



A Relativistic $O(a^2)$ Improved Action for Heavy Quarks

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We extend the Fermilab formalism for heavy quarks to develop an $O(a^2)$ improved relativistic action. We discuss our construction of the action, including the identification of redundant operators and the calculation of the improvement coefficients.

1. INTRODUCTION

A major source of uncertainty in numerical simulations of lattice QCD comes from finite lattice spacing effects. Since these effects arise at short distances, they can be analyzed using an effective field theory, as first proposed by Symanzik [1]. We may write the effective Lagrangian (\mathcal{L}) as

$$\mathcal{L}^{\text{eff}} = \mathcal{L}^{\text{cont}} + \sum_j c_j a^{s_j-4} O_j, \quad (1)$$

where $s_j = \dim[O_j]$. In this framework lattice spacing effects can be systematically removed by adding higher dimensional operators to the lattice action. The aim of this work is to design a lattice action such that the coefficients of $O(a)$ and $O(a^2)$ terms in Eq. (1) can be reduced.

For heavy quarks with $m_Q \gg \Lambda_{\text{QCD}}$, the mass introduces an additional short-distance scale into the problem, and the Symanzik formalism must be modified to separate the short-distance effects of both the lattice spacing and the heavy-quark mass from the long-distance physics. The Fermilab formalism [2] represents one approach to this problem. It allows the coefficients c_j in Eq. (1) to depend on the quark mass. Then it takes the Wilson action [3], modified to allow different coefficients for space-like and time-like operators. Improved lattice actions are constructed by adding higher-dimension operators, again with different coefficients for time- and space-like operators. Ref. [2] considers operators up to dimen-

sion five, to obtain a lattice action for fermions, improved to $O(a)$, and valid for quarks with arbitrary mass. In this work we extend the analysis of Ref. [2] to include operators of dimension six for $O(a^2)$ improvement.

2. THE ACTION

At dimension six we have to consider both fermion bilinears and four-fermion operators. For these proceedings, we concentrate on determining the bilinear interactions. The five dimension-six bilinears that satisfy axis-interchange symmetry are given in Ref. [4]. Without axis-interchange symmetry, these operators become the seventeen operators listed in Table 1. (The lower dimensional operators are given in Ref. [2].)

Following Refs. [4,2], we use field transformations to expose the redundant directions for on-shell improvement. Since we are considering dimension-six operators in the action we must include dimension-two operators in the transformations, and we must also consider gauge field transformations. Writing

$$\psi \rightarrow e^J \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{\bar{J}}, \quad (2)$$

we have

$$\begin{aligned} J = & a\epsilon_1(\not{D} + m) + a\delta_1\boldsymbol{\gamma}\cdot\mathbf{D} + a^2\epsilon_2(\not{D} + m)^2 \\ & - a^2\epsilon_F\frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu} + a^2\delta_2(\boldsymbol{\gamma}\cdot\mathbf{D})^2 + a^2\delta_B i\boldsymbol{\Sigma}\cdot\mathbf{B} \\ & + a^2\delta_u[\gamma_0 D_0, \boldsymbol{\gamma}\cdot\mathbf{D}], \end{aligned} \quad (3)$$

and \bar{J} is the same but with bars over the param-

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Table 1
Dimension-six bilinear interactions that could appear in the effective Lagrangian.

