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Analytical Study of the Incoherent Beam-Beam Resonances in the Tevatron Run II Lattice with the Beam-Beam Compensation

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1 Introduction

The beam-beam interaction is known to be one of the major limitations on the luminosity and particle lifetime which can be achieved in a collider. In the Tevatron Run II project [1] which is now under realization the beam-beam interaction will be significantly enhanced due to somewhat larger number of protons per bunch as well as larger number of bunches (see Table 1).

Table 1. Tevatron Run II parameters

Energy	E_0, GeV	1000
Bunches/beam	N_b	36
Protons/bunch	$N_p/10^{11}$	2.7
Antiprotons/bunch	$N_{\bar{p}}/10^{11}$	0.75
p emittance, r.m.s.	ε_p, (π) nm·rad	3.10
$\bar{p}$ emittance, r.m.s.	$\varepsilon_{\bar{p}}$, (π) nm·rad	2.35
Beta function at IP	β^*, m	0.35
Number of IPs	N_{IP}	2
Total $\bar{p}$ tuneshift	$\Delta Q_{\bar{p}}$	≤0.025
Bunch length	σ_s, m	0.37

A larger number of parasitic encounters - 70 as compared to 10 in the Run I configuration – can lead to a stronger excitation of the odd-order beam-beam resonances and increase the bunch-to-bunch tunespread due to the so called PACMAN effect posing a question of whether the simultaneous stability of all of the bunches can be ensured.

It has been proposed to reduce both in-bunch and bunch-to-bunch tunespreads in the weaker (antiproton) beam with the help of an electron beam [2]. However the electron beam itself can contribute to the resonance excitation so that a careful choice of the parameters of the compensation setup is needed.

The beam-beam effect has been under extensive study for about three decades, its nature is quite well understood, there is a number of tracking programs permitting to simulate it with a sufficient reliability. However, efficient analytical tools are highly desirable which could provide an insight into the tracking results and help in formulating the strategy for the search of the optimal solutions (certainly, with subsequent check of the found solutions by tracking).

In the present note an attempt is made to study analytically the incoherent beam-beam effect in the Tevatron 36 × 36 bunches operation and its compensation with the help of an electron beam. Some novel analytical formulae were used which had been implemented as the ***Mathematica*** notebook attached to the note as an Appendix. The notebook is publicly available on Fermilab Windows NT server beamssrv1 as file \\beamssrv1\bbcomp.bd\public\beambeam\inc_bb.nb or on the AFS as /afs/cern.ch/user/a/alexahin/public/mathem/inc_bb.nb

2 Review of the basic ideas

The beam-beam effect on the incoherent particle motion consists in: a) amplitude dependent tuneshift; b) excitation of the resonances.

In the vicinity of the resonance[1]

$$\Delta_{\underline{m}} = \underline{m}\cdot Q - n \equiv m_x Q_x + m_y Q_y - n = 0 \tag{1}$$

going into the rotating frame we can write the main terms in the Hamiltonian in the form

$$H \approx \int^{I_1} \Delta_{\underline{m}}(I_1', I_2)\, dI_1' + C_{\underline{m}}(I_1, I_2) \cos\psi_1 \tag{2}$$

[1] in the present note only the resonances of the transverse oscillations will be considered.

where the new action-angle variables were introduced (see e.g. Ref.[3])

$$I_1 = \frac{m_x I_x + m_y I_y}{m_x^2 + m_y^2}, \qquad I_2 = \frac{m_y I_x - m_x I_y}{m_x^2 + m_y^2} = \text{const},$$
$$\psi_1 = m_x\psi_x + m_y\psi_y - n\theta, \quad \psi_2 = m_y\psi_x - m_x\psi_y, \tag{3}$$

$I_{x,y}$ and $\psi_{x,y}$ being the original action-angle variables, θ being the generalized azimuth conventionally used as the independent variable.

With the help of the ***Mathematica*** notebook presented in the Appendix one can compute $\Delta_{\underline{m}}$ and $C_{\underline{m}}$ as functions of the action variables and find numerically the trajectories of the Hamiltonian (2). The major interest presents the separatrix width around the stable fixed points which gives the maximum swing of the betatron amplitudes on the resonance.

