a leading Z_N invariant term:

$$V = \frac{v^4}{4\pi^2} \sum_{q=1}^{\infty} (-1)^q \frac{(\eta^2 + \eta^{-2})^{2-Nq}}{q(qN-1)(qN-2)} \cos(q\sqrt{N}g\chi_0/v) \\ \times {}_2F_1[qN/2-1, (qN-1)/2; qN+1; 4/(\eta^2 + \eta^{-2})^2]. \quad (3.66)$$

For $\eta \ll 1$, the hypergeometric function goes to one, and the potential reduces to:

$$V \approx -\frac{v^4}{4\pi^2} \frac{\eta^{2(N-2)}}{N^2} \cos\left(g\sqrt{N}\chi_0/v\right) = -\frac{4e^{-(8\pi/\tilde{\alpha})|\ln(\eta)|}}{R^4\tilde{\alpha}^2} \cos(\chi_0/f_\chi). \quad (3.67)$$

The potential is now N -dependent, and corresponds to the result for a generalized Z_N theory (i.e., a “theory space” model). In the second expression we have swapped the explicit N dependence for the ratio of the unitarity bound, $\sim 4\pi$, to the low energy coupling $\tilde{\alpha} = \tilde{g}^2/4\pi$, i.e., setting $N = 4\pi/\tilde{\alpha}$. The expression is mysteriously reminiscent of a dilute gas-approximation instanton potential. We see that the small (or large) η limit produces an exponentially suppressed effective potential, and therefore an ultra-low-mass WLPNGB. In the parallel paper we show that this ultra-low-mass WLPNGB is immune to Planck scale breaking effects, and is a natural candidate for an axion, or a quintessence field [9].

4 Axial Anomaly

A final issue of importance is that of anomalies. The $U(1)^N$ theory in $1+3$ contains N NGB’s associated with the hopping terms. Eq. (2.39) with the $\Phi_n = (v/\sqrt{2}g)\exp(ig\tilde{\chi}_n/v)$ displayed takes the form:

$$\begin{aligned} & \sum_{n=1}^N \int d^4x \left[\bar{\tilde{\Psi}}_n(i\cancel{\partial} - g\cancel{A}_n)\tilde{\Psi}_n - \tilde{M}(\bar{\tilde{\Psi}}_{nL}\tilde{\Psi}_{nR} + h.c.) \right] \\ & + \sum_{n=1}^N \int d^4x \left[\left(-\eta v \bar{\tilde{\Psi}}_{nL} e^{ig\tilde{\chi}_n/v} \tilde{\Psi}_{(n+1)R} + h.c. \right) + \frac{1}{2}v^2(\partial\tilde{\chi}_n/v + \tilde{A}_n - \tilde{A}_{n+1})^2 \right]. \end{aligned} \quad (4.68)$$

Do the chiral fields, $\tilde{\chi}_n$ develop anomalous couplings to the gauge fields?

First, we must define the anomalies in the fundamental currents in a manner that is consistent with $U(1)^N$ and Z_N . The theory has $2N$ relevant currents, and each vectorial $U(1)$ must be anomaly free. Therefore we define:

$$\partial^\mu \tilde{\Psi}_n \gamma_\mu \tilde{\Psi}_{nL} = -\frac{g^2}{32\pi^2} F_{n\mu\nu} {}^*F_n^{\mu\nu} \quad \partial^\mu \tilde{\Psi}_n \gamma_\mu \tilde{\Psi}_{nR} = \frac{g^2}{32\pi^2} F_{n\mu\nu} {}^*F_n^{\mu\nu} \quad (4.69)$$

where ${}^*F_{n\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_n^{\rho\sigma}$ and $F_{n\mu\nu} = \partial_\mu\tilde{A}_{n\nu} - \partial_\nu\tilde{A}_{n\mu}$.