No	Operator	Parameter
1	$\bar{\psi}(\gamma_0 D_0)^3 \psi$	$\epsilon_2 + \bar{\epsilon}_2$
2	$\bar{\psi}\{\boldsymbol{\gamma}\cdot\mathbf{D}, \mathbf{D}^2\}\psi$	
3	$\bar{\psi}\{\gamma_0 D_0, (\boldsymbol{\gamma}\cdot\mathbf{D})^2\}\psi$	$\delta_2 + \bar{\delta}_2$
4	$\bar{\psi}[\gamma_0 D_0, (\boldsymbol{\gamma}\cdot\mathbf{D})^2]\psi$	$\delta_2 - \bar{\delta}_2$
5	$\bar{\psi}\{(\gamma_0 D_0)^2, \boldsymbol{\gamma}\cdot\mathbf{D}\}\psi$	$\delta_u - \bar{\delta}_u$
6	$\bar{\psi}[(\gamma_0 D_0)^2, \boldsymbol{\gamma}\cdot\mathbf{D}]\psi$	$\delta_u + \bar{\delta}_u$
7	$\bar{\psi}\boldsymbol{\gamma}\cdot[D_0, \mathbf{E}]\psi$	$\epsilon_F + \bar{\epsilon}_F$
8	$\bar{\psi}[\boldsymbol{\alpha}\cdot\mathbf{E}, \gamma_0 D_0]\psi$	$\epsilon_F - \bar{\epsilon}_F$
9	$\bar{\psi}\{\boldsymbol{\gamma}\cdot\mathbf{D}, \boldsymbol{\alpha}\cdot\mathbf{E}\}\psi$	
10	$\bar{\psi}[\boldsymbol{\alpha}\cdot\mathbf{E}, \boldsymbol{\gamma}\cdot\mathbf{D}]\psi$	*
11	$\bar{\psi}\{i\boldsymbol{\Sigma}\cdot\mathbf{B}, \gamma_0 D_0\}\psi$	$\delta_B + \bar{\delta}_B$
12	$\bar{\psi}\{i\boldsymbol{\Sigma}\cdot\mathbf{B}, \boldsymbol{\gamma}\cdot\mathbf{D}\}\psi$	
13	$\bar{\psi}[i\boldsymbol{\Sigma}\cdot\mathbf{B}, \boldsymbol{\gamma}\cdot\mathbf{D}]\psi$	
14	$\bar{\psi}[i\boldsymbol{\Sigma}\cdot\mathbf{B}, \gamma_0 D_0]\psi$	$\delta_B - \bar{\delta}_B$
15	$\bar{\psi}\gamma_0(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})\psi$	
16	$\bar{\psi}\boldsymbol{\gamma}\cdot(\mathbf{D}\times\mathbf{B} + \mathbf{B}\times\mathbf{D})\psi$	
17	$\bar{\psi}(\gamma_i D_i^3)\psi$	

eters. For the gauge fields

$$A_0 \rightarrow A_0 + \frac{1}{2}a^2\epsilon_A(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D}), \quad (4)$$

$$+ g_0^2 a^2 \epsilon_J (\bar{\psi} \gamma_0 t^a \psi) t^a$$

$$\mathbf{A} \rightarrow \mathbf{A} + \frac{1}{2}a^2(\epsilon_A + \delta_A)(\mathbf{D}\times\mathbf{B} + \mathbf{B}\times\mathbf{D})$$

$$- \frac{1}{2}a^2(\epsilon_A + \delta_E)[D_0, \mathbf{E}] \quad (5)$$

$$+ g_0^2 a^2 (\epsilon_J + \delta_J) (\bar{\psi} \boldsymbol{\gamma} t^a \psi) t^a.$$

Following the notation of Ref. [2], the ϵ (δ) coefficients label axis-interchange symmetric (asymmetric) operators, so that the axis-interchange symmetric analysis of Ref. [4] is recovered when all δ coefficients vanish. The transformations of Eqs. (2)–(5) generate the operators listed in Table 1 with coefficients which depend on the parameters in the transformations. The last operator of Table 1, $\bar{\psi}\gamma_i D_i^3\psi$, is not generated by the transformations, and hence cannot be eliminated. To remove it one must improve the

nearest-neighbor lattice derivative.