Simplified analytical formulas are useful which can be obtained assuming $C_{\underline{m}}$ to be constant across the separatrix and retaining just the linear term in the expansion

$$\Delta_{\underline{m}} \approx \Delta'_{\underline{m}} \cdot (I_1 - I_{\text{res}}) = (m_x^2 \frac{\partial Q_x}{\partial I_x} + 2 m_x m_y \frac{\partial Q_x}{\partial I_y} + m_y^2 \frac{\partial Q_y}{\partial I_y}) \cdot (I_1 - I_{\text{res}})\,. \tag{4}$$

Then the separatrix half-width in the phase space is

$$\delta I_1 \approx 2\left|C_{\underline{m}} / \Delta'_{\underline{m}}\right|^{1/2}, \tag{5}$$

or, in the tunes plane,

$$\delta\Delta_{\underline{m}} \approx 2\left|C_{\underline{m}} \cdot \Delta'_{\underline{m}}\right|^{1/2}. \tag{6}$$

There is a closely related value, *the island tune*, which is the tune of small amplitude libration w.r.t. the stable fixed point in the rotating frame:

$$\nu_{\text{isl}} \approx \left|C_{\underline{m}} \cdot \Delta'_{\underline{m}}\right|^{1/2}. \tag{7}$$

Let us remind the possible effects of the resonances on particle dynamics.

♦ An isolated resonance manifests itself as beatings in the betatron amplitudes which may dilute the beam core lowering the luminosity and, in the case of a large swing (5), lead to particle loss.

♦ A group of resonances can create dynamical chaos leading to particle diffusion to large amplitudes if the (refined) Chirikov overlap criterion is satisfied (see e.g. Ref.[4]), i.e. *if the distance between the resonances is less than 3/2 of the sum of the resonance islands half-widths.*
In a real system subject to external noise the safety factor should be even larger than 3/2 to avoid the global stochasticity.

♦ An adiabatic variation in the betatron tunes, $\delta Q_{x,y}$ (which may be called forth by the orbit deviations inside the sextupoles, current ripple in the quadrupoles etc.) makes the resonance islands move in- and outwards in the phase space transporting the trapped particles to larger amplitudes. This process, which we will call “sweeping”, may take place if the adiabaticity condition is satisfied

$$\nu_{\text{isl}} >> \nu_{\text{mod}}\,, \tag{8}$$

where ν_{mod} is the characteristic frequency (in units of the revolution frequency) of the variation. In such a case the amplitude of the tune variation should be added to the resonance island half-width in the tunes plane

$$\delta\Delta_{\underline{m}}^{\mathrm{eff}} \approx \delta\Delta_{\underline{m}} + \underline{m}\cdot\underline{\delta Q} \tag{9}$$

The experience with SPS shows that "sweeping" can be quite important [5]. Allegedly the tune variation as large as $\delta Q_{x,y}$ ~10^{-3} was seen in collision.

♦ The particular case of a harmonic tune modulation can be treated more rigorously in the terms of the Bessel satellites. The Hamiltonian in this case can be presented in the form (see e.g. Ref.[3])

$$H \approx \int^{I_1} \Delta_{\underline{m}}(I_1', I_2)\, dI_1' + C_{\underline{m}}(I_1, I_2) \sum_{m_s=-\infty}^{\infty} J_{m_s}(\underline{m}\cdot\underline{\delta Q}/\nu_{\mathrm{mod}})\, \cos(\psi_1 + m_s \nu_{\mathrm{mod}}\theta)\,. \tag{10}$$

The Bessel function in the r.h.s. of eq.(10) considered as a function of the order m_s reaches its maximum at

$$m_{s\,\max} \approx \underline{m}\cdot\underline{\delta Q}/\nu_{\mathrm{mod}} \tag{11}$$

The resonance is effectively widened by $m_{s\,\max}\nu_{\mathrm{mod}} \approx \underline{m}\cdot\underline{\delta Q}$, just as in the case of the arbitrary tune variation with time. This independence on ν_{mod} (below a certain value) was earlier observed in tracking and in a dedicated experiment at SPS [6].

Furthermore, now we can elaborate the coarse condition (8) of particles transport by requiring the satellites overlap, which for $m_{s\,\max}$ >>1 gives

$$\delta\Delta_{\underline{m}}\ (= 2\,\nu_{\mathrm{isl}}) \geq m_{s\,\max}^{1/4}\,\nu_{\mathrm{mod}} \tag{12}$$

Concluding the account of the basic physical mechanisms it is appropriate to point out one more beneficial effect of the beam-beam compensation. Namely, compensation of the nonlinear tunespread reduces the island tune (7) limiting to very low frequencies the spectral range of the betatron tunes variation which can lead to particles transport by the resonance islands. Certainly, for this being true the compensating electron beam should not enhance the particular resonances. A high stability of the electron beam current is also necessary.

3 Beam-beam effect in the absence of compensation

The beam-beam tuneshifts and resonance driving terms were computed within the framework of the first order perturbation theory by summing up the corresponding values over all 72 interaction points (2 nominal + 70 parasitic). Thus, we ignore the second (and higher) order terms in the beam-beam parameter which can be neglected except for the tunes in a close vicinity of half-integer values.

The optical functions and orbit separation at the IPs corresponding to the collision lattice *pbh15a.acol.nnpp2.bun3* were provided by P.Bagley [7]. Also, we assume the bare lattice tunes, i.e. the tunes in the absence of the beam-beam interaction, to be the same as in the Run Ib.