The key observation presently is that we can perform a sequence of N *vectorial* redefinitions of the fermions $\tilde{\Psi}_n$ and gauge transformations of the $\tilde{A}_{n\mu}$ which remove all but χ_0 from the mass terms. Because these are *vectorial* transformations on the fermions, they

have no associated anomalies. Let:

$$\begin{aligned}\tilde{\Psi}_n &\rightarrow \exp \left[-ig \sum_{k=1}^{n-1} \tilde{\chi}_k/v + ig(n-1)\chi_0/\sqrt{N}v \right] \tilde{\Psi}_n, \\ \tilde{A}_{n\mu} &\rightarrow \tilde{A}_{n\mu} + \sum_{k=1}^{n-1} \partial_\mu \tilde{\chi}_k/v - (n-1)\partial_\mu \chi_0/\sqrt{N}v,\end{aligned}\tag{4.70}$$

where, recall, $\chi_0 = \sum_{n=1}^N \tilde{\chi}_n/\sqrt{N}$.

These transformations remove all of the $\tilde{\chi}_n$ from all hopping terms of eq. (4.69), except for a residual χ_0 factor, and bring the vector potentials into the ‘‘unitary gauge’’

$$\begin{aligned}&\sum_{n=1}^N \int d^4x \bar{\tilde{\Psi}}_n (i\partial - g\tilde{A}_n) \tilde{\Psi}_n - \tilde{M}(\bar{\tilde{\Psi}}_{nL} \tilde{\Psi}_{nR} + h.c.) \\ &- \sum_{n=1}^N \int d^4x \left[\eta v \bar{\tilde{\Psi}}_{nL} e^{ig\chi_0/\sqrt{N}v} \tilde{\Psi}_{(n+1)R} + h.c. \right] + \frac{1}{2}(\partial\chi_0)^2 + \frac{1}{2}v^2(\tilde{A}_n - \tilde{A}_{n+1})^2.\end{aligned}\tag{4.71}$$

Hence we learn that *the only chiral anomalies involve the χ_0 zero mode*. The χ_0 zero mode can now be removed from the hopping terms, but this necessitates a *chiral* redefinition of the fermions, and it leads to a Wess-Zumino term.

The purely anomalous contribution to the effective Lagrangian occurs when there is a classical chiral symmetry. The Wilson term, which behaves like a bosonic kinetic term for the fermions, violates chirality. The model Lagrangian possesses the classical chiral symmetry when $\tilde{M} = M - \eta v = 0$.

In this limit we can redefine the fermions under a chiral transformation:

$$\begin{aligned}\tilde{\Psi}_{nL} &\rightarrow \exp \left[ig\chi_0/2\sqrt{N}v \right] \tilde{\Psi}_{nL}, \\ \tilde{\Psi}_{nR} &\rightarrow \exp \left[-ig\chi_0/2\sqrt{N}v \right] \tilde{\Psi}_{nR}.\end{aligned}\tag{4.72}$$

This produces, however, the Wess-Zumino term which is then added to the Lagrangian:

$$\frac{g\chi_0}{\sqrt{N}v} \sum_{n=1}^N \partial_\mu \bar{\tilde{\Psi}}_n \gamma^\mu \gamma^5 \tilde{\Psi}_n = \frac{g^3}{16\pi^2} \frac{\chi_0}{\sqrt{N}v} \sum_{n=1}^N F_{n\mu\nu} {}^* F^{n,\mu\nu}.\tag{4.73}$$

Now, we use the relationships to the low energy parameters, $v/N = 1/R$, and $\tilde{g} = g/\sqrt{N}$ and the Wess-Zumino term becomes:

$$\frac{\tilde{g}^2}{16\pi^2} \frac{\chi_0}{f_\chi} \sum_{n=1}^N F_{n\mu\nu} {}^* F^{n,\mu\nu}; \quad f_\chi = \frac{1}{\tilde{g}R}.\tag{4.74}$$

The decay constant of the WLPNGB is that obtained in the Coleman-Weinberg potential analysis of Section 3, and is given by the inverse of the product of the size R of the

extra dimension and the low energy coupling constant \tilde{g} . Thus the WLPNGB couples universally and anomalously to all KK modes, in analogy to the $\pi^0 \rightarrow 2\gamma$ coupling.

