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# Emittance Growth for the LHC Beams Due to Head-On Beam-Beam Interaction and Ground Motion

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**Abstract**

The influence of ground motion on the LHC beam is estimated applying the existing theories of particle diffusion due to a weak-strong beam-beam collision with random offset at the interaction point. Noise at odd harmonics of the betatron frequency contributes significantly to particle diffusion. The spectrum of the random offset, as obtained from the ground motion spectrum at the LHC site, shows a fast fall-off with frequency and the amplitude is very small even at the first harmonic. We find that the head-on beam-beam force in the weak-strong approximation and ground motion by themselves do not induce significant diffusion over the lifetime of the beam.

# 1 Introduction

It is known that ground vibration at frequencies higher than $f_m = 11$ Hz will cause uncorrelated quadrupole motion in the LHC ring[1] [1, 2]. Random quadrupole oscillations create distortions of the beam orbit and random beam-beam offset at the interaction point. The beam-beam interaction is the main nonlinearity at high energy and will be one of the dominant sources of emittance growth when the beams collide. Random fluctuations of the beam-beam offset at odd harmonics of the betatron frequency can strongly enhance the diffusion rate of the beam. Our aim is to estimate the effect of ground motion and the head-on beam-beam collision on the beam emittance.

The dynamical system under consideration is introduced in section 2. In order to evaluate the diffusion due to random beam-beam offset it is first necessary to characterize this stochastic process. In section 3 we present the expected spectral characteristics of the orbit offset which are extrapolated from the measured ground motion spectrum at the LHC site [3] and the amplification factor for the response of the closed orbit [4]. The spectral density of the ground motion decays very rapidly with frequency. The random process can be simulated as a discrete Ornstein-Uhlenbeck (OU) noise, with a given amplitude and correlation time [5], this stochastic process is also presented in this section. The analytical diffusion coefficients for action-variables are known [6], the results for a one dimensional model are summarized in section 4. In section 5 we compare the analytical diffusion coefficient with the results obtained by tracking. Knowing the diffusion coefficient, we integrate numerically the associated Fokker-Planck equation and evaluate for an initial Gaussian distribution the emittance doubling time. In section 6 these calculations are extended to a two-dimensional model. Finally the main conclusions of this study are summarized in section 7.

# 2 Dynamical system

We consider the dynamics of a test particle whose motion is followed over $N$ turns, assuming linear betatron motion and a weak-strong beam-beam collision at one interaction point (in the weak-strong approximation we only consider the influence of bunch A on bunch B, and ignore the influence of bunch B on A). At this interaction point, the particle experiences a deflection caused by the field of a counter rotating Gaussian beam. Our system of normalized variables in two transverse degrees of freedom is $(x, y) = (X/\sigma_x, Y/\sigma_y)$, $(v_x, v_y) = (\beta_x X'/\sigma_x, \beta_y Y'/\sigma_y)$, where $(X, Y)$ is the position of the particle, $(\sigma_x, \sigma_y)$ are the nominal rms sizes and $(\beta_x, \beta_y)$ the nominal betatron function. The prime denotes the derivative with respect to the longitudinal position $s$, so that *e.g.*, $X'$ is the slope of the horizontal trajectory.

We assume that the coupling between the transverse planes is negligible so that the linear map from

[1]The length of the LHC cell is about $l = 90$ m and the "average" velocity of the ground waves in the LEP tunnel is about 4000 m/s. The correlation between two probes at distance $l$ drops to zero for a frequency such that this distance is a quarter of wavelength $f_m = v/(4l)$. In practice the frequency dependence of the velocity leads to uncorrelated motion beyond frequencies somewhat lower than this limit.

At very low frequencies one expects the whole ring to move coherently. In fact it has been seen that these long powerful waves loose coherence and small blocks of the ring move independently. Their motion is not completely random, since the two dimensional covariance function is not equivalent to that of white noise, but the average squared distance between two of these blocks will grow linearly in time as it is expected from a typical diffusion process. This creates a drift of the closed orbit which is usually described by the "ATL-law". For LHC the rms orbit deformation has been estimated to be about 1% of $\sigma$ per second, at most. Adequate feedback systems should compensate this rate [7, 8].

one interaction point to the next is

$$\begin{pmatrix} x(n+1) \\ v_x(n+1) \\ y(n+1) \\ v_y(n+1) \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q_x) & \sin(2\pi Q_x) & 0 & 0 \\ -\sin(2\pi Q_x) & \cos(2\pi Q_x) & 0 & 0 \\ 0 & 0 & \cos(2\pi Q_y) & \sin(2\pi Q_y) \\ 0 & 0 & -\sin(2\pi Q_y) & \cos(2\pi Q_y) \end{pmatrix} \begin{pmatrix} x(n) \\ v_x(n)+\Delta v_x(n) \\ y(n) \\ v_y(n)+\Delta v_y(n) \end{pmatrix} \quad (1)$$

The beam-beam kick at turn $n$ depends on the distance from the test particle with position $(x, y)$ to the centroid of the counter rotating beam. The beam oscillates randomly due to the ground motion. The position (in units of $\sigma$) of the centroid of the counter rotating beam at turn $n$ is represented by the random variable $(\eta_x(n), \eta_y(n))$ whose spectral characteristics have to simulate those of the orbit offset spectrum. The kick due to the beam-beam interaction is

$$\begin{pmatrix} \Delta v_x(n) \\ \Delta v_y(n) \end{pmatrix} = 2C \begin{pmatrix} \beta_x^*(x-\eta_x(n))/\sigma_x^2 \\ \beta_y^*(y-\eta_y(n))/\sigma_y^2 \end{pmatrix} \frac{\left[1-\exp\left\{-\frac{1}{2}[(x-\eta_x(n))^2+(y-\eta_y(n))^2]\right\}\right]}{(x-\eta_x(n))^2+(y-\eta_y(n))^2} \quad (2)$$

with $C = N_p r_p/\gamma_p$, $N_p$ the number of protons per bunch in the opposing beam, $r_p$ the classical radius of the proton, $\gamma_p$ the relativistic kinematic factor of the protons, $(\beta_x^*, \beta_y^*)$ the beta functions at the interaction point(IP) and $(\sigma_x, \sigma_y)$ are the rms sizes of the opposing beam at the IP.