Not all the operators are independent from each other. Indeed, we have used the identities

$$2\gamma_0 D_0 \boldsymbol{\gamma}\cdot\mathbf{D} \gamma_0 D_0 = \quad (6)$$

$$-\boldsymbol{\gamma}\cdot[D_0, \mathbf{E}] + \{(\gamma_0 D_0)^2, \boldsymbol{\gamma}\cdot\mathbf{D}\},$$

$$2\boldsymbol{\gamma}\cdot\mathbf{D} \gamma_0 D_0 \boldsymbol{\gamma}\cdot\mathbf{D} = \quad (7)$$

$$\{\boldsymbol{\gamma}\cdot\mathbf{D}, \boldsymbol{\alpha}\cdot\mathbf{E}\} - \{\gamma_0 D_0, (\boldsymbol{\gamma}\cdot\mathbf{D})^2\},$$

$$2(\boldsymbol{\gamma}\cdot\mathbf{D})^3 = \{\boldsymbol{\gamma}\cdot\mathbf{D}, \mathbf{D}^2\} + \{\boldsymbol{\gamma}\cdot\mathbf{D}, i\boldsymbol{\Sigma}\cdot\mathbf{B}\}, \quad (8)$$

to remove the operators shown on the left hand side of these equations from Table 1. The interaction $\bar{\psi}[\boldsymbol{\alpha}\cdot\mathbf{E}, \boldsymbol{\gamma}\cdot\mathbf{D}]\psi$ (marked “*” in Table 1) does not obey particle-antiparticle interchange symmetry so it cannot be in the action. Nine independent parameter combinations appear in the coefficients of the transformed operators. They are $\epsilon_2 + \bar{\epsilon}_2$, $\delta_2 + \bar{\delta}_2$, $\delta_2 - \bar{\delta}_2$, $\delta_u + \bar{\delta}_u$, $\delta_u - \bar{\delta}_u$, $\delta_B + \bar{\delta}_B$, $\delta_B - \bar{\delta}_B$, $\epsilon_F + \bar{\epsilon}_F$ and $\epsilon_F - \bar{\epsilon}_F$.

The choice of redundant operators is not unique. Considerations, such as calculational convenience (or solving the fermion doubling problem) play a role in the choice. In this work, we want to avoid operators which contain higher order time derivatives, as they spoil the good properties of the transfer matrix of actions with Wilson-like time derivatives. There are nine such operators, all of which can be eliminated by the field transformations. Table 1 shows which parameter combination is used to eliminate a given operator. In summary, this analysis reduces the original seventeen operators to seven, and we can write the lattice fermion action as

$$S_F = S_0 + S_B + S_E + a^2 c_1 \int \bar{\psi}\{\boldsymbol{\gamma}\cdot\mathbf{D}, \mathbf{D}^2\}\psi$$

$$+ a^2 c_2 \int \bar{\psi}\gamma_i D_i^3 \psi + a^3 c_3 \int \bar{\psi}\{i\boldsymbol{\Sigma}\cdot\mathbf{B}, \boldsymbol{\gamma}\cdot\mathbf{D}\}\psi$$

$$+ a^2 c_4 \int \bar{\psi}\gamma_0(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})\psi \quad (9)$$

$$+ a^2 c_5 \int \bar{\psi}\boldsymbol{\gamma}\cdot(\mathbf{D}\times\mathbf{B} + \mathbf{B}\times\mathbf{D})\psi$$

$$+ a^2 c_6 \int \bar{\psi}\{\boldsymbol{\gamma}\cdot\mathbf{D}, \boldsymbol{\alpha}\cdot\mathbf{E}\}\psi + a^2 c_7 \int \bar{\psi}[i\boldsymbol{\Sigma}\cdot\mathbf{B}, \boldsymbol{\gamma}\cdot\mathbf{D}]\psi.$$

where S_0 , S_B and S_E are given in Ref. [2].

