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Gauge Invariant Effective Lagrangian for Kaluzsa-Klein Modes

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Abstract

We construct a manifestly gauge invariant Lagrangian in $3+1$ dimensions for N Kaluza-Klein modes of an $SU(m)$ gauge theory in the bulk. For example, if the bulk is $4+1$, the effective theory is $\Pi_{i=1}^{N} SU(m)_i$ with $N-1$ chiral $(\overline{m}, m)$ fields connecting the groups sequentially. This can be viewed as a contiuum limit of a Wilson action for a transverse lattice in x^5, and is shown explicitly to match the continuum $4+1$ bulk Lagrangian truncated in momentum space. Scale dependence of the gauge couplings is described by the standard renormalization group technique with threshold matching, leading to effective power law running. We also discuss the unitarity constraints, and chiral fermions.

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1 Introduction

It is widely believed that the main low energy signature of extra dimensions [1] is the appearance of the tower of Kaluza-Klein (KK) modes. For example, if QCD lived in the bulk, experimentalists would see massive spin-1 degenerate color octet vector bosons (colorons) appearing at large mass scales corresponding to (inverse) compactification scales. As these new massive KK particles begin to emerge in accelerator experiments, we might ask how would we describe them in an effective four-dimensional renormalizeable Lagrangian that is an extension of QCD, without an *a priori* knowledge of the existence of extra dimensions? The main goal of the present paper is to give a manifestly gauge invariant effective Lagrangian description of KK modes in $1+3$ dimensions.

It is important to realize at the outset that there is an implicit dynamical assumption underlying such a theory of extra-dimensions and KK modes. This is the assumption that there is a meaningful separation of scales between the compactification scale, $M_c \sim 1/R_c$ and the "string" or "fundamental scale" M_s at which the extra-dimensional theory breaks down as a perturbative local field theory. To have $N >> 1$ KK modes in a $4+1$ theory we require $M_s/M_c \gtrsim N >> 1$. It is not obvious how such a separation of scales occurs in the theory. The separation of scales is a requirement of very low mass, or infrared states, in an essentially strong-dynamical theory at the scale M_s.

Now, in a strong theory like QCD we know how, in principle, to engineer far infrared physics, i.e., states with masses much less than the fundamental scale Λ_{QCD}. To do so we introduce approximate or exact chiral symmetries, e.g., we take m_u and m_d very small. Then the pions, the pseudo-Nambu-Goldstone bosons associated with the spontaneous breaking of chiral symmetry become light as $m_\pi^2 \sim (m_u + m_d)\Lambda_{QCD}$ (and, we must switch off explicit breaking of the chiral symmetries due to electromagnetism which produce $m_{\pi+}^2 - m_{\pi 0}^2 \sim \alpha\Lambda_{QCD}^2$, etc.). We say, then, that the chiral symmetry is the "custodial symmetry" of the hierarchy m_π^2/Λ_{QCD}^2. Now, all meson couplings are suppressed on scales $\mu << \Lambda_{QCD}$ by factors of $1/f_\pi$ where $f_\pi \sim \Lambda$. Hence, there is another approximate symmetry on scales $m_\pi << \mu << \Lambda_{QCD}$, which is classical scale invariance. Classical scale invariance is always present when we have a hierarchy, though it may be accidental as we have just seen.

Indeed, another profound separation of scales occurs in nature, again in QCD, which is the smallness of the ratio Λ_{QCD}/M_{GUT} or Λ_{QCD}/M_{Planck}. Here we have a remarkably different phenomenon in that the scale Λ_{QCD} is generated by quantum mechanics itself, i.e., by dimensional transmutation of the running coupling constant $\alpha_s(\mu)$ into the scale Λ_{QCD} when $\alpha_s(\mu) \to \infty$ as $\mu \to \Lambda_{QCD}$. The running of $\alpha_s(\mu)$ is a quantum phenomenon, determined to leading order in $\hbar$ by the one-loop $\beta(\alpha_s)$ function. It represents an explicit breaking of scale invariance (i.e., no dilaton associated with spontaneous breaking occurs) because the trace anomaly $T^\mu_\mu \propto \beta G^a_{\mu\nu} G^{a\mu\nu}$ is large and nonzero as $\mu \to \Lambda_{QCD}$. The custodial symmetry in this case is the approximate classical scale invariance associated with the limit $\beta(\alpha_s) \approx 0$, for pertubative values of $\alpha_s(\mu)$. Indeed, in the limit of an exact fixed point $\beta(\alpha_s) = 0$, where we have exact classical scale invariance, we have $\Lambda_{QCD}/M_{Planck} \to 0$.

The separation of the compactification scale and the fundamental scale in extra-dimensional models would seem similarly to involve, at least accidentally, approximate classical scale invariance. To arrange that N can be taken arbitrarily large implies that the theory must have a slowly running dimensionless couplng constant (remniscent of "walking technicolor") in $D = 4$ on scales well below M_s, so it does appear that quantum scale breaking effects are under control. However, the trace of the stress-tensor in $D = 5$ is nonzero classically, and the theory has explicit scale breaking, owing to the $D = 5$ dimensional coupling constant. The nonzero trace, $T^\mu_\mu \propto G^a_{\mu\nu} G^{a\mu\nu}$ in $D = 5$ must clearly match onto the KK masses as seen in in $D = 4$, since the KK masses are seen as explicit sources of scale breaking on all scale from M_c to M_s. It is therefore quite puzzling as to what is the custodial symmetry of a scale hierarchy in extra dimensions.