When nonzero \widetilde{M} is considered, we have explicit breaking of the chiral symmetry and the coupling of χ_0 to $F_{n\mu\nu}^* F^{n,\mu\nu}$ is modified. For further discussion of this, in application to axion and quintessence physics, see [9].

5 Conclusions

The theory we have described, QED in $1+4$, has been periodically compactified and latticized to produce an equivalent $U(1)^N$ theory describing physics for a finite set of KK-modes, in the low energy limit $n \ll N$.

In general a zero-mode χ_0 , occurs which may be interpreted as the 5th component of the vector potential, or the Wilson line around the compact 5th dimension. This possesses only discrete symmetries under gauge transformations, i.e., $\chi_0 \rightarrow \chi_0 + 2\pi n v / \tilde{g}$, and the $U(1)$ action is generally broken to the Z_N subgroup. In the effective $1+3$ Lagrangian χ_0 appears as a pseudo-Nambu Goldstone Boson (WLPNGB). In the absence of matter fields, the WLPNGB is massless. In general, however, the WLPNGB acquires a mass with a periodic potential, consistent with the discrete Z_N invariance.

We introduced matter fields into the latticized theory and encounter the Wilson flavor doubling problem. We have seen that the Wilson doubling has a remarkable diagrammatic interpretation, as two independent zig-zag chiral hopping threads through the lattice when the Dirac mass $M = 0$. One intriguing model building possibility, which we have not presently explored, would be to allow the Wilson doubling phenomenon to be the *fundamental* origin of flavor. This requires, however, dealing with the issues of imbedding the structure into the Standard Model and its associated flavor-chiral gauge structure. It is not clear that sensible models exist, but we are presently investigating this possibility [16].

If one desires, however, a simple lattice description that is faithful to the continuum theory, one must eliminate the flavor doubling. This necessitates including the Wilson term. The Wilson term improved action is studied and utilized in our present analysis. One obvious consequence of the fermionic flavor doubling problem is that faithfully representing latticized SUSY theories is subtle, and may be problematic [16]. To our knowledge, deconstructed SUSY models discussed in the literature to date, [12], have not addressed this problem. The SUSY kinetic terms implemented in these analyses are

typically superpotentials, and are not deconstructed Kahler potentials. This problem is fundamental to the deconstruction approach since the Poincare group is modified by the lattice, the extra dimensional continuous translational symmetries are replaced by Z_N .

We obtain the Coleman-Weinberg potential for the WLPNGB, which is finite for large N . The finiteness is a consequence of the Z_N symmetry. Such Z_N finite potentials have been discussed previously, e.g., in the schizon model of Hill and Ross, [6], which remarkably has the equivalent structure to the present scheme in the extra-dimension's momentum space. Moreover, when recast in terms of the physical variables, R (the size of the extra dimension) and \tilde{g} (the low energy 3 + 1 QED coupling constant) all dependence upon the high energy scales, N , or v ($v = 1/a$ where a is the lattice spacing), completely disappears in the Coleman-Weinberg potential. Therefore, the Coleman-Weinberg potential for the Wilson line is reliably determined and is independent of the regulator scheme.

When we calculate with antiperiodic (periodic) fermions, we find that the Coleman-Weinberg potential is minimized for the special case $\chi_0 = 0$ ($\chi_0 = \pi f_\chi$) (modulo the periodicity). Upon absorbing the Wilson line into the fermionic wave-function, this corresponds to the fermions always having dynamically preferred *antiperiodic boundary conditions* in traversing the extra dimension.

We study the anomaly structure of the deconstructed theory and find that only χ_0 develops a Wess-Zumino term. This WZ-term is the analogue of the $\pi \rightarrow 2\gamma$ anomaly in electrodynamics. In the present case the anomaly universally couples χ_0 to all KK modes.

By tweaking the parameters in the theory we are able to exponentially suppress the Coleman-Weinberg potential and generate χ_0 as an ultra-low-mass spin-0 particle. This suggests a number of phenomenological applications. The effect of Planck-scale breaking of global symmetries is highly suppressed by the Z_N symmetry. The construction of natural models of the axion and quintessence will be described elsewhere [9].

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