Fermi National Accelerator Laboratory

FERMILAB-Pub-00/031-T

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February 2000

Submitted to *Physical Review D*

Operated by Universities Research Association Inc. under Contract No. DE-AC02-76CH03000 with the United States Department of Energy

FERMILAB-PUB-00/031-T
hep-lat/0002008

Application of heavy-quark effective theory to lattice QCD: I. Power Corrections

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(February 7, 2000)

Abstract

Heavy-quark effective theory (HQET) is applied to lattice QCD with Wilson fermions at fixed lattice spacing a. This description is possible because heavy-quark symmetries are respected. It is desirable because the ultraviolet cutoff $1/a$ in current numerical work and the heavy-quark mass m_Q are comparable. Effects of both short distances, a and $1/m_Q$, are captured fully into coefficient functions, which multiply the operators of the usual HQET. Standard tools of HQET are used to develop heavy-quark expansions of lattice observables and, thus, to propagate heavy-quark discretization errors. Three explicit examples are given, namely the mass, decay constant, and semileptonic form factors of heavy-light mesons.

PACS numbers: 12.38.Gc, 12.39.Hg, 13.20.-v

I. INTRODUCTION

One of the most vital parts of high-energy physics is the study of heavy quarks. Several large experimental data sets of hadrons with beauty or charm are available now, or will be soon. These data are valuable, because the decay properties of these hadrons depend on poorly known elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. A broad range of measurements can be used to determine the CKM matrix with many cross checks and, thus, to test the flavor structure of the standard model, including the origin of CP violation.

In this enterprise numerical lattice QCD plays the role of providing hadronic matrix elements, ideally with controllable, transparent uncertainties. The two sources of uncertainty that attract the greatest concern are discretization effects and the quenched approximation, in which the feedback of (light) quark loops on the gluons is omitted. Both can be eliminated with ever larger computing resources. Effects of the lattice discretization also can be studied theoretically. They appear at short distances, so they can be disentangled from long-distance physics with field theoretic methods. This paper uses effective field theory to separate the short-distance scales of the lattice spacing and the heavy quark mass from the long-distance QCD scale. It is an extension of work with El-Khadra and Mackenzie [1] on massive fermions in lattice gauge theory. The effective field theory approach yields several concrete results, which illustrate strategies for reducing cutoff effects of heavy quarks. Numerical calculations may then focus computer resources on incorporating the light quark loops.

The idea of using effective field theory to study cutoff effects goes back to Symanzik [2]. For any *lattice* field theory his idea was to introduce a local effective Lagrangian, which is the Lagrangian of the corresponding *continuum* field theory, augmented with higher-dimension operators. Coefficients of the operators depend on the underlying lattice action, and their dimensions are balanced by powers of the lattice spacing a, which is the only short distance in Symanzik's analysis. For small enough a one should be able to treat the higher-dimension terms as perturbations and express observables of the lattice theory as an expansion in terms of continuum observables. For a pedagogical introduction, see Ref. [3].

The aim of this paper is to understand numerical data generated using lattice actions with Wilson fermions [4,5] for the heavy quarks.[1] Many papers, starting with the work of Gavela *et al.* [6] and Bernard *et al.* [7], have attempted to calculate properties of heavy-quark systems in this way. In practice, the bottom quark's mass in lattice units is large, $m_ba \approx 1$, and even the charmed quark's mass is not especially small, $m_ca \approx \frac{1}{3}$. Thus, these calculations are hardly in the asymptotic regime $m_Qa \to 0$ (m_Q fixed) for which these actions were originally devised. In particular, any expansion in small m_Qa, as is usually assumed in analyses based on Symanzik's work, fails. This does not imply that heavy-quark cutoff effects in these calculations are large, but it does mean that a different analysis is needed.

The heavy quark masses are larger than Λ_{QCD}, so they introduce additional short-distance scales. One is free to seek an effective theory that lumps the effects of all short distances—the

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[1] In this paper the term "Wilson fermions" encompasses any action with Wilson's solution of the doubling problem [4]. These include the Sheikholeslami-Wohlert ("clover") action [5], the actions of Ref. [1], and—of course—the Wilson action.

lattice spacing and the heavy quarks' Compton wavelengths—into coefficients. This effective theory does not have to be continuum QCD. The crucial observation [1] is that lattice actions with Wilson fermions satisfy the same heavy-quark symmetries [8] as continuum QCD. For heavy-light systems, therefore, a version of the heavy-quark effective theory (HQET) is appropriate. Similarly, for quarkonia the same Lagrangian applies, but with the power-counting of non-relativistic QCD (NRQCD). The operators in such a description of lattice gauge theory are the same as in the usual NRQCD [9–11] or HQET [12–15] descriptions of continuum QCD, but the coefficients differ because the lattice modifies the dynamics at short distances.