Now, we know that we can take a strongly coupled theory and tune it's coupling close to a critical value (provided the critical value is associated with a second order phase transition). For example, in the Nambu-Jona-Lasinio model, where a four-fermion interaction is postulated at a scale Λ, by tuning the coupling constant close to a critical value we can produce boundstate scalars with masses that are arbitrarily small. Unfortunately, however, there is nothing natural about the occurence of this hierarchy. This is usually viewed as a fine-tuning of the coupling constant. Hence, while technically natural, there appears to be a fine-tuning associated with extra-dimensional theories in arranging a KK-tower with a large number of distinct KK modes.

Having tuned a hierarchy, by analogy with critical behavior in a second order phase transition in condensed matter physics, there must exist a wide range, or universality class, of theories that have identical behavior in the infra-red, but are radically different in detail at the scale M_s. In the present paper we exploit universality. We treat the physics at M_s not as a "string theory," but rather as a "transverse lattice gauge theory"

[2]. For us, the normal $1 + 3$ dimensions of space-time are continuous, but the extra dimensions are latticized (nothing prevents us from adopting a full lattice theory, but it is convenient for our presnt purposes to use the transverse lattice). This theory will have a well-defined finite short-distance behavior for arbitrarily large coupling and will be manifestly gauge invariant, reflecting the full gauge invariance of the higher dimensional theory. It will have the same infra-red behavior as the usual KK-mode description, but will illuminate how the gauge invariance is maintained.

As a result, we will understand something implicitly puzzling about KK modes. Longitudinal KK mode scattering is essentially the scattering of Nambu-Goldstone Bosons in a nonlinear chiral Lagrangian. As such it violates perturbative unitarity, i.e., there is a Lee-Quigg-Thacker bound on the applicability of the conformal theory. We will see that this hapens at the scale M_s in our effective Lagrangian.

The main reason for desiring such a description is that it is difficult to treat nonabelian gauge theories in loop expansions with momentum space cut-offs. Normally, the momentum space cut-off is not compatible with gauge invariance, and this causes the loop expension to become non-gauge invariant. However, the usual treatment of extra-dimensional gauge theories involves a truncation on KK modes, which is a de facto momentum space cut-off. With gauge fields in the bulk, a $d+1$ theory with $d > 3$ has infinitely more gauge invariance than the $3+1$ theory since there is more space in which to perform local gauge transformations. Clearly the gauge invariance of $3 + 1$ QCD must be maintained, but how does the expanding local gauge invariance of the theory manifest itself as the extra dimension begins to open up with the emergence of KK modes? How does the power-law running of the coupling constant emerge and what is the correct renormalization group for such a description?

2 Manifestly Gauge Invariant Effective Lagrangian

The KK modes of the vector potential of QCD, i.e., the colorons, are heavy matter fields and must transform linearly under the adjoint representation of $SU(3)$ (in contrast to the zero-mode gluon which transforms nonlinearly by the Yang-Mills gauge transformation). References [3] have argued that vector fields in linear adjoint representations of a local gauge group $SU(m)$ will always contain a "hidden" local symmetry, which is a copy of $SU(m)$. The gluon plus one massive octet vector multiplet corresponds to the local symmetry $SU(m) \times SU(m)$, each factor having the same coupling constant (our present discussion is classical; we'll worry about running couplings below). This is broken diagonally by an effective Higgs field, Φ, which transforms as a $(\overline{m}, m)$, to a local $SU(m)$ and an $SU(m)$ global symmetry. Only the chiral components of Φ are relevent here so we can replace $\Phi \to v \exp(\phi^a \lambda^a / 2v)$ (we can always arrange a potential which sends the Higgs field and any $U(1)$ components to arbitrarily large mass, but we must then worry about perturbative unitarity). The ϕ^a are eaten to give the coloron mass. Hence, in describing one massive octet this way it is the low energy hidden local symmetry due to the spontaneous breaking that reflects the expanded gauge invariance of the extra-dimensional theory as the space of the extra dimension is opening up.

As experiments go to higher energies, one starts to see more KK masstive gauge bosons. It is obvious that one requires more "hidden" local $SU(3)$ symmetries and more Higgs fields as in the previous case to construct an effective Lagrangian to discribe these massive gauge bosons. Hence, we propose that the effective Langrangian for the first n KK modes would contain N $SU(3)$'s with $N-1$ Φ's. The interconnections between the gauge symmetries and the Higgs could become completely arbitrary, and resolve into different hydrocarbon-like chain molecules.

It would seem that we can, therefore, guess the effective Lagrangian for N KK modes. When an experimentalist detects the first KK mode of $SU(3)$, we can describe this by $SU(3) \times SU(3)$, each factor having a common coupling constant, with a single $\Phi \subset (\overline{3}, 3)$ "straddling" the two gauge groups. I f a second isolated KK mode is discovered we require an $SU(3)_1 \times SU(3)_2 \times SU(3)_3$ with two Φ_i representations. The Φ_i must transform as $\Phi_1 \sim (\overline{3}_1, 3_2)$ and $\Phi_2 \sim (\overline{3}_2, 3_3)$. Indeed, up to three modes, i.e., the gluon plus two KK modes, there is no other possibility for arranging the Φ_i modes, which must interconnect all of the gauge groups (unless the two modes are degenerate, as we consider below). However, for four or more modes we require $SU(3)^4$ and three Φ_i and we begin to encounter ambiguities as to how to assign the Φ_i representations. With N modes we must interconnect the N gauge groups pairwise with $N-1$ Φ's, and there are $\frac{1}{2}N(N-1)$ pairs. The interconnections become completely arbitrary, and resolve into different hydrocarbon-like chain molecules.