Due to the orbit separation inside the sextupoles there is a tunesplit between the two beams amounting to $\Delta Q_{x0} = 0.0008,\ \Delta Q_{y0} = 0.0026$. Assuming the highest peaks observed in the Schottky spectra of
.595 Tevatron beams in collision [8] being the proton bare lattice tunes we obtain then for the antiproton bare lattice tunes
.590
.585
.580
.575

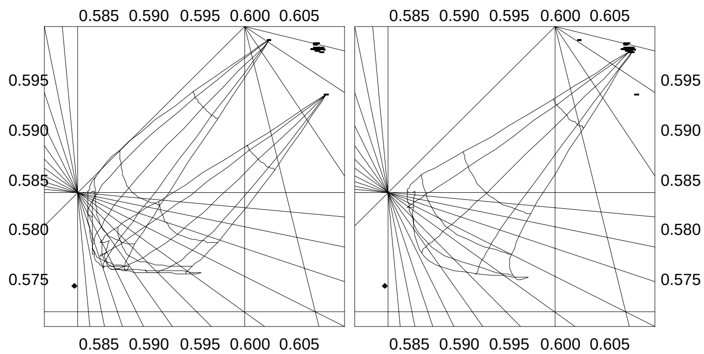

Figure 1. Beam-beam footprints in the Q_x, Q_y plane for antiproton bunches #1 and #12 (left) and #6 (right). See the text for details.

$$Q_{x0}^{\bar{p}} = 20.583,\ Q_{y0}^{\bar{p}} = 20.574 \tag{13}$$

which we will further refer to as the "nominal" working point (WP). Position of the calculated with this WP beam-beam footprints w.r.t. the 5th, 7th and 12th order sum resonance lines[2] is shown in Fig.1. The arc lines in the footprints correspond to equidistant with step 2 values of the transverse amplitude

$$a = \left(a_x^2 + a_y^2\right)^{1/2} = \left(2\frac{I_x + I_y}{\varepsilon_{\bar{p}}}\right)^{1/2},\quad a_{x,y} = \left(2\frac{I_{x,y}}{\varepsilon_{\bar{p}}}\right)^{1/2}. \tag{14}$$

The radial lines are equidistant in arctan(a_x/a_y) value. Dots in the upper right corners show small-amplitude tunes of all the antiproton bunches.

As can be seen the 5th order resonances lie well within the antiproton tunespread. These resonances posed no severe problem during Run I since they: i) can not be excited by head-on collisions (as any odd-order resonance) and ii) were reached (if at all) at too small amplitudes owing to smaller tuneshifts. However in the Run II configuration they can become quite strong due to numerous parasitic long range interactions.

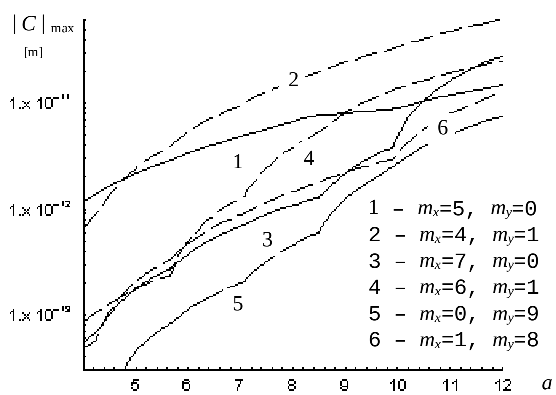

Figure 2. Maximum values of some odd-order resonance driving terms over the range 0<tan(a_x/a_y)<π/2 vs. amplitude a for bunch #6.

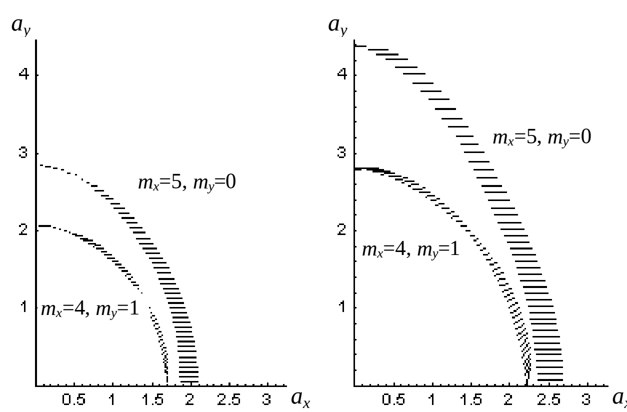

Figure 3. Beatings of the betatron amplitudes on the 5th order resonances at the nominal working point (left) and with the tunes shifted to Q_{x0} = 20.586, Q_{x0} = 20.576 (right)

The maximum values of the strength of some odd-order resonances taken over the range 0<arctan(a_x/a_y)<π/2 for bunch #6 are plotted against the amplitude a in Fig.2.