This paper focuses on hadrons with one heavy quark and, consequently, on HQET. It uses tools of the usual HQET to derive formulae of the form

$$B_{\rm lat} = z_1(m_Qa)B_\infty(\Lambda_{\rm QCD}a) + \frac{1}{m_2(m_Qa)}B'_\infty(\Lambda_{\rm QCD}a) + \cdots, \tag{1.1}$$

where $B_{\rm lat}$ is a physical observable calculated in lattice gauge theory. The quantities $z_1(m_Qa)$ and $1/m_2(m_Qa)$ are short-distance coefficients of mass dimension 0 and -1, respectively.[2] They do not depend on the light degrees of freedom. The quantities B_∞ and B'_∞ describe the long-distance physics. They are matrix elements in the infinite-mass limit and do not depend on m_Q. Thus, the heavy-quark mass is entirely isolated into the coefficients.

The logic to derive formulae like Eq. (1.1) parallels that of the standard HQET. In both cases the deviations from the infinite-mass limit are expressed as a series fo small corrections. Each term consists of a short-distance coefficient multiplying a long-distance matrix element of the infinite-mass limit. From this structure a simple picture of cutoff effects emerges. The heavy-quark cutoff effects lie in the difference between the short-distance coefficient functions and their values in continuum QCD. On the other hand, matrix elements of the infinite-mass limit, such as B_∞ and B'_∞, suffer from discretization effects only of the light degrees of freedom.

This paper is organized as follows: Sec. II clarifies the non-relativistic interpretation of Wilson fermions introduced in Ref. [1], by giving more direct, though also more abstract, reasoning to relate lattice gauge theory to HQET. Section III establishes some general notation and introduces the HQET Lagrangian. As in Sec. II the emphasis is on symmetries. The leading, heavy-quark symmetric, effective Lagrangian is shown to be the same for lattice gauge theory as for continuum QCD. This static Lagrangian is the foundation of the heavy-quark expansion, so some of its properties are recalled in Sec. IV. The next task is to propagate deviations from the static limit to observables, so Sec. V develops a suitable form of perturbation theory. Applications of the formalism are in Sec. VI–VIII. Section VI works out the heavy-quark expansion for hadron masses to second order. Semileptonic form factors, at the so-called zero-recoil point, are addressed in Sec. VII. As with continuum QCD, the first order vanishes. The technical details of the second order are considerable and appear in Appendix A, correcting some minor errors in the literature. Section VIII derives

[2] Here m_Q the heavy quark mass in some scheme, for example the bare mass. When there is more than one heavy quark in the problem, Eq. (1.1) is schematic, and the coefficients depend on all heavy quark masses.

the first-order expansion for decay constants. In all three cases the analysis follows work on the usual HQET, but keeping careful track of the HQET coefficients. Some implications of these concrete results are discussed in Sec. IX.

II. HQET FOR LATTICE QCD

In this section lattice gauge theory with a general action for Wilson fermions is related to HQET. A derivation starting from the path integral of lattice QCD and making field redefinitions has been given previously [1]. That procedure is analogous to derivations of HQET from the path integral of continuum QCD [18–20], and it yields the coefficients at the tree level. Reference [1] used heavy-quark symmetry only to show that the approach to the infinite-mass limit is smooth and stable in the presence of radiative corrections. Here the argument is reversed: owing to heavy-quark symmetry, there must be a version of HQET describing lattice QCD. This is so whenever momentum transfers are much smaller than m_Q. Whether $m_Q a \ll 1$, $m_Q a \gg 1$, or $m_Q a \sim 1$, the reasoning is the same. The concepts are spelled out in this section, and the mathematical formalism is developed in Sec. III.