We might guess that the simplest linear interconnection for N modes having $\Phi_i \subset (\overline{3}_i, 3_{i+1})$ is somehow relevant. We'll follow the organic chemistry nomenclature and call this an "aliphatic" $(SU(3)^{N+1}, \Phi^N)$ model. The Lagrangian for this scheme is:[1]

$$\mathcal{L} = -\sum_{i=0}^{N} \frac{1}{4g^2} F^a_{i\mu\nu} F^{i\mu\nu a} + \sum_{i=1}^{N} D_\mu \Phi_i^\dagger D^\mu \Phi_i \tag{2.2}$$

Upon substituting,

$$\Phi_i \to v \exp(i\phi_i^a \lambda^a / 2v) \tag{2.3}$$

the Φ kinetic terms lead to a mass matrix for the gauge fields:

$$\sum_{i=1}^{N} \frac{1}{2} g^2 v^2 (A^a_{(i-1)\mu} - A^a_{i\mu})^2 \tag{2.4}$$

This mass matrix has the structure of a nearest neighbor coupled oscillator Hamiltonian. We can diagonalize the mass matrix to find the eigenvalues (which corresponds to the dispersion relation for the coupled oscillator-system):

$$M_n = \sqrt{2} g v \sin\left[\frac{\gamma_n}{2}\right] \qquad \gamma_n = \frac{(n-1)\pi}{N}, \qquad n = 0, 1, \ldots, N. \tag{2.5}$$

Thus we see that for small n this system has a KK tower of masses given by:

$$M_n \approx \frac{g v \pi (n-1)}{\sqrt{2} N} \qquad\qquad n << N \tag{2.6}$$

and $n = 0$ corresponds to the zero-mode gluon.

To match on to the spectrum of the KK modes, we require

$$\frac{gv}{\sqrt{2}N} = \frac{1}{R}. \tag{2.7}$$

Hence, the aliphatic system with $SU(3)^N$ and $N-1$ Φ_i provides a gauge invariant description of the first n KK modes by generating the same mass spectrum. It is thus crucial to examine the interactions from the aliphatic model.

In a geometric picture, the aliphatic model corresponds to a "transverse lattice" description of a full $4+1$ gauge theory. We construct a transverse lattice in the x^5 dimension where the lattice size is R and short-distance lattice cut-off is a, so $N-1 = R/a$. This is a

[1] A renormalizable potential can be constructed for the Higgs fields,

$$V(\Phi_j) = \sum_{j=1}^{N} \left[-M^2 \mathrm{Tr}(\Phi_j^2) + \lambda_1 \mathrm{Tr}(\Phi_j^4) + \lambda_2 \mathrm{Tr}(\Phi_j^2)^2 + M' det(\Phi_j) \right], \tag{2.1}$$

We can always arrange the parameters in the potential such that the diagonal components of each Φ_j develop a vacuum expectation value v, and the Higgs and $U(1)$ PNGB are heavy.

foliation of N parallel branes, each spaced by a lattice cut-off a (Fig.(1)). On the ith brane we have an $SU(m)$ gauge theory denoted by $SU(m)_i$. The $SU(m)_i$ automatically have a common coupling constant g. Each brane $SU(m)_i$ theory can be viewed as predefined in the continuum limit of a fine-grained Wilson plaquette action, and a hypothetical $3+1$ lattice spacing a_4. The lattice spacing in the x^5 dimension can be viewed as relatively coarse with $a >> a_4$.

The theory thus has $N-1$ links in the x^5 direction that are continuous functions of x_μ. These correspond to the continuum limit Wilson lines:

$$\Phi_n(x^\mu) = \exp\left[ig\int_{na}^{(n+1)a} dx^5 A_5(x^\mu, x^5)\right] \to \exp\left[igaA_5(x^\mu, (n+\frac{1}{2})a)\right] \qquad (2.8)$$

The N Φ_n therefore transform as an $(\overline{m}, m)$ representation of $SU(m)_n \times SU(m)_{n+1}$ as in the aliphatic model (straddling the nearest neighbor $SU(m)_n$ and $SU(m)_{n+1}$ gauge groups). Φ_n is a unitary matrix and may be parameterized as in eq.(1.3). The theory is a spline approximation to the configurations in the continuum x^5 dimension.

3 Compare the Continuum Theory

(i) Definition of the Continuum Theory

A $d+1$ ($d>3$) field theory becomes ill-defined at energy scale $M_s >> 1/R$. Presumably it matches onto a string theory at M_s, and we usually refer to M_s as the "string scale." While the exact structure of the theory on scales $\mu \sim M_s$ is unknown, its symmetries, e.g., local gauge invariances, must remain intact at lower scales.

A Wilson transverse lattice Lagrangian is a reasonable candidate for a well-defined short distance definition of the nonperturbative higher dimensional theory. This manifestly preserves local gauge invariance and permits, in principle, a nonperturbative treatment. Presumably a continuum $d+1$ Yang-Mills Lagrangain is an equivalent valid description below M_s. How, then, does the aliphatic $(SU(3)^{N+1}, \Phi^N)$ model match in detail to the perturbative $4+1$ continuum theory at lower energies, yet still above the mass of the lowest KK mode?