[2] analysis shows that the difference resonances are less important at the considered WPs.

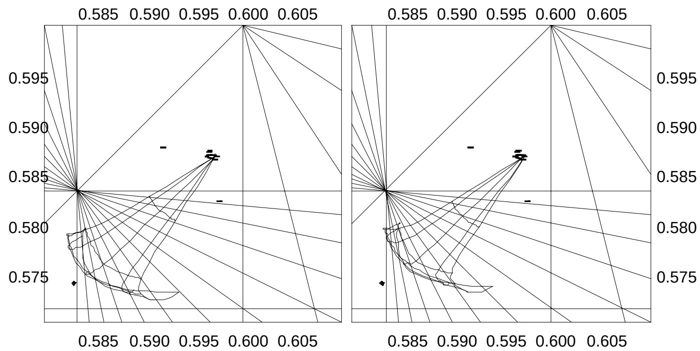

Figure 4. Beam-beam footprints in the Q_x, Q_y plane for antiproton bunche #6 with partial compensation by electron beams of two different profiles (see the text).

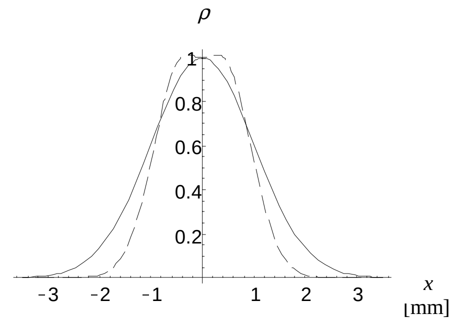

Figure 5. Electron beams profiles.

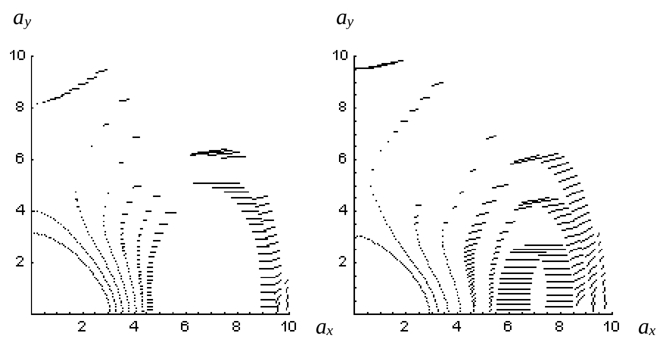

Figure 6. Beatings of the betatron amplitudes on the 12th order resonances for the two electron beams profiles.

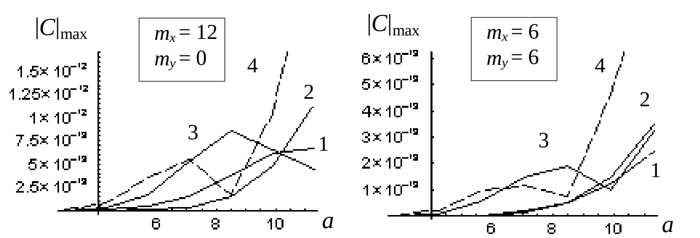

Figure 7. Maximum values of contributions to the 12th order resonance driving terms due to: 1- long-range, 2 - Gaussian e-beam with σ=1mm, 3,4 - bi-Gaussian e-beam with σ=1mm (solid) and 0.8mm (dashed).

To assess the influence of a resonance we compute the amplitude width of the separatrix taking into account the dependence of the resonance driving terms and the tunes on the amplitude across the resonance islands. This value shows the maximum beatings of the betatron amplitude in the single-resonance approximation.

According to Fig.3 the 5th order resonances appear to be strong enough to manifest themselves even at small amplitudes $a \sim 2$. The island tune for $\underline{m} = (5,0)$ resonance is as large as $\nu_{isl} \sim 10^{-3}$ ($\nu_{isl} f_0 \sim 50$Hz). One may expect the 5th order resonances to be real "killers" at higher proton intensities due to increase in the resonance strength.

Compensation of the beam-beam tuneshift with the help of an electron beam is therefore a prospective option.

4 Beam-beam compensation

Fig.4 demonstrates the effect of a partial beam-beam compensation ($\Delta Q_{x,y} = -0.01$) by round electron beams with two different shapes shown in Fig.5: a Gaussian beam of σ=1mm (left) and a flat-top bi-Gaussian beam of σ=0.8mm (right). The charge density of the latter is described by the formulas

$$\rho(r)/\rho_0 = (1+\kappa)\exp(-\frac{r^2}{2\sigma_1^2}) - \kappa\exp(-\frac{r^2}{2\sigma_2^2}),$$
$$\sigma_1 = \sigma/(1+\kappa-\kappa\zeta^2)^{1/2}, \quad \sigma_2 = \zeta\sigma_1, \qquad (15)$$

with particular values κ =1.2, ζ =0.7. The dependence of $\sigma_{1,2}$ in eqs.(15) is chosen so as to obtain the same total current as in a Gaussian beam of the given σ. The electron beam was placed between the nominal IPs at a location where the optical functions were β_x = 70.77m and β_y = 72.52m (the corresponding $\sigma_{\bar{p}} \approx 0.4$ mm).