The action for Wilson fermions [4] (including improvements [5,1] with Wilson's solution of the doubling problem) can be written

$$S = \sum_x \bar{\psi}_x \psi_x - \kappa \sum_{x,y} \bar{\psi}_x M_{xy} \psi_y, \tag{2.1}$$

where x and y run over all lattice sites and M_{xy} has support only for y near x. To maintain gauge invariance M_{xy} includes parallel transport along some path from x to y. The hopping parameter κ controls the fermion's movement through the lattice, and small κ corresponds to large $m_Q a$. For $\kappa \to 0$ any action of the form (2.1) clearly has the heavy-quark spin and, for more than one quark, flavor symmetries of continuum QCD in the heavy-quark limit [8].

To sharpen this point, consider for now fixed lattice spacing a. Expanding the quark propagator

$$S(x - z) = T\psi(x)\bar{\psi}(z) \tag{2.2}$$

in small κ, gives a leading term which is the shortest and straightest path permitted by iterating M_{xy}. If $\boldsymbol{x} = \boldsymbol{z}$ this is nothing but the propagator of the static theory [12]. Thus, the heavy-quark limit of lattice QCD is the static theory, plus small flavor- and spin-dependent contributions. Therefore, one should look for a heavy-quark effective theory to describe processes with all momentum exchanges small compared to m_Q.

Because the symmetries are the same as in the continuum, the operators of this HQET are the *same* as those of the usual HQET describing continuum QCD. To define the operators the main issue is to regulate divergences. There is no need choose the same ultraviolet regulator for the effective theory as for the underlying theory. One is free to regulate the ultraviolet with, say, dimensional regularization and either a physical or a minimal renormalization scheme. On the other hand, because the effective and underlying theories are supposed to describe the same long-distance physics, the same infrared regulator, when needed, should be chosen.

Because the details of the short-distance dynamics are those of the lattice theory the coefficient functions of HQET must be modified. They can be calculated by computing the observables in lattice QCD and the modified HQET and matching. The lattice breaks some rotational and translational symmetries, so coefficients of corresponding operators need not vanish, as they would in the usual HQET. The explicit form of the coefficients is not needed in this paper, but it is helpful to have an idea how they might be calculated. With Feynman diagrams, for example, one would expand lattice amplitudes around the static limit in small momentum transfers, keeping the full dependence on $m_Q a$.

In summary, at any given lattice spacing a observables of the action (2.1) can be described with the usual operators of HQET but modified coefficients. As a varies the short-distance properties change, and so the coefficients must change to compensate. Eventually, when $a \to 0$, lattice QCD becomes (indeed, defines) continuum QCD, so the coefficients of the modified HQET smoothly turn into those of the usual HQET.

In Eq. (2.1) the hopping matrix M_{xy} is not specified in detail. In general it contains many free couplings, which are irrelevant in the sense of the renormalization group. In the usual improvement program [16] they are chosen to accelerate the approach to the continuum limit. In the HQET analysis advocated here, a similar principle holds. The irrelevant couplings of the lattice action alter the short-distance coefficients of the modified HQET. Thus, they can be adjusted so that the HQET expansion of lattice QCD systematically reproduces more and more of the HQET expansion of continuum QCD.

For a generic lattice action, the heavy-quark symmetries hold only in the rest frame. On a superficial glance this is a drawback, because much of the power of HQET comes from boosting heavy-light hadrons to arbitrary frames. On a second glance, it may be a blessing in disguise. By combining heavy-quark symmetry, Lorentz covariance, and reparametrization invariance [17], it may be possible to develop a non-perturbative improvement program.

III. NOTATION AND FORMALISM

This section reviews the main ingredients of HQET in a notation well-suited to Euclidean space-time. The details are slanted to Euclidean space-time because the aim of the paper is to understand the output of Monte Carlo calculations of lattice QCD. All results, however, are for matrix elements defined at a fixed (Euclidean) time, so they apply equally well to the Minkowski theory. Indeed, with the conventions introduced here, the formulae in this paper hold for both kinds of time, unless specifically noted.