We define the continuum theory in $4+1$ and expand in modes in the compact x^5. We truncate this theory after N terms. Now, momentum space truncations in Yang-Mills theories are notoriously awkward at best. The expansion is usually done in a particular gauge. Then, with truncation of the theory in momentum space we lose track of the full gauge invariance of the theory. However, we will see, remarkably, that this truncation can be matched identically onto the aliphatic theory which is manifestly gauge invariant. Since the aliphatic model is manifestly gauge invariant and defined in $3+1$ dimensions this completely justifies the treatments of running coupling constants of Dienes, et. al, [5] and Dobrescu etal. [6], and indicates how to proceed to answer more difficult field theory questions.

First, we consider a simple well-defined compactification scheme. We define QCD in $4+1$ dimensions between two parallel branes.[2]. The branes are respectively located at I: $x^5 = R_I = 0$ and II: $x^5 = R_{II} = R$, with a constant interbrane separation R. The covariant derivative $D_M = \partial_M + ig_0 \hat{A}_M$, with field strengths $ig_0 \hat{F}_{MN} = [D_M, D_N]$, where the canonical mass dimension of the vector potential $\hat{A}_M$ in $4+1$ dimensions is $3/2$, and the coupling constant g_0 must therefore have dimension $-1/2$.

The five-dimensional theory is locally gauge invariant but non-renormalizable. In addition to the compactification radius R, it is defined by the fundamental short-distance cut-off scale M_s. It is then nature to define a dimensionless g by $g_0 = g/\sqrt{M_s} = 1/M$. The $4+1$ Lagrangian takes the form:

$$\mathcal{L}_5 = -\frac{1}{4} Tr(\hat{F}_{MN}\hat{F}^{MN}), \qquad F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_0 f^{abc} A^b_M A^c_N; \tag{3.9}$$

where a is the gauge index and f^{abc} is the structure constant.

(ii) Momentum Space Expansion and Truncation

A necessary gauge-covariant boundary conditions on the experimentalist's physical brane is:

$$F^{5N} = F^{N5} = 0 \ , \qquad \text{at} \ \ x^5 = R_{I,II} \tag{3.10}$$

This removes unwanted gauge invariant vector field strengths that transform as a 4-vector in the $3+1$ theory. The simplest gauge choice realizing these boundary conditions is to impose Neumann conditions for $\hat{A}_\mu$ with $\mu = 0,1,2,3$, i.e. $\partial \hat{A}_\mu / \partial x^5 = 0$, at $x_5 = R_{I,II}$, and Dirichlet conditions for the $3+1$ "scalars" $\hat{A}_5$, i.e. $\hat{A}_5 = 0$ at $x_5 = R_{I,II}$. The lowest energy physical $\hat{A}_\mu$ modes are massless, independent of x_5, and form the usual $3+1$ gauge field. We can further choose an axial gauge $\chi^A A_A = 0$ where χ^A is a 5-vector normal to the branes. This sets $A^5 = 0$. We will adopt this gauge choice after the momentum space expansion.

[2]The ordinary spacetime coordinates are labeled by x^μ, $\mu = 0,1,2,3$, and the fifth dimension by x^5 to avoid confusion with $x^4 = ict$; Capital letters denote the bulk coordinates, $M, N = 0,1,2,3,5$.

We thus can expand the 4-vector potential $\hat{A}_\mu(x_\mu, x_5)$ in a Fourier cosine series,

$$\hat{A}_\mu = \frac{1}{\sqrt{R}}\left[A^0_\mu + \sqrt{2}\sum_{n=1}^{+\infty} A^n_\mu(x_\mu)\cos(n\theta)\right] , \qquad \theta = \frac{\pi x_5}{R}, \tag{3.11}$$

where we have suppressed the gauge index a and A^0 is the $n = 1$ zero-mode. The fifth component $\hat{A}_5(x_\mu, x_5)$ is given by a Fourier sine series,

$$\hat{A}_5 = \sqrt{\frac{2}{R}}\sum_{n=1}^{+\infty} A^n_5(x_\mu)\sin(n\theta) \ . \tag{3.12}$$

and this has no zero-mode. The coefficients of the expansions are:

$$\begin{aligned}
A^0_\mu &= \frac{1}{2\sqrt{R}}\int_0^R dx_5 \hat{A}_M(x_\mu, x_5); && (3.13)\\
A^n_\mu &= \frac{1}{\sqrt{2R}}\int_0^R dx_5 \hat{A}_M(x_\mu, x_5)\cos(n\theta) \ ; \qquad n = 1, \cdots, +\infty \\
A^n_5 &= \frac{1}{\sqrt{2R}}\int_0^R dx_5 \hat{A}_5(x_\mu, x_5)\sin(n\theta) \ ; \qquad n = 1, \cdots, \infty.
\end{aligned}$$

The non-tilde vector field A^n_M has mass dimension +1.

The field strengths read,

$$\begin{aligned}
\hat{F}_{\mu\nu}(x_\alpha, x_5) &= \frac{1}{\sqrt{R}}\left\{\left[\partial_{[\mu}A^0_{\nu]} + \sum_{n=1}^{+\infty}\cos(n\theta)\partial_{[\mu}A^n_{\nu]}\right]\right. && (3.14)\\
&\left. + \frac{g}{\sqrt{M_s R}} f\left[A^0_\mu + \sqrt{2}\sum_{n=1}^{+\infty} A^n_\mu\cos(n\theta)\right]\left[A^0_\mu + \sqrt{2}\sum_{m=1}^{+\infty} A^m_\mu\cos(m\theta)\right]\right\},
\end{aligned}$$

Intergrating over x^5 we obtain the effective $3 + 1$ theory.