As seen in Fig.4 with the beam-beam compensation the footprints fold at smaller amplitude ($a \sim 5$) due to relative increase of the long-range contribution. Since the width of resonances encountered at the folding amplitude may become too large it is dangerous to further increase the degree of compensation.

Without beam-beam compensation the 12th order resonances do not present a serious danger: the long-range contribution is not large whereas the contribution from the nominal IPs is almost completely integrated out due to a large bunch length $\sigma_s \sim \beta^*$ [9].

With compensation the tune dependence on the betatron amplitudes is reduced and one may expect an increase in the resonance widths, the more so that the e-beam itself contributes to the excitation of even order resonances.

Fig.6 presents the resonance widths under the joint action of the long-range interactions and e-beam with profiles shown in Fig.5. It can be seen that the e-beam profile can have a drastic effect on the p-bar stability. At large horizontal amplitudes, $a_x \geq 5$, a narrow flat-top e-beam definitely deteriorates the particle stability. At intermediate amplitudes the situation is less clear since the increase in the resonance strength is counteracted by larger detuning with amplitude in the case of a narrow e-beam (carrying smaller current).

To clarify the situation the contribution of different sources to the 12th order resonance driving terms is shown separately in Fig.7. It is interesting to note a characteristic "dip" in the cases of bi-Gaussian (flat-top) profiles. The island tune of $\underline{m}$ = (12, 0) resonance in the case of a Gaussian beam is rather low: $\nu_{isl} \sim 5\cdot10^{-5}$.

5 Shifted working point

The probable complications with the 12th order resonances make worthwhile to look for another working point, $Q_{x0}^{\bar{p}} = 20.566$, $Q_{y0}^{\bar{p}} = 20.556$ being one of the possibilities [10].

The beam-beam footprint with partial compensation by a Gaussian beam of σ=1mm and the amplitude beatings on the 7th and 9th order resonances are shown in Figs. 8 (left) and 9 (left). One can see that there is a potential problem associated with the 9th order resonances which can get into the folding part of the footprint where the stabilizing dependence of tunes on amplitudes is weak. One can avoid them by shifting the vertical tune up by an amount of ~0.003. The 7th order resonances appear to be rather weak since they are encountered at small amplitudes; however the island tunes are rather high, e.g. on $\underline{m}$ = (7, 0) resonance $\nu_{isl} \sim 2\cdot10^{-4}$, so that a good betatron tune stability is necessary to avoid "sweeping".

The 7th order resonances can become stronger in the case of a large offset of the e-beam with respect to the antiproton closed orbit. Such a situation may take place on the initial stage of operation of the beam-beam compensation setup.

Figs. 8 (right) and 9 (right) show the footprint and the width of the 7th and 9th order resonances for an offset as large as 1σ=1mm in the case of a Gaussian e-beam. The width of the 7th order resonances is now comparable with the distance between the resonances which may lead to a (weakly) chaotic behavior, especially in the presence of the external noise. However this phenomenon is beyond the scope of the present simplified analytical considerations.

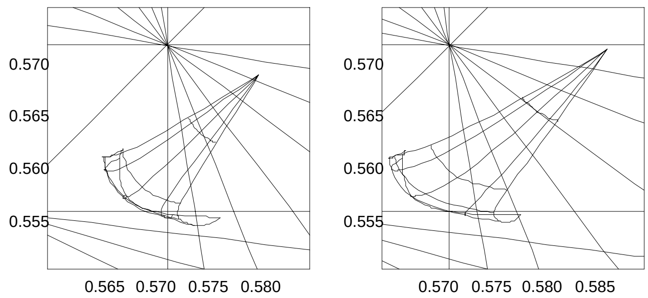

Figure 8. Beam-beam footprints with compensation by a Gaussian e-beam of σ=1mm with no offset (left) and with an offset of 1mm (right).