We shall use for our study the LHC beam parameters: $\sigma = 0.0159$ mm, $\gamma_p = E_p/E_0$, $E_0 = 0.93827$ GeV, $E_p = 7000$ GeV, $N_p = 1.05 \times 10^{11}$, $r_p = 1.5347 \times 10^{-15}$ mm, $\beta_x^* = \beta_y^* = 500$ mm, which correspond to a beam-beam parameter of $\xi = 0.003355$.

# 3 Ground motion and closed orbit spectrum

In order to evaluate the diffusion due to the combined effect of the beam-beam interaction and the random offset we need to estimate the spectral density of the offset in the vicinity of the betatron tune.

The spectral density of the ground motion measured at 10 Hz in the LEP tunnel is $S_{gm} = 5 \times 10^{-15}\mathrm{mm}^2/\mathrm{Hz}$ while the logarithmic slope with frequency at low frequencies is about -2.5. Assuming that this fall-off rate continues at high frequencies, we can then expect the spectral density of the ground motion in the vicinity of the betatron tune to be about $S_{gm} = 10^{-20}\mathrm{mm}^2/\mathrm{Hz}$ [1]. It should be pointed out that ground motion measurements at various accelerators show that above approximately 400Hz, the motion is indistinguishable from electronic noise, even with the most sensitive piezo-electric accelerometers available. Any measurable noise above a few hundred Hz has its origin in other sources including power supplies, water flow, liquid helium flow in superconducting magnets etc. Hence approximating the ground motion noise spectrum by a $f^{-2.5}$ law over the whole range probably over-estimates the contribution of ground-motion.

The effect of plane ground waves on the closed orbit of LHC has been studied for the collision configuration of the LHC lattice Version 4.3 [4], using MAD for computation of the closed orbit. Selected elements suffer vertical displacements which are computed for plane ground motion waves with given harmonic number $h$ and phase. The vertical amplification factor R (*i.e.* the ratio between the closed orbit offset and the ground motion amplitude) is evaluated at the experimental pits for $0 \le h \le 200$. R rises quickly reaching a maximum for wavelengths of the order of the betatron wavelength. The mean square response for LHC in the vicinity of the betatron frequency is $R^2 = 10$.

Therefore we can estimate the spectral density of the orbit offset in the vicinity of the betatron tune to be about $S_o(Q_{beta}) = R^2 \times S_{gm}(Q_{beta}) = 10^{-19}\mathrm{mm}^2/\mathrm{Hz}$. One has to keep in mind that in addition magnet support can also enhance the motion. For instance at HERA proton ring, measurements show an amplification factor of approximately 2 and 4 for the vertical and horizontal quadrupole motion respectively [9].

### 3.1 Modelling the stochastic process

We have seen that the orbit offset spectrum decays very rapidly with frequency, having a logarithmic slope of -2.5. We can model these fluctuations by an OU process whose spectrum has a $1/f^2$ dependence [5]. It is well known that this is the only stationary Gaussian Markov process. If $\eta(t)$ is a stochastic variable of zero mean following an OU process, then its correlation function decays exponentially with time

$$K(t_1, t_2) \equiv < \eta(t_1)\eta(t_2) > = |\eta|^2 e^{-|t_1-t_2|/t_c} \tag{3}$$

where $t_c$ is the correlation time and $|\eta|$ the amplitude of the fluctuations in units of $\sigma$. We measure time in units of the revolution period $T_{rev}$ and it is more appropriate to label time by a discrete turn number. It can be shown that for the discrete time process, the stochastic differential equation describing an OU process is transformed to the following difference equation for $\eta_n$ defined as

$$\eta_{n+1} = (1 - \frac{1}{\tau_c})\eta_n + \sqrt{\frac{2}{\tau_c}}|\eta|\xi_{n+1} \tag{4}$$

where $n$ is the number of turns, $\tau_c$ is the correlation time measured in number of turns, and $\xi$ is a Gaussian white noise process of zero mean and unit variance. The spectral density of this process is (see Appendix A)

$$S_{OU}(Q) = (|\eta|\sigma)^2 \frac{T_{rev}}{2\pi} \frac{\sinh\theta}{(1 - 1/(2\tau_c))[\cosh\theta - \cos(2\pi Q)]} \tag{5}$$

where $\theta = -\ln(1 - 1/\tau_c)$. The fall in noise power with increasing frequency is characterized by the correlation time $\tau_c$. For instance for a correlation time $\tau_c = 100$ (in units of turns), $S_{OU}(0.28)/S_{OU}(0) \approx 4 \times 10^{-5}$ which is similar to the expected ratio in the LHC offset spectrum $S_o(Q_{beta})/S_o(10) \approx 2 \times 10^{-5}$.

A discrete OU process with oscillation amplitude of $|\eta| = 10^{-4}$ (in units of $\sigma$) and correlation time $\tau_c = 100$ (turns) has a spectral density at $Q_\beta = 0.28$ of about $10^{-19}\mathrm{mm}^2/\mathrm{Hz}$ which is the expected spectral density of the orbit offset in the vicinity of the betatron tune.