If we now impose the axial gauge $A_5(x_\mu, x_5) \equiv 0$, the effective Lagrangian after intergrating over x_5 and truncating at the Nth KK mode takes the form:

$$\begin{aligned}
\mathcal{L}_4 &= (\partial_\mu A^0_\nu - \partial_\nu A^0_\mu + \frac{g}{\sqrt{M_s R}} f^{abc} A^0_\mu A^0_\nu)^2 + \sum_{n=1}^{N}(\partial_\mu A^n_\nu - \partial_\nu A^n_\mu)^2 && (3.15)\\
&+ \frac{2g}{\sqrt{M_s R}} f^{abc}\sum_{n=1}^{N}\left[\partial_{[\mu}A^0_{\nu]}A^{n\ \mu}A^{n\ \nu} + \partial_{[\mu}A^n_{\nu]}(A^{0\ \mu}A^{n\ \nu} + A^{n\ \mu}A^{0\ \nu})\right] \\
&+ \frac{g}{\sqrt{2M_s R}} f^{abc}\sum_{n,m,l=1}^{N}\partial_{[\mu}A^n_{\nu]}A^{m\ \mu}A^{l\ \nu}\Delta_1(n, m, l) \\
&+ \frac{g^2}{M_s R} f^{abc} f^{ade}\sum_{n=1}^{N}\left(A^0_\mu A^0_\nu A^{n\mu}A^{n\nu} + \text{all permutations}\right)
\end{aligned}$$

$$+\frac{g^2}{2M_sR}f^{abc}f^{ade}\sum_{n,m,l,k=1}^{N}A_\mu^nA_\nu^mA^{l\ \mu}A^{k\ \nu}\Delta_2(n,m,l,k)$$
$$+\sum_{n=1}^{N}(\frac{n\pi}{R})^2A_\mu^nA^{n\mu},$$

where the Δ_i are defined as:

$$\Delta_1 = \delta(n+m+l)+\delta(n+m-l)+\delta(n-m+l)+\delta(n-m-l) \tag{3.16}$$
$$\Delta_2 = \delta(n+m+l+k)+\delta(n+m-l-k)+\delta(n+m+l-k)+\delta(n+m-l+k)$$
$$+\delta(n-m+l+k)+\delta(n-m-l-k)+\delta(n-m+l-k)+\delta(n-m-l+k).$$

The zero mode has the canonical $3+1$ kinetic term with field strength:

$$F_{\mu\nu}^{0\ (a)} = \partial_\mu A_\nu^{0\ (a)} - \partial_\nu A_\mu^{0\ (a)} + gf^{abc}A_\mu^{0\ (b)}A_\nu^{0\ (c)}, \tag{3.17}$$

Hence, $\tilde{g} \equiv g/\sqrt{M_sR}$ is the dimensionless low-energy $3+1$ coupling constant. If the truncation $N = M_sR$ on the number of the KK modes is introduced then $\tilde{g} \equiv g/\sqrt{N}$.

(iii) Comparison to Aliphatic Theory

Now, consider again the aliphatic theory with the gauge structure $SU(3)_0 \times SU(3)_2 \times \ldots \times SU(3)_N$, where the vector potentials are $A_\nu^{j\ a}$. In addition, there are a set of chiral Φ_i fields which straddle the ith and $i+1$th $SU(3)$ gauge groups. The Lagrangian takes the form as in eq.(1.1), and the mass spectrum as in eq.(1.3). The gauge fields A_μ^j can be expressed as linear combinations of the mass eigenstates $\tilde{A}_\mu^n$ as:

$$A_\mu^j = \sum_{n=0}^{N} a_{jn}\tilde{A}_\mu^n. \tag{3.18}$$

The a_{nj} form a normalized eigenvector $(\vec{a}_n)$ associated with the nth $n \neq 1$ eigenvalue and has the following components:

$$a_{nj} = \sqrt{\frac{2}{N+1}}\cos{(\frac{2j+1}{2}\gamma_n)}\ , \qquad j = 0,1,\ldots,N, \tag{3.19}$$

The eigenvector for the zero-mode, $n = 1$, is always $\vec{a}_1 = \frac{1}{\sqrt{N+1}}(1,1,\ldots,1)$. The orthorgonality between the eigenvectors is due to:

$$\sum_{j=0}^{N}\cos{(\frac{2j+1}{2}\gamma_n)}\cos{(\frac{2j+1}{2}\gamma_m)} = \delta(n-m)\frac{N+1}{2}\ , \qquad n,m \neq 0 \ll N \tag{3.20}$$

with $\gamma_n = \frac{n\pi}{N+1}$. We can now rewrite the Lagrangian eqn.(1.1) in the mass eigenstates of the vector bosons ($\tilde{A}^n_\mu$) and derive the interactions between them.

Let us now compare the KK reduction of the five-dimensional theory, eqn.(3.16), and the aliphatic $(SU(3)^{N+1}, \Phi^N)$ theory at the level of interactions. The masses of the KK modes in the truncated theory are $n\pi/R$, the mass eigenstates of the $SU(3)^{N+1}$ theory have the spectrum $gvn\pi/\sqrt{2}(N+1)$, in the limit of $n \ll N$. In the aliphatic theory, as far as the mass spectrum is concerned, there are three free parameters, namely, the gauge coupling constant g, the total number of $SU(3)$ groups N and the VEV of the Higgs field v. The spacing of the linear mass spectrum is completely determined by the ration between gv and N. Hence, one can arrange the parameters of the $SU(3)^{N+1}$ theory to satisfy $gv/\sqrt{2}(N+1) = \frac{1}{R}$, such that the mass spectrum of the two theories matches for $n << N$. The departure from linearity as $n \to N$ is not surprising since we are then approaching the string scale and exiting the low energy universal theory.