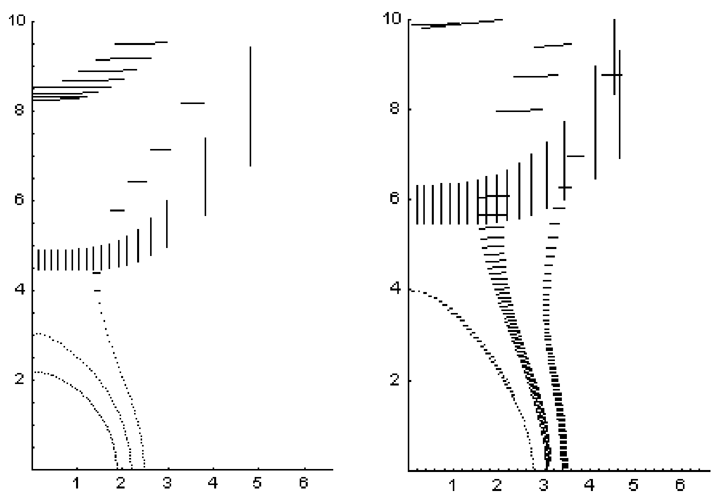

Figure 9. Beatings of the betatron amplitudes in the cases of no e-beam offset (left) and of a 1mm offset (right)

6 Summary and outlook

Although the present analysis is a preliminary one, some conclusions can already be made:

♦ in the Run II configuration the odd-order resonances, especially those of the 5th order which are close to the standard WP, become quite strong due to numerous parasitic long range interactions;

♦ the beam-beam compensation permits to avoid the strongest resonances by reducing both in-bunch and bunch-to-bunch tunespreads in the antiproton beam;

♦ the in-bunch tunespread reduction should not be too large (50% seems quite reasonable) in order to avoid the footprint "folding" at small amplitudes (a < 5) due to relative increase of the long-range contribution;

♦ the compensating electron beam can heavily contribute to excitation of even-order resonances (and odd-order ones in the case of a large offset); the 12th order resonances being a potential problem at the standard WP;

♦ a wide Gaussian electron beam is preferable as compared to a narrow flat-top beam from the point of view of reducing the resonance excitation;

♦ with the beam-beam compensation on it is beneficial to shift the working point away from the 12th order resonances (e.g. to $Q_{x0}^{\bar{p}} = 20.566$, $Q_{y0}^{\bar{p}} = 20.556$);

♦ the beam-beam compensation reduces the frequency range in which the sweeping of particles by moving resonance islands may occur, however the e-beam current should be sufficiently stable ($\Delta I_e/I_e<10^{-3}$ at frequencies below 100Hz seems to be a reasonable limitation) in order not to produce the betatron tune modulation itself.

The study along the lines of the present note can be continued in a systematic search for the optimum working point, parameters of the compensating electron beam (profile, current) and extended to the TEV33 configuration (140×121 bunches, crossing angle etc.). It is desirable also to make a careful comparison of the analytics with the predictions of the ongoing tracking simulations [10] and with the results of a dedicated experiment on the beam-beam compensation.

On a wider scope such problems should be addressed as well as the effect of the incoherent tunespread reduction on the coherent beam-beam modes, tolerances for high-frequency fluctuation of the e-beam parameters from the point of view of heating the antiproton beam.

In conclusion the author would like to express his gratitude for helpful discussions to P.Bagley, T.Sen and especially to V.Shiltsev, who also carefully read the manuscript and made many suggestions for improving it.

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Appendix. Incoherent Beam-Beam Tuneshifts & Resonance Coefficients

(***Mathematica*** notebook)

■ Introduction

The present notebook can be used for computing tuneshifts and resonance coefficients as functions of action variables of all three degrees of freedom taking into account finite offsets and crossing angle in one plane but assuming bunch length sufficiently small to ignore the "hour-glass" effect. The aspect ratio is arbitrary, however in the extreme cases it is better to use simplified formulas.

■ Formulae

Assuming horizontal crossing we start with the beam-beam potential in the form

$$H_{\mathrm{bb}} = \frac{2 r_p N}{\gamma} \delta_p(\theta - \theta_{\mathrm{IP}}) \int_0^1 \frac{1}{t\sqrt{1+(r^2-1)t^2}} \exp\left\{-\frac{(d_x + x_\beta + \alpha z)^2}{2\sigma_{\mathrm{eff}}^2} t^2 - \frac{(d_y + y_\beta)^2}{2\sigma_y^2[1+(r^2-1)t^2]} t^2\right\} dt$$

where

r_p − classical radius ($r_p > 0$ for $p-p$ beams),

$\delta_p(\theta)$ − periodic δ − function (hereafter $\theta_{\mathrm{IP}} = 0$ for simplicity),