# 4 Analytical evaluation of the diffusion coefficient

In this section we will consider the theoretical predictions for the one degree of freedom case. This has been analytically studied, in the case of tunes far from resonances, using action angle variables ($x = \frac{\sqrt{2J_x\beta^*}}{\sigma}\cos\psi_x$, $v_x = -\frac{\sqrt{2J_x\beta^*}}{\sigma}\sin\psi_x$ ) [5, 6]. For completeness, we include the details of the derivation here.

The 1D Hamiltonian is

$$H = Q_x J_x + U(x)\delta_p(\theta) \tag{6}$$

where $\theta$ is the azimuthal variable. $U(x)$ is the beam-beam potential that can be expressed as a Fourier series

$$U(x) \quad = \quad C\sum_{k=0}^{\infty} U_k(a)\cos(2k\psi_x) \tag{7}$$

$$U_k = \int_0^a \frac{1}{w}[\delta_{0k} - (2-\delta_{0k})(-1)^k e^{-w} I_k] dw \quad (8)$$

$$a = \frac{\beta^* J_x}{2\sigma^2}, \quad (9)$$

with $I_k$ the modified Bessel functions. The one turn map in action angle variable to first order in the beam-beam parameter reads

$$\Delta\psi_x = 2\pi Q_x + \frac{\partial U}{\partial J_x} \quad (10)$$

$$\Delta J_x = -\frac{\partial U}{\partial \psi_x} \quad (11)$$

For small closed orbit offsets $\eta$, we can expand the potential in a Taylor series

$$U(J_x, \psi_x) = U(x) + U'(x)\eta + O(\eta^2) \quad (12)$$

$$\begin{aligned} f(J_x, \psi_x) &\equiv U'(x) = \frac{\partial J_x}{\partial x}\frac{\partial U_x}{\partial J_x} + \frac{\partial \psi_x}{\partial x}\frac{\partial U_x}{\partial \psi_x} \quad (13) \\ &= C\sum_{k=0}^{\infty} G_k(J_x)\cos((2k+1)\psi_x) \end{aligned}$$

where $G_k$ are the Fourier coefficients of the beam-beam force given by

$$G_k(a) = \frac{\sqrt{a}}{\sigma}(U'_{k+1} + U'_k) + \frac{1}{\sqrt{a}\sigma}((k+1)U_{k+1} - kU_k) \quad (14)$$

$$(15)$$

We wish to calculate the change in action due to the fluctuating offset alone, given that we know that in the absence of fluctuation the change in action is negligible. To first order in $\eta$ the change at turn $m$ is

$$\Delta J_x(m) = -\frac{\partial}{\partial \psi_x} f(J_x(m), \psi_x(m))\eta(m). \quad (16)$$

If $J(0)$ is the initial value of the action of a particle and $J(N)$ the particle action at turn $N$, the total change at turn $N$ is obtained by summing over all previous turns

$$\begin{aligned} \Delta J_x^2(N) &= (J(N) - J(0))^2 \quad (17) \\ &= \sum_{l=0}^{N-1}\sum_{m=0}^{N-1} \frac{\partial}{\partial \psi_x(l)} f(J_x(l), \psi_x(l)) \frac{\partial}{\partial \psi_x(m)} f(J_x(m), \psi_x(m))\eta(l)\eta(m) \end{aligned}$$

The diffusion coefficient is defined as

$$D_{\text{off}}(J) = \lim_{N\to\infty} \frac{< (J(N) - J(0))^2 >}{N} \quad (18)$$

where the average is over many noise realizations. Extracting the dominant terms, and introducing the correlation function of the offset $K(n)$

$$< \eta(l)\eta(n+l) >= \sigma^2 K(n) \quad (19)$$

one gets

$$\lim_{N\to\infty} \Delta J_x^2(N) = N\frac{1}{8}C^2\sigma^2 \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}(2k+1)^2 G_k^2 \cos[(2k+1)\psi_n]K(n) \quad (20)$$

Hence the diffusion coefficient due to collisions at a single IP is

$$D_{\rm off}(J_x) = \frac{\pi C^2\sigma^2}{4T_{rev}}\sum_{k=0}^{\infty}(2k+1)^2 G_k^2(a) S_{off}[(2k+1)Q_x] \tag{21}$$

In the expression for the diffusion coefficient, $S_{off}[(2k+1)Q_x]$ is the spectral density of the fluctuating offsets at odd harmonics of the betatron frequency.

When the noise is described by an Ornstein-Uhlenbeck process, the expression can be simplified to

$$D_{\rm off}^{O-U}(J_x) = \frac{1}{8}\frac{(C\sigma|\eta|)^2\sinh\theta}{[1-1/(2\tau_c)]}\sum_{k=0}^{k=\infty}\frac{(2k+1)^2 G_k^2(a)}{\cosh\theta - \cos[2\pi(2k+1)Q_x]} \tag{22}$$

Assuming that the fluctuating offsets at the two IPs are uncorrelated, we may infer that the diffusion coefficients at two IPs add in quadrature. We assume that the amplitude and spectral characteristics of the offsets are similar at the two IPs so the effective diffusion coefficient from collisions at two IPs is $\sqrt{2}$ times that given by Equation (22).

# 5 Diffusion coefficient and emittance growth

The parameters of the random process are set to $\tau_c = 100$ and $|\eta| = 0.01$. Notice that the amplitude of the random offset $|\eta|$ is $10^2$ times stronger than the one expected at the site of the LHC ring. We have chosen this strong noise in order to get a stronger diffusion in the tracking simulation. Since the diffusion coefficient scales with $|\eta|^2$ for a realistic parameter we expect the diffusion coefficient $D_{\rm off}(J)$ to be $10^4$ times smaller.