The Euclidean action can be written $S = -\int d^4x\,\mathcal{L}$, where $\mathcal{L}$ is the Lagrangian, and the weight factor in the functional integral is then e^{-S}. The metric is $\delta^{\mu\nu}$, Greek indices run from 1 to 4, and Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. A convenient basis is given in Ref. [1], in particular

$$\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

As usual, we take real (Minkowski) time to be $t = x^0$. Then Euclidean time $x^4 = ix^0$, and the general rule relating the fourth component to the zeroth component of a four-vector q is

$$q^4 = iq^0. \tag{3.2}$$

Because the spatial components are the same, it is convenient to put all modifications into the time component. Therefore, this paper uses the metric $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, where Greek indices run from 0 to 3, so $q_0 = -q^0$ and $q^2 = -(q^0)^2 + \boldsymbol{q}^2$. And the Dirac matrices $\gamma^0 = -i\gamma^4$ and γ^j differ by a factor of $-i$ from those of the most common Minkowski convention.

The four-volume element is defined to be

$$d^4x := dx^1dx^2dx^3dx^4 = i\, dx^0dx^1dx^2dx^3. \tag{3.3}$$

The factor of i in Eq. (3.3) is the most unusual convention introduced here, but it allows many formulae given below look the same in both Euclidean and Minkowski space-time. For example the weight factor of the path integral is always $e^{\int d^4x\, \mathcal{L}}$.

The foregoing conventions can be used in any field theory. In HQET one introduces a velocity v, with $v^2 = -1$. Although heavy-quark symmetry of lattice gauge theory is only guaranteed in the rest frame $\boldsymbol{v} = \mathbf{0}$, it is convenient to keep v arbitrary. The projectors

$$P_\pm(v) = \tfrac{1}{2}(1 \mp i\not{v}) \tag{3.4}$$

project onto "upper" and "lower" components of spinors. For any vector q the components orthogonal to v,

$$q_\perp^\alpha = q^\alpha + v^\alpha v \cdot q, \tag{3.5}$$

play a special role. In the rest frame they are the spatial directions. It is also convenient to introduce

$$\eta^\alpha{}_\beta = \delta^\alpha{}_\beta + v^\alpha v_\beta \tag{3.6}$$

to project out orthogonal components of a tensor, *e.g.*, $q_\perp^\alpha = \eta^\alpha{}_\beta q^\beta$.

In HQET heavy quarks are represented by a heavy-quark field $h_v^{(+)}$ satisfying

$$h_v^{(+)} = P_+(v)h_v^{(+)}. \tag{3.7}$$

The anti-quarks are represented by $h_v^{(-)} = P_-(v)h_v^{(-)}$. As in the usual HQET one can either consider the anti-quarks to be decoupled [18,19] or integrated out [20]. But in this paper, having shown that the heavy-quark symmetries hold in lattice QCD, the effective Lagrangian is developed principally on the basis of symmetry. The heavy-quark Lagrangian is written

$$\mathcal{L}_{\mathrm{HQET}} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \cdots, \tag{3.8}$$

where the leading term is

$$\mathcal{L}^{(0)} = \bar{h}_v^{(+)}(iv \cdot D - m_1)h_v^{(+)}. \tag{3.9}$$

A non-zero *rest mass* m_1 is introduced to describe the exponential fall-off of Euclidean Green functions, $e^{-E|x_4|}$ with energies $E \approx m_1$. The further interactions $\mathcal{L}^{(s)}$ contain operators of dimension $4+s$. By dimensional analysis their coefficients, of dimension $-s$, contain powers of the short-distance scales $1/m_Q$ or a.

The Lagrangian $\mathcal{L}^{(0)}$ is the unique scalar of dimension four satisfying the Isgur-Wise symmetries [8]. The heavy-quark spin symmetry is manifest, but with $m_1 \neq 0$ the flavor symmetry is not. It is, however, there. In Eq. (3.9) let the field $h_v^{(+)}$ to be a column vector for all the flavors of velocity v, and let m_1 denote a mass matrix. For example, for two flavors

$$m_1 = \begin{pmatrix} m_{1c} & 0 \\ 0 & m_{1b} \end{pmatrix}. \tag{3.10}$$

Let $\theta = (m_{1c} - m_{1b}) v \cdot x$ and consider the generators

$$\tau^1 = \frac{i}{2}\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}, \quad \tau^2 = \frac{i}{2}\begin{pmatrix} 0 & -ie^{i\theta} \\ ie^{-i\theta} & 0 \end{pmatrix}, \quad \tau^3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.11}$$

satisfying the SU(2) algebra $[\tau^d, \tau^e] = \varepsilon^{dfe}\tau^f$. Then the flavor symmetry is

$$h_v^{(+)} \mapsto e^{\tau^a\omega_a} h_v^{(+)}, \quad \bar{h}_v^{(+)} \mapsto \bar{h}_v^{(+)} e^{-\tau^a\omega_a}. \tag{3.12}$$