$\sigma_{\mathrm{eff}}^2 = \sigma_x^2 + \alpha^2 \sigma_s^2$,

$r = \sigma_y / \sigma_{\mathrm{eff}}$,

$d_{x,y}$ − full separation between the beams in the x − and y − directions

To compute the tuneshifts and resonance coefficients we introduce the action-angle variables via the relations (normalizing the action variables to the corresponding emittances):

$$x_\beta = \sigma_x \sqrt{2 I_x} \cos\psi_x,\ y_\beta = \sigma_y \sqrt{2 I_y} \cos\psi_y,\ z = \sigma_s \sqrt{2 I_s} \cos\psi_s,$$

and denote

$$\kappa = \alpha\sigma_s/\sigma_x,\ \sigma_{\mathrm{eff}} = \sigma_x\sqrt{1+\kappa^2},\ r = \sigma_y/\sigma_{\mathrm{eff}},\ \delta_x = d_x/\sqrt{2}\,\sigma_{\mathrm{eff}},\ \delta_y = d_y/\sqrt{2}\,\sigma_y,$$

We make use of the general relation (see e.g. Gradshteyn & Ryzhik)

$$\int_0^{2\pi} \cos(m\psi) F(\cos\psi)\, d\psi = \frac{1}{(2m-1)!!} \int_0^{2\pi} \sin^{2m}\psi\, F^{(m)}(\cos\psi)\, d\psi$$

and Rodrigues' formula for the Hermite polynomials

$$\frac{\partial^m}{\partial x^m} e^{-x^2} = (-1)^m e^{-x^2} H_m(x)$$

to perform expansion in the Fourier series in the phase angles

$$H_{\mathrm{bb}} = \frac{2 r_p N}{\gamma\pi^3} \delta_p(\theta) \sum_{m_{x,y,s}=0}^{\infty} \frac{(-1)^{m_x+m_y+m_s} \epsilon_{m_x,0} \epsilon_{m_y,0} \epsilon_{m_s,0}}{(2m_x-1)!!\,(2m_y-1)!!\,(2m_s-1)!!} \frac{r^{m_y} \kappa^{m_s}}{(1+\kappa^2)^{(m_x+m_s)/2}} \times$$

$$\cos(m_x\psi_x)\cos(m_y\psi_y)\cos(m_s\psi_s)\ I_x^{m_x/2} I_y^{m_y/2} I_s^{m_s/2} \int_0^1 \frac{t^{m_x+m_y+m_s-1}\, dt}{[1+(r^2-1)t^2]^{(m_y+1)/2}} \times$$

$$X\left[0, m_x, m_s, \delta_x t, \sqrt{\frac{I_x}{1+\kappa^2}}\, t, \sqrt{\frac{\kappa^2 I_s}{1+\kappa^2}}\, t\right] Y\left[0, m_y, \frac{\delta_y r t}{\sqrt{1+(r^2-1)t^2}}, \frac{\sqrt{I_y}\, rt}{\sqrt{1+(r^2-1)t^2}}\right],$$

where $\varepsilon_{m,0} = 2 - \delta_{m,0}$ and

$$X[l, m_x, m_s, d, x, z] = \int_0^\pi \int_0^\pi \sin^{2 m_x}\psi_x \sin^{2 m_s}\psi_s e^{-(d+x\cos\psi_x+z\cos\psi_s)^2} H_{l+m_x+m_s}(d + x\cos\psi_x + z\cos\psi_s)\, d\psi_x\, d\psi_s,$$

$$Y[l, m, d, y] = \int_0^\pi \sin^{2m}\psi\, e^{-(d+y\cos\psi)^2} H_{l+m}(d + y\cos\psi)\, d\psi,$$

One can easily include a finite dispersion (in the same plane with the crossing angle) in the above formulas by defining

$$\kappa = \sqrt{(\alpha\sigma_s)^2 + (D_x\,\sigma_E)^2} \Big/ \sigma_x$$

and properly taking into account the shift in the synchrotron phase.

■ Implementation

In the above integrals a sufficient precision (4-5 digits) at large m (up to 30) and y (up to 10) and for d up to 10 is achieved with the maximum number of sample points

```
npt[m_Integer, y_] :=
2 Ceiling[(86 + 10 m + (63.5 - .35 m) Abs[y] + (3.5 + .25 m) y^2) / 36] + 1;
```

At small m and y this number is somewhat excessive though computation is rather fast anyway.
Mathematica can complain when performing integration but the warnings may be discarded:

```
Off[General::spell]; Off[General::spell1];
Off[General::dupsym]; Off[NIntegrate::mccnv]; Off[NIntegrate::ncvb]; Off[NIntegrate::"slwcon"];
SetOptions[NIntegrate, Compiled → False];
```

The functions entering the expression for H_{bb} are computed as follows

```
nX[l_Integer, m_Integer, n_Integer, d_, x_, z_]:=
NIntegrate[Sin[q]^(2 m) nY[l + m, n, d + x *Cos[q], z], {q, 0, Pi},
PrecisionGoal → 3, MaxPoints → npt[m, x]];

nY[l_Integer, n_Integer, d_, y_]:=
NIntegrate[Sin[p]^(2 n) HermiteH[l + n, d + y *Cos[p]]/Exp[(d + y *Cos[p])^2], {p, 0, Pi},
PrecisionGoal → 3, MaxPoints → npt[l + n, y]];
```