In Fig. 1-left we show the diffusion coefficient $D_{\rm off}(J)$ of Eq. (22), evaluated with these parameters, as a function of $x = \frac{\sqrt{2\beta J}}{\sigma}$ and for different values of the nominal tune $Q$. The diffusion rate for particles with amplitudes less than $1\sigma$ seems to be independent of the tune for the four tunes studied. For particles with amplitudes bigger than 1 $\sigma$ the diffusion rate depends strongly on the tune.

In Fig. 1-right, we compare this analytical expression with the diffusion rate obtained from tracking for the cases $Q = 0.28$ and $Q = 0.32$. We follow the dynamics of a set of 50 initial conditions with action $J_0$ and random angle $\psi$ (distributed with a random uniform distribution in $[0, 2\pi]$), subject to the one dimensional version of the maps (1) and (2) and the OU process of Eq. (4). We evaluate the diffusion coefficient as defined in Eq. (18) in the limit of $N = 10^7$ turns. The effects of resonances have not been included in the analytical expression. In the simulations, effects which break the symmetry of the beam-beam force, such as constant offsets between the beams, are not included. Hence only even order resonances driven by the beam-beam interaction can be observed in the simulation. None of the tunes chosen are close to an even low order resonance so we presume that the emittance growth we observe is not driven by resonances. For the case of $Q = 0.32$ the tune is close to the third order resonance and this in practice could lead to a stronger diffusion rate due to effects which drive this resonance.

Particle diffusion in amplitude causes emittance growth over the period of stored beam (typically 24 to 30 h). The emittance evolution can be followed by solving the Fokker-Planck equation. Assuming that the diffusion in action is a Markov process and the drift coefficient is half the derivative of the diffusion coefficient (as is usually the case for a Hamiltonian system [10]), the density distribution $\rho(J)$ evolves according to the Fokker-Planck equation

$$\frac{\partial\rho(J)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial J}\left(D(J)\frac{\partial\rho(J)}{\partial J}\right). \tag{23}$$

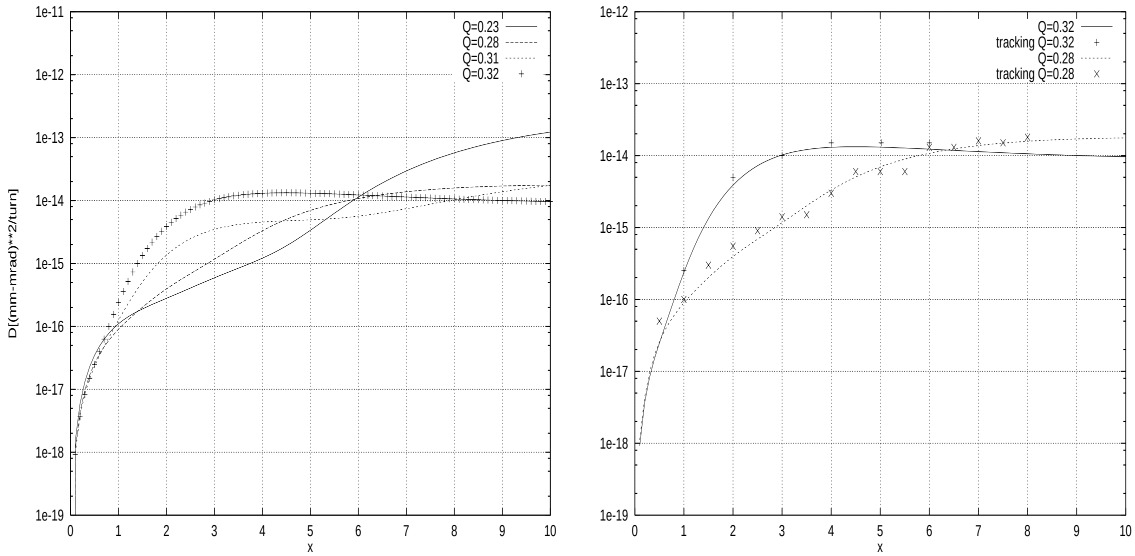


Figure 1: Diffusion coefficient due to beam-beam interaction and random offset. Horizontal axis particle amplitude in units of $\sigma$, vertical axis diffusion coefficient in units of [(mm-mrad)$^2$/turn]. Left: diffusion for different nominal tunes. Right: comparison with tracking. The lines are the diffusion coefficient as evaluated from the analytical expression (22). Crosses are the tracking results for $Q = 0.28$ and pluses the tracking results for $Q = 0.32$. Theory and simulation are in perfect agreement.

with $D(J) = D_{off}\sqrt{2}$ being the effective diffusion coefficient due to head-on collision with random offset at the two IPs.

We integrate this one-dimensional Fokker-Planck equation using a finite-difference implicit scheme, with absorbing boundary at the action corresponding to $J_{max} = 10\sigma$ and reflecting boundary at $J = 0$ [11]. The initial beam distribution in phase space is

$$\rho_0(x, v_x) = A \times \exp\left(-\frac{(x^2 + v_x^2)}{2}\right) \tag{24}$$

where $v_x = \beta^* x'$ and $x$ and $x'$ are in units of $\sigma$ and $A$ is a normalization constant. In terms of the action $J = (x^2 + v_x^2)\frac{\sigma^2}{2\beta^*}$ the initial density is

$$\rho_0(J) = \frac{1}{2J_0} \times \exp\left(-\frac{J}{2J_0}\right) \tag{25}$$

with $J_0 = \frac{\sigma^2}{2\beta^*}$ and $\int_0^\infty \rho_0(J)dJ = 1$.

The average action over the beam distribution is a measure of the beam emittance. It is given by the expression

$$\langle J \rangle = \int_0^{J_{max}} J\rho(J)dJ. \tag{26}$$

We integrate the evolution of the density over 30 h. In Fig. 2 we show the relative emittance growth for different tunes. The emittance doubling time for $\eta = 0.01$ and $\tau_c = 100$ is about 6 hours for $Q = 0.32$, 16hours for $Q = 0.31$, 40 hours for $Q = 0.28$ and 60 hours for $Q = 0.23$.