3. THE COEFFICIENTS

Conditions on the coefficients of the improvement operators are obtained by matching on-shell quantities in the lattice theory to their continuum counterparts. From the lattice fermion propagator we derive the dispersion relation

$$\cosh E = 1 + \frac{(\mu(\mathbf{p}) - 1)^2 + (f(\mathbf{p})\mathbf{S} + 2c_2\mathbf{Q})^2}{2\mu(\mathbf{p})} \quad (10)$$

where (in lattice units)

$$\mu(\mathbf{p}) = 1 + m_0 + \frac{1}{2}r_s\hat{\zeta}\hat{\mathbf{p}}^2 \quad (11)$$

$$f(\mathbf{p}) = \zeta - 2c_2 - 2c_1\hat{\mathbf{p}}^2 \quad (12)$$

$S_i = \sin p_i$, $\hat{p}_i = 2 \sin(p_i/2)$ and $2Q_i = \sin 2p_i$. Expanding the energy in powers of \mathbf{p} , we obtain

$$E = M_1 + \frac{\mathbf{p}^2}{2M_2} - \frac{1}{6}w_4 \sum_{i=1}^3 p_i^4 - \frac{(\mathbf{p}^2)^2}{8M_4^3} + \dots \quad (13)$$

where M_1 is the rest mass and M_2 is the kinetic mass. As in Ref. [2] we impose $M_2 = m_q$. For a relativistic LE \mathcal{L} we also impose $M_1 = M_2$. The improvement condition $M_2 = M_4$ yields a relation for the coefficient c_1 . Rotational invariance ($w_4 = 0$) gives a condition for c_2 . These relations were already derived in Ref. [2].

For the remaining operators, c_3 through c_7 , we calculate temporal and spatial matrix elements with one gluon exchange at tree-level. Using the lattice spinors of Ref. [2], we expand the temporal matrix element up to and including $O(p^2)$. The lattice matrix element is

$$\langle q(\xi', \mathbf{p}') | V_0^G | q(\xi, \mathbf{p}) \rangle_{\text{lat}} = u^\dagger(\xi', \vec{\mathbf{0}}) \left[1 - \frac{(\mathbf{p}'^2 + \mathbf{p}^2 - 2\mathbf{p}' \cdot \mathbf{p})}{8M_E^2} + \frac{i\epsilon_{ijl} \sum_l p'_i p_j}{4M_{E'}^2} \right] u(\xi, \vec{\mathbf{0}}), \quad (14)$$

where

$$\frac{1}{8M_E^2} = \frac{\zeta^2}{2m_0^2(2+m_0)^2} + \frac{c_E\zeta^2}{2m_0(2+m_0)} + \frac{c_6 - c_4}{(1+m_0)}, \quad (15)$$

$$\frac{1}{4M_{E'}^2} = \frac{1}{4M_E^2} + \frac{2c_4}{1+m_0}. \quad (16)$$

The right-hand side of Eq. (14) must be matched to the continuum matrix element, which has M_E

and $M_{E'}$ replaced with m_q . One matching condition is to set $M_{E'} = M_E$, which requires $c_4 = 0$. Another matching condition sets $M_E = M_2$, yielding a condition on c_6 and c_E :

$$\frac{4\zeta^2}{m_0^2(2+m_0)^2} + \frac{4c_E\zeta^2}{m_0(2+m_0)} + \frac{8c_6}{1+m_0} = \frac{1}{M_2^2}. \quad (17)$$

Eq. (17) is in agreement with a result from Ref. [2] obtained in the Hamiltonian formalism. (The expression for M_2 and the definition of c_E can be found in Ref. [2].)

The determination of improvement conditions on c_3 , c_5 and c_7 , requires the calculation of the spatial matrix element including terms up to $O(p^3)$. This is currently in progress.

4. CONCLUSIONS

We propose a relativistic $O(a^2)$ improved action for heavy quarks. We use the redundant directions to eliminate all operators with higher order time derivatives. As a result, our action keeps the Wilson time derivative. We have determined the mass dependent coefficients of four improvement operators at tree-level. The determination of all remaining coefficients is in progress.

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