To compare the Lagrangian's couplings we substitute eqn.(3.18) into the gauge part of the Lagrangian eq.(1.2):

$$\mathcal{L}_{gauge} = -\frac{1}{4}\sum_{j=0}^{N}(\sum_{n=0}^{N} a_{jn}\partial_{[\mu}A^n_{\nu]} + g_L f^{abc}\sum_{n=0}^{N}\sum_{m=0}^{N} a_{jn}a_{jm}A^n_\mu A^m_\nu)^2 \tag{3.21}$$

We isolate the zero-mode, $\tilde{A}^{0a}_\mu$ and, using orthonormality, the kinetic terms take the canonical form:

$$\mathcal{L}_{g,kin} = -\frac{1}{2}(\partial_{[\mu}\tilde{A}^0_{\nu]} + \frac{g_L}{\sqrt{N+1}}f^{abc}\tilde{A}^0_\mu\tilde{A}^0_\nu)^2 + \sum_{n=1}^{N}(\partial_{[\mu}\tilde{A}^n_{\nu]})^2. \tag{3.22}$$

The trilinear gauge coupling takes the form:

$$\mathcal{L}_{g,3A} = -\frac{1}{4}\sum_{n,m,l\neq(0,0,0)}(\sum_{j=0}^{N} a_{jn}a_{jm}a_{jl})g_L f^{abc}\partial_{[\mu}\tilde{A}^n_{\nu]}\tilde{A}^{m\ \mu}\tilde{A}^{l\ \nu}. \tag{3.23}$$

Using pairwise summations and orthogonality:

$$\sum_{j=0}^{N} a_{jn}a_{jm}a_{jl} = \begin{cases} \sqrt{\frac{1}{N+1}}\,[\delta(n)\delta(m-l) + \delta(m)\delta(n-l) + \delta(l)\delta(n-m)]\ , & \\ \sqrt{\frac{1}{2(N+1)}}\Delta_1(n,m,l)\ , & n,m,l \neq 0; \end{cases} \tag{3.24}$$

where Δ_1 is defined previously. Similarly, the quadrilinear couplings take the form:

$$\mathcal{L}_{g,4A} = -\frac{1}{4}\sum_{n,m,l,k\neq(0,0,0)}(\sum_{j=0}^{N} a_{jn}a_{jm}a_{jl}a_{jk})g_L f^{abc}g_L f^{ade}\tilde{A}^n_\mu\tilde{A}^m_\nu\tilde{A}^{l\ \mu}\tilde{A}^{k\ \nu}, \tag{3.25}$$

with the coefficients,

$$\sum_{j=0}^{N} a_{jn}a_{jm}a_{jl}a_{jk} = \begin{cases} \frac{1}{N+1}, & \text{two of (n, m, l, k) are zero, remaindersare equal;} \\ \frac{1}{2(N+1)}\Delta_2(n,m,l,k)\ , & n,m,l,k \neq 0; \end{cases} \tag{3.26}$$

We see that $\Delta_2(n,m,l,k)$ is exactly the same function defined in the discussion of truncated momentum space expansion.

Thus, we see that, defining the gauge coupling constant in the aliphatic theory to satisfy $\overline{g} = g_L/\sqrt{N+1} = g/\sqrt{M_s R}$, the couplings and Feynman rules in the two theories agree perfectly. This completes the demonstration of the equivalence.

4 Incorporation of Fermions

The models we presented for the gauge bosons in the bulk can easily accommondate fermions and bosons in the bulk.

Consider $N+1$ fermions Ψ_i $(i = 0 \cdots N)$, each of which is charged under the coresponding $SU(3)_i$ symmetry. The Higgs fields Φ_i which is $(\overline{3}, 3)$ under the two neighboring $SU(3)$ symmetries provides the nearest neighbor couplings between the fermion fields. The effective Lagarangian takes the form

$$\mathcal{L}_{fermion} = \sum_{i=0}^{N} \overline{\Psi}_{i,L/R} \not{D}\, \Psi_{i,L/R} + M_f \left[\overline{\Psi_{i,L}} \left(\frac{\Phi_{i+1}^{\dagger}}{v} \Psi_{i+1,R} - \Psi_{i,R} \right) - \overline{\Psi_{i,R}} \left(\Psi_{i,L} - \frac{\Phi_i}{v} \Psi_{i-1,L} \right) \right], \tag{4.27}$$

where $\not{D}$ is defined as the four dimensional covariant derivative.

One can impose different boundary conditions on the left-handed and right-handed components of the 5D fermions Ψ

$$\frac{\partial}{\partial x_5} \Psi_L|_{x_5=0,R} = 0; \qquad \Psi_R|_{x_5=0,R} = 0, \tag{4.28}$$

such that the 4D effective theory has one massless left-handed fermions with a tower of massive KK modes, while all the right-handed fermions are massive. Equivalently, in our models, the boundary conditions translate into $\Psi_{0,R} = \Psi_{N,R} = 0$ and $\Psi_{L,N} - \Psi_{L,N-1} = 0$. As a result, in the vacuum where Φ_i has non-zero VEV v, the mixed mass terms for the left-handed and right-handed fermions are

$$\begin{aligned} \mathcal{L}_{mass} &= M_f \left\{ \overline{\Psi_{0,L}} \Psi_{1,R} + \textstyle\sum_{i=1}^{N-1} \left[\overline{\Psi_{i,L}} (\Psi_{i+1,R} - \Psi_{i,R}) - \overline{\Psi_{i,R}} (\Psi_{i,L} - \Psi_{i-1,L}) \right] \right\} \\ &= (\overline{\Psi_{0,L}}, \cdots, \overline{\Psi_{N-1,L}}) M (\Psi_{1,R}, \cdots, \Psi_{N-1,R})^T; \end{aligned} \tag{4.29}$$

where the $N \times (N-1)$ mass matrix M takes the form

$$M = M_f \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ & & \cdots & \\ 0 & \cdots & -1 & 1 \\ 0 & \cdots & 0 & -1 \end{pmatrix}. \tag{4.30}$$