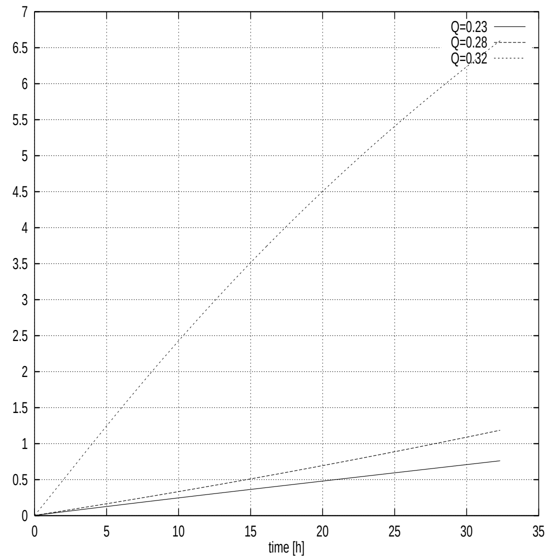


Figure 2: Relative growth of the average action $(< J > - < J_0 >)/J_0$ as a function of time for different betatron tunes. The emittance doubling time for these parameters is about 6 hours for $Q = 0.32$, 16 for $Q = 0.31$, 40 hours for $Q = 0.28$ and 60 hours for $Q = 0.23$.

Notice that the emittance growth is greater for $Q = 0.32$ and $Q = 0.31$ since the diffusion coefficient for particles in the core of the Gaussian distribution (region $1 - 3\sigma$) is stronger.

One can conclude that the spectral density of offset fluctuations at frequencies in the range of betatron frequencies must be below $10^{-16}\text{mm}^2/\text{Hz}$ in order to keep the emittance doubling time to more than 1 day.

We can extrapolate this result to the case of random offset with amplitude $|\eta| = 10^{-4}$ (diffusion coefficient $10^4$ smaller). For the LHC parameters, with nominal tune $Q = 0.32$ and using the weak-strong approximation for the head-on beam-beam interaction, we expect an emittance doubling time of about $6 \times 10^4$ hours.

A strong-strong beam-beam collision by itself is enough to excite size fluctuations which induce a much stronger diffusion. For instance for a particle with initial amplitude $x = 1$ (in units of $\sigma$) and random OU offset caused by ground motion with amplitude $|\eta| = 10^{-4}$ the weak-strong approximation predicts a diffusion coefficient of $D \approx 5 \times 10^{-21}\text{mm-mrad}^2/\text{turn}$ which is four orders of magnitude smaller than the diffusion due to the variations in the size induced by the strong-strong interaction of the two beams: $D \approx 5 \times 10^{-17}(\text{mm-mrad})^2/\text{turn}$ (evaluated for the SSC parameters $Q = 0.285$, $\xi = 0.0021$). These results were found in a self consistent way, solving the linearized Vlasov equation by the method of characteristics [12].

The transverse blow-up of the emittance due to dipole errors generated by the uncorrelated motion of quadrupoles along the circumference is also much faster. It has been estimated that in order to keep the growth rate associated with these errors below 40 hours the ground motion spectrum should be smaller than $10^{-18}\text{mm}^2/\text{Hz}$ [1].

Finally, the long range interactions have not been included in this calculation. For the LHC, the long range interactions dominate the head-on in determining the tune shifts at large amplitude and dynamic aperture. Orbit fluctuations induced by ground motion will probably have a stronger effect on the emittance growth when the long range collisions are included.

## 6 Two-dimensional case

This approach can be extended to a two dimensional model of the beam with $x = \frac{\sqrt{2\beta_x^* J_x}}{\sigma_x}\cos\phi_x$ and $y = \frac{\sqrt{2\beta_y^* J_y}}{\sigma_y}\cos\phi_y$. Let us express the beam-beam potential as a two-dimensional Fourier series

$$U(x,y) = C\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} U_{k,l}(J_x, J_y)\cos(2k\phi_x)\cos(2l\phi_y) \tag{27}$$

where the coefficients are

$$\begin{aligned}
U_{k,l}(J_x,J_y) &= 4\int_0^\infty \frac{dq}{(2\sigma_x^2+q)^{1/2}(2\sigma_y^2+q)^{1/2}}(-1)^{(k+l+1)}e^{-(w_x+w_y)}I_k(w_x)I_l(w_y) && (28)\\
U_{0,l}(J_x,J_y) &= 2\int_0^\infty \frac{dq}{(2\sigma_x^2+q)^{1/2}(2\sigma_y^2+q)^{1/2}}(-1)^{(l+1)}e^{-(w_x+w_y)}I_0(w_x)I_l(w_y) && (29)\\
U_{k,0}(J_x,J_y) &= 2\int_0^\infty \frac{dq}{(2\sigma_x^2+q)^{1/2}(2\sigma_y^2+q)^{1/2}}(-1)^{(k+1)}e^{-(w_x+w_y)}I_k(w_x)I_0(w_y) && (30)\\
U_{0,0}(J_x,J_y) &= \int_0^\infty \frac{dq}{(2\sigma_x^2+q)^{1/2}(2\sigma_y^2+q)^{1/2}}\left(1-e^{-(w_x+w_y)}\right)I_0(w_x)I_0(w_y) && (31)\\
w_x &= \frac{\beta_x^* J_x}{2\sigma_x^2+q}, \quad w_y = \frac{\beta_y^* J_y}{2\sigma_y^2+q}. && (32)
\end{aligned}$$