The symbol $\mathcal{D}^\mu = D^\mu - i m_1 v^\mu$, which was introduced in Ref. [21], satisfies $[\mathcal{D}^\mu, \tau^d] = 0$ and is, thus, trivially covariant under the transformation (3.12). Therefore, flavor-symmetric operators take the form

$$O_\Gamma^{\mu_1 \cdots \mu_n} = \bar{h}_v^{(+)} \Gamma \mathcal{D}^{\mu_1} \cdots \mathcal{D}^{\mu_n} h_v^{(+)}, \tag{3.13}$$

where $\Gamma = P_+(v)\Gamma P_+(v)$. Spin-symmetric operators have $\Gamma = 1$ (or $\not{v}$, since $\not{v} h_v^{(+)} = i h_v^{(+)}$). The only flavor- and spin-symmetric scalar at dimension four is $\bar{h}_v^{(+)} i v \cdot \mathcal{D} h_v^{(+)}$, which is $\mathcal{L}^{(0)}$. Thus, the symmetries of HQET with non-zero rest masses are the same as without.

In the following anti-quarks are not considered further, so from now on the heavy quark field is written h_v instead of $h_v^{(+)}$.

To describe deviations from the symmetry limit, one introduces the higher-dimension interactions $\mathcal{L}^{(s)}$, which are built from operators like O_Γ. These are general enough to include the gluon field strength, because $F^{\mu\nu} = [D^\mu, D^\nu] = [\mathcal{D}^\mu, \mathcal{D}^\nu]$. One may omit operators that would vanish by the equations of motion of $\mathcal{L}^{(0)}$, $-iv \cdot \mathcal{D} h_v = 0$. Such operators make no net contribution on the HQET mass shell, so they do not appear in on-shell matching calculations. At dimension five there can be two $\mathcal{D}$s, so

$$\mathcal{L}^{(1)} = \frac{\mathcal{O}_2}{2m_2} + \frac{\mathcal{O}_B}{2m_B}, \tag{3.14}$$

where

$$\mathcal{O}_2 = \bar{h}_v D_\perp^2 h_v, \tag{3.15}$$
$$\mathcal{O}_B = \bar{h}_v s_{\alpha\beta} B^{\alpha\beta} h_v, \tag{3.16}$$

with $s_{\alpha\beta} = -i\sigma_{\alpha\beta}/2$ and $B^{\alpha\beta} = \eta_\mu^\alpha \eta_\nu^\beta F^{\mu\nu}$. In the rest frame, $\mathcal{O}_2$ gives the kinetic energy and $\mathcal{O}_B$ the chromomagnetic interaction. At dimension six, with three $\mathcal{D}$s,

$$\mathcal{L}^{(2)} = \frac{\mathcal{O}_D}{8m_D^2} + \frac{\mathcal{O}_E}{8m_E^2}, \tag{3.17}$$

where

$$\mathcal{O}_D = \bar{h}_v [D_\perp^\alpha, iE_\alpha] h_v, \tag{3.18}$$
$$\mathcal{O}_E = -\bar{h}_v i\sigma_{\alpha\beta} \{D_\perp^\alpha, iE^\beta\} h_v, \tag{3.19}$$

with $E^\beta = -v_\alpha F^{\alpha\beta}$.[3] In the rest frame, $\mathcal{O}_D$ gives the Darwin term and $\mathcal{O}_E$ the spin-orbit interaction. The complete list of dimension-six interactions includes four-quark operators, such as $\bar{q}\gamma^\mu q \bar{h}_v v_\mu h_v$, but their coefficients all vanish at the tree level.

Distinct inverse masses $1/m_2$, $1/m_B$, $1/m_D^2$, and $1/m_E^2$ are introduced as a notation for the coefficients of the modified HQET. One could have equally well written z_B/m_2 instead of $1/m_B$, and so on, but to trace the effects of the higher-dimension operators on physical observables the notation of inverse masses is adequate. The numerical factors and powers of the inverse masses have been chosen so that all masses become the same in the tree-level continuum limit. At non-zero lattice spacing and in the presence of radiative corrections, this is no longer guaranteed.