To calculate the mass eigenvalues and eigenstates for the right-handed components, one can diagonalize the $(N-1) \times (N-1)$ matrix $M^\dagger M$,

$$M^\dagger M = |M_f|^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{4.31}$$

Therefore, the eigenvalues of the right-handed fermions are

$$M_{R,n} = 2M_f \sin(\frac{n\pi}{2N}), \qquad n = 1, 2, \cdots, N-1. \tag{4.32}$$

In terms of the mass eigenstates $\tilde{\Psi}_{n,R}$,

$$\Psi_{i,R} = \sqrt{\frac{N}{2}} \sum_{n=1}^{N-1} \sin(i\frac{n\pi}{N}) \tilde{\Psi}_{n,R}. \tag{4.33}$$

The mass eigenvalues of the left-handed fermions can be calulated from the $N \times N$ matrix $MM^\dagger$, which takes the following form,

$$MM^\dagger = |M_f|^2 \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}. \tag{4.34}$$

Hence, the eigenvalues of the left-handed fermions are similar to those of the gauge bosons,

$$M_{i,L} = 2M_f \sin(\frac{i\pi}{2N}), \quad i = 0 \cdots N-1. \tag{4.35}$$

Hence, the left-handed fermions have a massless zero mode. The massive modes have the same mass as those of the right-handed fermions, thus form massive vector pairs.

The eigenvectors of the LH fermions also have the same structure as that of the gauge bosons, namely, in terms of the mass eigenstates $\tilde{\Psi}_{n,L}$,

$$\Psi_{i,L} = \sqrt{\frac{N}{2}} \sum_{n=0}^{N-1} \cos\left(\frac{2i+1}{2}\frac{n\pi}{N}\right) \tilde{\Psi}_{n,L}. \tag{4.36}$$

Note that left-handed fermions have a cos expansion, while the right-haned fermions assume a sin expansion.

In the limit that $i \ll N$, a linear massive spectrum is recovered for both right-handed and left-haned fermions, in which $M_i = M_f \frac{i\pi}{N}$. It is well known that the masses of the KK modes for a 5D fermion are $M_i = \frac{i\pi}{R}$, where R is the size of the fifth dimension. Hence, by choosing $M_f = \frac{N}{R}$, one reproduces the linear spectrum for the KK theory.

The coupling between the fermions and the gauge field takes the following form in their mass eigenstate basis,

$$\begin{aligned} \mathcal{L}_{ffA} \;\; = \;\; & \textstyle\sum_{n,m,l \neq (0,0,0)} g_L \bar{\tilde{\Psi}}_{n,L} \gamma^\mu \tilde{A}_{\mu m} \tilde{\Psi}_{l,L} \Delta_{n,m,l} + g_L \bar{\tilde{\Psi}}_{0,L} \gamma^\mu \tilde{A}_{\mu 0} \tilde{\Psi}_{0,L} \\ & + \textstyle\sum_{n,m,l \neq 0,N} g_L \bar{\tilde{\Psi}}_{n,R} \gamma^\mu \tilde{A}_{\mu m} \tilde{\Psi}_{l,R} \Delta_{n,m,l}, \end{aligned} \tag{4.37}$$

in which $\Delta_{n,m,l}$ is defined as the sum in Eq.(3.24).

One can also write down the effective Lagarangian for a massless complex boson in the bulk in our frame work. Consider $N+1$ 4D complex scalar with the following Lagrangian,

$$\mathcal{L}_{boson} = \sum_{i=0}^{N} |D_\mu \phi_i|^2 - M_b^2 \sum_{i=1}^{N} |\phi_{i-1} - \frac{1}{v} \Phi_i \phi_i|^2. \tag{4.38}$$

In the vacuum in which $\langle \Phi_i \rangle = v$, the scalars have the mass terms $-M_b^2 \sum_{i=1}^N |\phi_{i-1} - \phi_i|^2$. It can diagonalized by

$$\phi_j = \frac{1}{N+1} \sum_{n=1}^{N} e^{i2\pi nj/(N+1)} \tilde{\phi}_n, \tag{4.39}$$

with the mass spectrum

$$M_{n,b} = 2M_b \sin\gamma_n, \quad n = 0, 1, \cdots N. \tag{4.40}$$

Each level with $n \neq 1$ is degenerate with the level $N-n$, while the zero mode is a singlet. This doubling of energy levels corresponds to the mode expension in x^5 in terms of 1, $\sin(k_n x^5)$ and $\cos(k_n x^5)$, where the sine and cosine terms are degenerate modes.

5 Renormalization of gauge coupling constant

The spontaneously broken gauge theory $(SU(3)^{N+1},\ \Phi^N)$ is a renormalizable field theory. Thus, we can discuss the scale dependence of the coupling strength $\overline{g}(\mu)$ of the unbroken $SU(3)$ via the radiative corrections to the triple gluon coupling, etc. In principle, this vertex can be calculated in any order of perturbation in the full spontaneously broken $(SU(3)^{N+1},\ \Phi^N)$ theory, such calculation faces the usual problem of the resummation of large logarithms generated by the spectrum of the massive vector bosons. It is, therefore, necessary to systemetically discuss the decoupling procedure in the evolution from high energy scale to the low energy scale. However, the resummation of the leading logarithms (1-loop) can be carried out with the standard renormalization group approach.