We assume that the offset fluctuations can be described by a stationary random process and that the fluctuations in the horizontal and vertical planes are independent. Thus the correlation functions are

$$\begin{gathered}
K_{xx}(n) \equiv \langle\eta_x(0)\eta_x(n)\rangle, K_{yy}(n) \equiv \langle\eta_y(0)\eta_y(n)\rangle,\\
K_{xy}(n) \equiv \langle\eta_x(0)\eta_y(n)\rangle = 0 = K_{yx}(n)
\end{gathered} \tag{33}$$

We define the two-dimensional diffusion coefficients

$$DJ_{x,\,\mathrm{off}}(J_x,J_y) = \lim_{N\to\infty}\frac{<(J_x(N)-J_x(0))^2>}{N}, \tag{34}$$

$$DJ_{y,\,\mathrm{off}}(J_x,J_y) = \lim_{N\to\infty}\frac{<(J_y(N)-J_y(0))^2>}{N} \tag{35}$$

A calculation similar to that in the one degree of freedom case shows that [13]

$$\begin{aligned}
DJ_{x,\,\mathrm{off}} &= \frac{1}{4}(C\sigma_x)^2\sum_{k=0}^{k=\infty}\sum_{l=0}^{\infty}(2k+1)^2F_{kl}^2\sum_n K_{xx}(n)\cos[2\pi(2k+1)Q_x n]\cos[4\pi l Q_y n]\\
&+ \frac{1}{4}(C\sigma_y)^2\sum_{k=0}^{k=\infty}\sum_{l=0}^{\infty}(2k)^2G_{kl}^2\sum_n K_{yy}(n)\cos[4\pi k Q_x n]\cos[2\pi(2l+1)Q_y n] && (36)\\
DJ_{y,\,\mathrm{off}} &= \frac{1}{4}(C\sigma_y)^2\sum_{k=0}^{k=\infty}\sum_{l=0}^{\infty}(2l+1)^2G_{kl}^2\sum_n K_{yy}(n)\cos[4\pi k Q_x n]\cos[2\pi(2l+1)Q_y n]\\
&+ \frac{1}{4}(C\sigma_x)^2\sum_{k=0}^{k=\infty}\sum_{l=0}^{\infty}(2l)^2F_{kl}^2\sum_n K_{xx}(n)\cos[2\pi(2k+1)Q_x n]\cos[4\pi l Q_y n] && (37)
\end{aligned}$$

When the noise is described by an Ornstein-Uhlenbeck process, the above expressions can be written as

$$
\begin{aligned}
DJ^{O-U}_{x,\,\mathrm{off}} &= \frac{1}{4}\frac{(C\sigma_x|\eta_x|)^2\sinh\theta_x}{[1-1/(2\tau_{c,x})]}\sum_{k=0}^{k=\infty}\sum_{l=0}^{\infty}(2k+1)^2F^2_{kl}(A^+_{kl}(\theta_x)+A^-_{kl}(\theta_x)) \\
&+ \frac{1}{4}\frac{(C\sigma_y|\eta_y|)^2\sinh\theta_y}{[1-1/(2\tau_{c,y})]}\sum_{k=0}^{k=\infty}\sum_{l=0}^{\infty}(2k)^2G^2_{kl}(B^+_{kl}(\theta_y)+B^-_{kl}(\theta_y)) \qquad (38)
\end{aligned}
$$

$$
\begin{aligned}
DJ^{O-U}_{y,\,\mathrm{off}} &= \frac{1}{4}\frac{(C\sigma_x|\eta_x|)^2\sinh\theta_x}{[1-1/(2\tau_{c,x})]}\sum_{k=0}^{k=\infty}\sum_{l=0}^{\infty}(2l)^2F^2_{kl}(A^+_{kl}(\theta_x)+A^-_{kl}(\theta_x)) \\
&+ \frac{1}{4}\frac{(C\sigma_y|\eta_y|)^2\sinh\theta_y}{[1-1/(2\tau_{c,y})]}\sum_{k=0}^{k=\infty}\sum_{l=0}^{\infty}(2l+1)^2G^2_{kl}(B^+_{kl}(\theta_y)+B^-_{kl}(\theta_y)) \qquad (39)
\end{aligned}
$$

$$
A^{\pm}_{kl}(\theta) = \frac{1}{\cosh\theta-\cos\left[2\pi((2k+1)Q_x\pm(2l)Q_y)\right]} \qquad (40)
$$

$$
B^{\pm}_{kl}(\theta) = \frac{1}{\cosh\theta-\cos\left[2\pi((2k)Q_x\pm(2l+1)Q_y)\right]} \qquad (41)
$$

The $F,G$ coefficients are expressed in terms of the Fourier harmonics $U_{k,l}$ as

$$
F_{kl} = \frac{1}{\sqrt{2\beta^*}}\left[\sqrt{J_x}(U_{k+1,l;J_x}+U_{k,l;J_x})+\frac{1}{\sqrt{J_x}}((k+1)U_{k+1,l}-kU_{k,l})\right] \qquad (42)
$$

$$
G_{kl} = \frac{1}{\sqrt{2\beta^*}}\left[\sqrt{J_y}(U_{k,l+1;J_y}+U_{k,l;J_y})+\frac{1}{\sqrt{J_y}}((l+1)U_{k,l+1}-lU_{k,l})\right] \qquad (43)
$$

$$
F_{0l} = \frac{1}{\sqrt{2\beta^*}}\left[\sqrt{J_x}(U_{1,l;J_x}+2U_{0,l;J_x})+\frac{1}{\sqrt{J_x}}U_{1,l}\right] \qquad (44)
$$

$$
G_{k0} = \frac{1}{\sqrt{2\beta^*}}\left[\sqrt{J_y}(U_{k,1;J_y}+2U_{k,0;J_y})+\frac{1}{\sqrt{J_y}}U_{k,1}\right]. \qquad (45)
$$

where $U_{k,l;J_x} \equiv \partial U_{k,l}/\partial J_x$ etc. As in the one degree of freedom case, the diffusion is enhanced near the odd harmonics of the betatron tunes.