Concrete expressions for the coefficients lie beyond the scope of this paper. They depend on couplings of the lattice action, the velocity $\boldsymbol{v}$, and the HQET renormalization scheme. Ideally one would like to devise a non-perturbative scheme for computing the coefficients, but so far they have been studied only in perturbation theory. For the lattice actions in common use, expressions are available at the tree level for m_1, $1/m_2$, and $1/m_B$ [1], and at the one-loop level for m_1 and $1/m_2$ [22].

Through dimension six the effective heavy-quark Lagrangian is rotationally invariant. Starting with dimension seven, this is no longer the case. For example, consider the term

$$\mathcal{L}^{(3)} = \cdots + a^3 w_4 \sum_{i=1}^{3} \bar{h}_v^{(+)} D_i^4 h_v^{(+)}, \tag{3.20}$$

written in the rest frame, $\boldsymbol{v} = \boldsymbol{0}$. In the usual HQET, rotational invariance of continuum QCD implies $w_4 = 0$. With lattice QCD, however, w_4 does not vanish unless the lattice action has been improved accordingly.

To describe electroweak transitions among hadrons containing a single heavy quark, HQET introduces effective operators for the interactions mediating the transitions. Even in simple cases, such as the vector and axial vector currents examined below, the number of operators in the heavy-quark expansion is large, and the details of the construction are different for heavy-to-heavy and heavy-light transitions. The notation for currents is postponed, therefore, to Secs. VII and VIII.

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[3] The chromoelectric field of Ref. [1] is related (in the rest frame) to the one here by $\boldsymbol{E}_{[1]} = i\boldsymbol{E}$.

IV. PROPERTIES OF $\mathcal{L}^{(0)}$

The previous two sections establish that the heavy-quark limit of lattice QCD can be described by the effective Lagrangian $\mathcal{L}^{(0)}$, with small corrections from

$$\mathcal{L}_I = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \cdots . \tag{4.1}$$

This means that the eigenstates of lattice QCD are not very different from the eigenstates of the quantum field theory defined by $\mathcal{L}_{\text{light}} + \mathcal{L}^{(0)}$, where $\mathcal{L}_{\text{light}}$ is the (Symanzik effective) Lagrangian of the light quarks and gluons. Apart from the rest mass m_1 and lattice artifacts of $\mathcal{L}_{\text{light}}$, this is the same lowest-order Lagrangian that is used to describe heavy quarks in continuum QCD.

To use HQET to connect lattice QCD to continuum QCD, one must understand how the rest mass and the higher-dimension interactions influence observables. This section shows that the eigenstates of the Hamiltonian corresponding to $\mathcal{L}_{\text{light}} + \mathcal{L}_{\text{HQET}}$ are independent of the rest mass. In particular, the eigenstates of $\mathcal{L}_{\text{light}} + \mathcal{L}^{(0)}$ do not depend on the heavy flavor at all. The remainder of the paper then develops perturbation theory in $\mathcal{L}_I$ around these flavor-independent states and studies how the perturbations affect several observables.

To show that the rest mass m_1 decouples from non-perturbative observables, it is convenient to switch to the Hamiltonian formalism of HQET. The canonical conjugate to the field h_v is [23]

$$\pi_v = iv^0\bar{h}_v \tag{4.2}$$

so at equal times ($x^0 = z^0$)

$$\{h_v(x), \pi_v(z)\} = \{h_v(x), iv^0\bar{h}_v(z)\} = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{z})P_+(v). \tag{4.3}$$

The Hamiltonian $H = \int d^3x\, \mathcal{H}$ has the density

$$\mathcal{H} = \mathcal{H}_{\text{light}} + \mathcal{H}^{(0)} - \mathcal{L}_I, \tag{4.4}$$

including a term for the light degrees of freedom. The leading heavy-quark Hamiltonian density

$$\mathcal{H}^{(0)} = \pi_v\partial_0 h_v - \mathcal{L}^{(0)} \tag{4.5}$$

$$= m_1\bar{h}_v h_v + iv^0\bar{h}_v A^0 h_v - i\bar{h}_v \boldsymbol{v}\cdot\boldsymbol{D}h_v. \tag{4.6}$$

From Eq. (4.3) one can see that $\int d^3x\, \bar{h}_v h_v$ commutes with all other terms in H, including with H_{light} and $L_I = \int d^3x\, \mathcal{L}_I$. Thus, the eigenstates of H are independent of m_1. This result is well known in other approaches to the heavy-quark limit [10,24], but the general proof within HQET does not seem to be widely appreciated.[4]

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[4]In specific examples, a small rest mass, called a *residual mass*, has been shown to drop out of the $1/m_Q$ corrections [21].