At 1-loop, the running of the gauge coupling constant $\overline{g}$ between the scales $(M_n,\, M_{n-1})$ only involves the $n-1$ massive modes which are lighter then M_n, as a result the running can be described by the 1-loop β function of a $SU(3)^{n-1}$ theory,

$$\frac{d\overline{g}}{d\log\mu} = -[(n-1)\frac{\beta}{4\pi^2}]\ \overline{g}^3, \tag{5.41}$$

in which β is the 1-loop RGE coefficient of a pure $SU(3)$ theory. Hence, given the measured coupling constant at low energy, the gauge coupling constant at a scale μ is

$$\alpha^{-1}(\mu) = \alpha^{-1}(M_Z) - \frac{\beta}{4\pi}\left[\ln(\frac{M_1}{M_Z}) + \sum_{n=2}^{n_{max}} n\ln(\frac{M_n}{M_{n-1}}) + (n_{max}+1)\ln(\frac{\mu}{M_{n_{max}}})\right], \tag{5.42}$$

where $n_{max} \le \mu/M_1 < n_{max}+1$. When the series is summed up, the running of $\overline{g}$ exhibits power law behavior,

$$\alpha^{-1}(\mu) = \alpha^{-1}(M_Z) - \frac{\beta}{4\pi}\ln(\frac{\mu}{M_Z}) - \frac{\beta}{4\pi}\left((n_{max}\ln(\mu) - \ln(n_{max}!)\right). \tag{5.43}$$

The same results are presented in Deines et al, Cheng et al [4,5]. Our calculation is however done is a well-defined gauge invariant theory compared to the momentum space truncated KK theory in which the earlier calculations are performed. In principle, we can systemetically go beyond 1-loop to any higher orders in the renormalization group analysis. What we ultimaetly seek is a block-spin renormalization group, or a decimation approach more well-suited to the lattice construction [8].

6 Further discussion and Conclusion

Suppose we had a bulk $5+1$ theory. Then we would have a different structure for the low energy effective theory, and we would have a correspondingly different lattice theory. No longer would the theory be an aliphatic model, and would appear then as a more complex closed structure.

The simplest case is the limit of a single plaquette in the two compact dimensions of $5+1$, the analogue of an Eguchi-Kawai model. The low energy theory would contain the gluon zero-mode, a doubly degenerate pair of colorons as the first KK modes, and a third heavy singlet. This requires $SU(m)^4$ with 4 Φ_i fields. The zero mode gluon is just the rotational zero-mode of such a molecule. As more KK modes are excited we see that we need a lattice structure of $SU(m)^{NM}$, corresponding to $N \times M$ sites in the plane, linked together with the corresponding chiral-Higgs fields.

It is interesting that ultimately the lattice structure must also reflect the homotopy of the extra dimensions. If there is a "hole" in the space of the extra dimensions, there must be corresponding nontrivial paths through the Higgs field links that match the non-contractable loops in that space.

The fact that the links in the fifth dimension are in the broken phase, i.e., a chiral Lagrangian, suggests a unitarity problem in the 4 dimensional theory. Essentially, longitudinal KK mode scattering must violate perturbative unitarity when $s \gtrsim 4\pi v^2$. This is the Lee-Quigg-Thacker bound which applies to electroweak symmetry breaking [7]. It occurs when we lift the Higgs mass above the cut-off scale M_s, which then decouples from the the low energy theory. We see, however that this corresponds to energy scales approaching M_s at which the 5-dimensional coupling constant is large. Hence the perturbative unitarity violation inherent in the large coupling constant of the parent 5-d theory is matched by the unitarity breakdown in the effective $3+1$ theory.

In conclusion, We have constructed a manifestly gauge invariant description of N KK modes for an $SU(m)$ gauge theory in the bulk. We showed in this letter the four-dimensional KK theory deducted from a compactified five-dimensional $SU(3)$ theory can be considered as a $(SU(3)^N,\ \Phi^{N-1})$ theory, in which the $SU(3)^N$ gauge symmetry is spontaneously broken to $SU(3)$. This theory owes its structure to a transverse lattice theory with one extra dimension. The three dimensionful parameters of the original KK theory, the string (cut-off) scale M_s, the compactification radius R and the five-dimensional gauge coupling $g_0 \equiv M^{-1}$, determine the structure of the $(SU(3)^N,\ \Phi^{N-1})$

theory: $N = M_s R$, the coupling constant of the unbroken $SU(3)$ $\overline{g} \equiv g/\sqrt{(}N) = 1/\sqrt{MR}$, and the scale $V = \sqrt{2M_s M}$ of the spontaneous symmetry breaking $SU(3)^N \rightarrow SU(3)$.

The approach maintains manifest gauge invariance. Is it possible to construct analogous effective Lagrangians which maintain SUSY and general covariance for yielding KK modes of gravity? And how are the topological aspects of extra dimensional gauge theories [6] expressed in an effective Lagrangian such as this?

(Note added:) Upon completion of this work the preprint of Arkani-Hamed, Cohen and Georgi, [9], appeared which obtains essentially the identical construction. Georgi's moose notation, used in [9], is a useful way to extend to higher dimensions such as $5 + 1$ with 2 compact dimensions. The theory may be graphically represented as a "moose lattice," and the anomaly free incorporation of fermions is automatic.

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