The corresponding Fokker-Planck equation reads

$$
\frac{\partial\rho(J_x,J_y)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial J_x}\left(DJ_x(J_x,J_y)\frac{\partial\rho(J_x,J_y)}{\partial J_x}\right)+\frac{1}{2}\frac{\partial}{\partial J_y}\left(DJ_y(J_x,J_y)\frac{\partial\rho(J_x,J_y)}{\partial J_y}\right). \qquad (46)
$$

with $DJ_x(J_x,J_y)=\sqrt{2}DJ_{x,\,\mathrm{off}}$ and $DJ_y(J_x,J_y)=\sqrt{2}DJ_{x,\,\mathrm{off}}$ to consider collisions at two IPs.

In a pessimistic approximation we assume the horizontal ground motion spectra and response factor to be the same as in the vertical plane. In Fig. 3 we see the new diffusion coefficients evaluated from Eqs. (38,39) using in each plane the same parameters of the one dimensional model. The betatron tunes are set to $Q_x = 0.31$, $Q_y = 0.32$ which is one of the proposed working points for LHC. We observe that diffusion in $J_x$ has a weak dependence on the vertical amplitude, and similarly $DJ_y$ has a weak dependence on $J_x$.

Integrating the Fokker-Planck Eq. (46) with an initial Gaussian distribution, absorbing boundaries at the action corresponding to $10\sigma$ and reflecting boundaries at $J = 0$ and assuming the same parameters for both planes (this is a pessimistic approximation since the ground motion will have mainly an effect on the vertical plane) we evaluate the relative increment of the mean action in each plane as a function of time, see Fig. 4. The emittance doubling time is about 11 hours for the horizontal plane and 5 hours for the vertical plane. For realistic offset amplitudes of $|\eta_x| = |\eta_y| = 10^{-4}$ (in units of $\sigma$), we expect an emittance doubling time of $11 \times 10^4$ hours horizontally and $5 \times 10^4$ hours vertically.

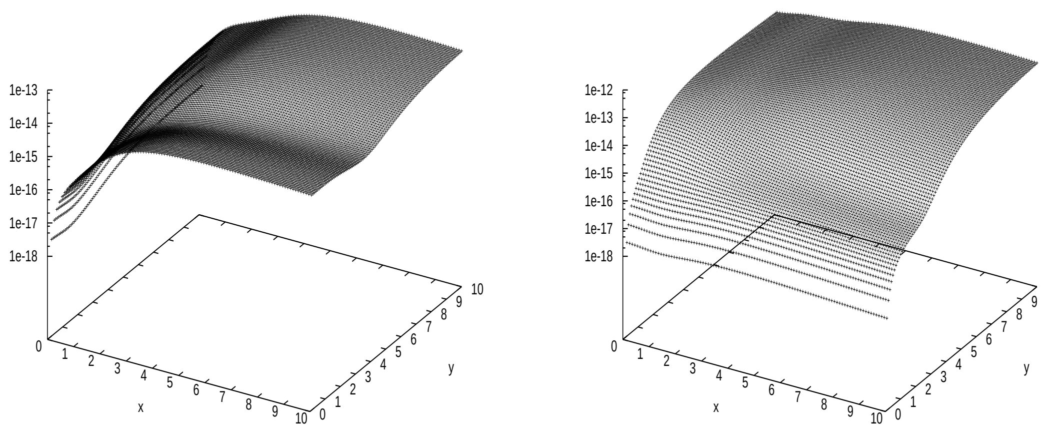


Figure 3: Diffusion coefficient in the two-dimensional case. Left: $DJ_{x,\,\mathrm{off}}(J_x, J_y)$, right: $DJ_{y,\,\mathrm{off}}(J_x, J_y)$. Vertical axis: diffusion rate in $[(\text{mm-mrad})^2/\text{turn}]$, $x$ and $y$ are, respectively, the horizontal and vertical particle amplitudes in units of the rms beam size.

# 7 Conclusions

We have estimated the influence of ground motion on the LHC beam applying the theory of particle diffusion induced by the beam-beam head-on collision with random offset at the interaction point. We have found that the analytical expression of the one dimensional diffusion coefficient is in perfect agreement with the results of tracking. These calculations have been extended to a two-dimensional model. In these calculations we have used an Ornstein-Uhlenbeck spectrum for the noise. However the theory developed can also be applied to a measured noise spectrum by direct use of the expressions (36) and (37) which require a knowledge of the correlation functions.

We have integrated the Fokker-Planck equation in a one- and two-dimensional case predicting for the LHC beam an emittance doubling time of about $5 \times 10^4$ hours for $Q = 0.32$. In order to keep the emittance doubling time larger than 1 day the spectral density of the offset fluctuations in the neighbourhood of the betatron frequency should be below $10^{-16}\text{mm}^2/\text{Hz}$ which is three orders of magnitude below the expected density. We conclude that, under the weak-strong approximation and considering only head-on collisions, the ground motion alone has a negligible influence on the emittance of the beam. Several factors not included in this calculation may increase the emittance growth rate beyond the above estimates. As already mentioned, the several long-range interactions and ground motion may increase the emittance growth. In addition, machine nonlinearities and other effects which drive the nearby third order resonances have not been included. These can also be expected to have an impact on the observed emittance growth.