● Tuneshifts

Beam-beam tuneshifts are defined by the formula

$$\Delta Q_\mu = \frac{\partial}{\epsilon_\mu \partial I_\mu} < H_{bb} >, \quad \mu = x, y, s$$

where brackets denote averaging over all angles. Longitudinal tuneshift is (usually) negligible due to large value of the corresponding emittance. For the transverse tuneshifts normalized to the beam-beam parameters

$$\xi_x = \frac{r_p N}{2\pi\gamma\epsilon_x (1+\kappa^2)(1+r)}, \quad \xi_y = \frac{r_p N}{2\pi\gamma\epsilon_y (1+1/r)},$$

we have

$$\Delta Qx[r\_, \kappa\_, \delta x\_, \delta y\_, Ix\_, Iy\_, Is\_] := \frac{1+r}{\pi^3} \text{NIntegrate}\Big[nX\Big[1, 1, 0, \delta x\, t, \sqrt{\frac{Ix}{1+\kappa^2}}\, t, \sqrt{\frac{\kappa^2 Is}{1+\kappa^2}}\, t\Big]$$

$$nY\Big[0, 0, \frac{\delta y\ r\ t}{\sqrt{1+(r^2-1)t^2}}, \frac{\sqrt{Iy}\, r\, t}{\sqrt{1+(r^2-1)t^2}}\Big] \frac{t}{\sqrt{1+(r^2-1)t^2}}, \{t, 0, 1\}, \text{PrecisionGoal} \to 3\Big];$$

$$\Delta Qy[r\_, \kappa\_, \delta x\_, \delta y\_, Ix\_, Iy\_, Is\_] := \text{If}\Big[r < 10^{-6}, 0, \frac{r(1+r)}{\pi^3} \text{NIntegrate}\Big[$$

$$nX\Big[0, 0, 0, \delta x\, t, \sqrt{\frac{Ix}{1+\kappa^2}}\, t, \sqrt{\frac{\kappa^2 Is}{1+\kappa^2}}\, t\Big] nY\Big[1, 1, \frac{\delta y\ r\ t}{\sqrt{1+(r^2-1)t^2}}, \frac{\sqrt{Iy}\, r\, t}{\sqrt{1+(r^2-1)t^2}}\Big] \frac{t}{\sqrt{(1+(r^2-1)t^2)^3}},$$

$$\{t, 0, 1\}, \text{PrecisionGoal} \to 3\Big]\Big];$$

Resonance coefficients

In the vicinity of the resonance

$\overline{m}\,\overline{Q} = n$, $\overline{m} = \{m_x, m_y, m_s\}$, $\overline{Q} = \{Q_x, Q_y, -|Q_s|\}$,

one may retain only the corresponding Fourier harmonics of the beam-beam potential

$H_{bb}{}^{(res)} = C_{\overline{m},n} \cos(\overline{m}\overline{\psi} - n\theta)$,

$C_{\overline{m},n} = \dfrac{r_p N}{2\pi\gamma} R_c[|m_x|, |m_y|, |m_s|, r, \kappa, \delta_x, \delta_y, I_x, I_y, I_s]$,

$$rC[mx\_, my\_, ms\_, r\_, \kappa\_, \delta x\_, \delta y\_, Ix\_, Iy\_, Is\_] :=$$

$$\frac{4}{\pi^3} (-1)^{mx+my+ms}\, \text{If}\Big[mx == 0, 1, \frac{1}{(2\,mx-1)!!} \Big(\frac{Ix}{1+\kappa^2}\Big)^{mx/2}\Big] \text{If}\Big[my == 0, 1, \frac{1}{(2\,my-1)!!} (r^2\, Iy)^{my/2}\Big]$$

$$\text{If}\Big[ms == 0, 1, \frac{1}{(2\,ms-1)!!} \Big(\frac{\kappa^2 Is}{1+\kappa^2}\Big)^{ms/2}\Big] \text{NIntegrate}\Big[nX\Big[0, mx, ms, \delta x\, t, \sqrt{\frac{Ix}{1+\kappa^2}}\, t, \sqrt{\frac{\kappa^2 Is}{1+\kappa^2}}\, t\Big]$$

$$nY\Big[0, my, \frac{\delta y\ r\ t}{\sqrt{1+(r^2-1)t^2}}, \frac{\sqrt{Iy}\, r\, t}{\sqrt{1+(r^2-1)t^2}}\Big] \frac{t^{mx+my+ms-1}}{(1+(r^2-1)t^2)^{(my+1)/2}}, \{t, 0, 1\},$$

$$\text{PrecisionGoal} \to 3\Big];$$