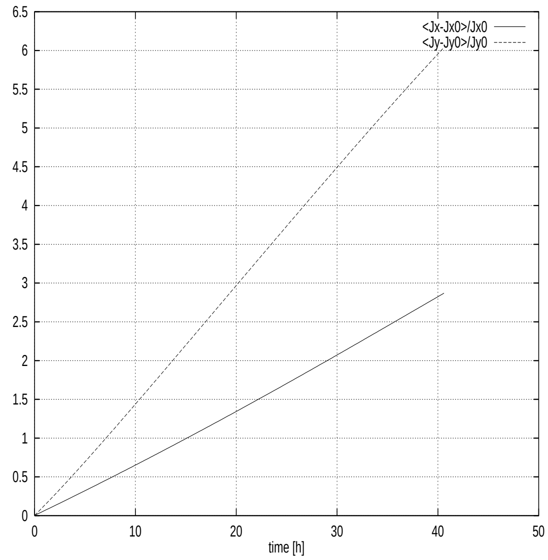


Figure 4: Relative increment of the average action in each plane ($< J_x - J_{x,0} > /J_{x,0}$, $< J_x - J_{x,0} > /J_{x,0}$) as a function of time. Evaluated using a two-dimensional model with $Q_x = 0.31$ and $Q_y = 0.32$

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# Appendix A: Discrete time Ornstein-Uhlenbeck process

The difference equation describing the discrete O-U process at turn $n$ is

$$\Gamma_{n+1} = (1 - \frac{1}{\tau_c})\Gamma_n + \sqrt{\frac{2}{\tau_c}}\xi_{n+1} \tag{A.1}$$

where $\Gamma$ is the random variable obeying the O-U statistics, $\xi$ is a Gaussian distributed random variable with zero mean and unit variance and $\tau_c$ is a (dimensionless) correlation time measured in numbers of turns. Writing

$$\alpha = 1 - \frac{1}{\tau_c} \leq 1 \ , \qquad z_n = \sqrt{\frac{2}{\tau_c}}\xi_n \ , \quad \langle z_n z_m \rangle = \frac{2}{\tau_c}\delta_{nm} \tag{A.2}$$

we have

$$\Gamma_n = \alpha\Gamma_{n-1} + z_n \tag{A.3}$$

By definition $\Gamma_0 = z_0$. Iterating backwards in time

$$\begin{aligned} \Gamma_n &= \alpha^2\Gamma_{n-2} + \alpha z_{n-1} + z_n \\ &= \alpha^n\Gamma_0 + \alpha^{n-1}z_1 + \alpha^{n-2}z_2 + \ldots + z_n = \sum_{j=0}^{n}\alpha^j z_{n-j} \end{aligned} \tag{A.4}$$

Using the relations in Equation (A.2), the correlation function is

$$\begin{aligned} \langle \Gamma_n \Gamma_{n+m} \rangle &= \sum_{j=0}^{n}\sum_{k=0}^{n+m}\alpha^{j+k}\langle z_{n-j} z_{n+m-k} \rangle \\ &= \frac{2}{\tau_c}\alpha^m \frac{1-\alpha^{2(n+1)}}{1-\alpha^2} \end{aligned} \tag{A.5}$$

Particular cases of this are

$$\langle \Gamma_0 \Gamma_m \rangle = \frac{2}{\tau_c}\alpha^m, \quad \langle \Gamma_0^2 \rangle = \frac{2}{\tau_c} = \langle z_0^2 \rangle \tag{A.6}$$

The stationary limit of the correlation function is obtained by taking the limit $n \to \infty$. We find

$$K(m) \equiv \lim_{n\to\infty}\langle \Gamma_n \Gamma_{n+m} \rangle = \frac{2}{\tau_c}\frac{\alpha^m}{1-\alpha^2} = \frac{(1-\frac{1}{\tau_c})^m}{1-\frac{1}{2\tau_c}} = \frac{\exp[m\ln(1-\frac{1}{\tau_c})]}{1-\frac{1}{2\tau_c}} \tag{A.7}$$

where $m > 0$.

The spectral density is the Fourier transform of the correlation function.

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}K(t)dt \tag{A.8}$$

When the process is sampled at discrete intervals of some sampling time $T_{sample}$, we have $t = nT_{sample}$ and the spectral density is written as the discrete time Fourier transform,

$$S(\omega) = \frac{T_{sample}}{2\pi}\sum_{n=-\infty}^{\infty}K(nT_{sample})\exp\{-i[n\omega T_{sample}]\} \tag{A.9}$$

When $K(m)$ is an even function, as in our case, this can be replaced by

$$S(\omega) = \frac{T_{sample}}{2\pi}\left[1 + 2\sum_{n=1}^{\infty}K(n)\cos(n\omega T_{sample})\right] \tag{A.10}$$

Let $r$ represent the ratio $r = \frac{T_{sample}}{T_{rev}}$. Substituting the first expression for the correlation function from Eq. (A.7),

$$\begin{aligned} S(2\pi f_{rev}Q) &= \frac{T_{sample}}{\pi\tau_c(1-\alpha^2)}\left[1 + 2\sum_{n=1}^{\infty}\alpha^n\cos(2\pi rQn)\right] \\ &= \frac{T_{sample}}{\pi\tau_c}\frac{1}{[1 - 2\alpha\cos(2\pi rQ) + \alpha^2]} \end{aligned} \tag{A.11}$$

Another expression is obtained by substituting the last expression for the correlation function from Eq. (A.7),

$$\theta = -\ln(1 - 1/\tau_c), \quad K(m) = \frac{\exp(-m\theta)}{(1 - 1/2\tau_c)} \tag{A.12}$$

$$S(2\pi f_{rev}Q) = \frac{T_{sample}}{2\pi(1 - 1/2\tau_c)}\frac{\sinh\theta}{\cosh\theta - \cos(2\pi rQn)} \tag{A.13}$$

The correlation time $\tau_c$ is measured in units of the sampling time $T_{sample}$. If the sampling time is the revolution period $T_{rev}$, then $r = 1$ in the above expressions.

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