This result has a very important consequence. In the HQET description of lattice QCD, lattice-spacing dependence appears in three places: the rest mass, the short-distance coefficients of $\mathcal{L}_I$, and the light degrees of freedom. Because the rest mass drops out of physical observables, it is acceptable—perhaps even advisable—to tolerate a discrepancy of the rest mass from the physical mass. Genuine lattice artifacts of the heavy quark stem from deviations of the *higher*-dimension short-distance coefficients from their continuum limit, and the couplings of the lattice action should be tuned to minimize them. To make this more point concrete, the effects of $\mathcal{L}_I$ can be propagated to observables with tools developed for the usual HQET, as shown in the rest of this paper.

V. PERTURBATION THEORY IN $\mathcal{L}_I$

The previous sections have established that lattice gauge theory with heavy quarks can be described by the effective Lagrangian $\mathcal{L}_{\mathrm{HQET}}$, whose eigenstates are close to those of the leading-order theory with Lagrangian $\mathcal{L}^{(0)}$. To trace the effects of the higher-dimension operators in $\mathcal{L}_I$ on observables, they can be treated as perturbations. A formalism for perturbation theory that exploits heavy-quark symmetry is reviewed in this section.

When proceeding to second order in $\mathcal{L}_I$, as in Secs. VI and VII below, one must be careful to be consistent, for example about the normalization of states. Thus, the discussion starts (Sec. V A) with a careful setup of time-ordered perturbation theory, to generate heavy-quark expansions based on the eigenstates of $\mathcal{L}^{(0)}$. These states are desirable not only because they form mass-independent multiplets under heavy-quark symmetry, but also because they are affected by the lattice only through the light degrees of freedom. The formalism makes no explicit reference to the short-distance coefficients of the modified HQET, so it applies equally well to the usual HQET and could be used there as well. The heavy-quark expansion becomes a series of terms consisting of short-distance coefficients multiplying matrix elements of time-ordered products in the eigenstates of $\mathcal{L}^{(0)}$. Many relations among these matrix elements follow from heavy-quark symmetry, and Sec. V B reviews the trace formalism, a technique for deriving such relations.

A. Time-ordered perturbation theory

The perturbative series can be generated by generalizing the interaction picture for vacuum expectation values to transition matrix elements. There are three quantum field theories to consider: the underlying theory [here lattice QCD with action (2.1)]; the full HQET with Lagrangian (3.8); and the leading HQET with Lagrangian (3.9). The states treated here are hadrons with one heavy quark. The (lattice) QCD state with a heavy quark of flavor b (c) is denoted $|B\rangle$ ($|D\rangle$). The analogous full HQET state is denoted $|B_v\rangle$, where the subscript labels the chosen velocity. Finally, the infinite-mass states are denoted $|b_vJ;j\alpha\rangle$, where b is the heavy flavor in the HQET with velocity v, J is the hadron's spin, j is the spin of the light degrees of freedom, and α encompasses all other quantum numbers of the light degrees of freedom. By heavy-quark flavor and spin symmetry, the spatial wave-functions of these states do not depend on b or J.

By the Gell-Mann–Low theorem [25] the lowest-lying (*i.e.*, $\alpha = 0$) spin-J hadron is related to the corresponding infinite-mass state by

$$|B_v\rangle = \lim_{T\to\infty(1-i0^+)} \frac{Z_B^{1/2}\, U(0,\mp T)|b_vJ;j0\rangle}{\langle b_vJ;j0|U(0,\mp T)|b_vJ;j0\rangle}, \tag{5.1}$$

where Z_B is a state renormalization factor and

$$U(t,t_0) = T\,\exp\int_{t_0}^{t} d^4x\,\mathcal{L}_I \tag{5.2}$$

is the familiar interaction-picture propagator. The state renormalization factor has the usual interpretation of the overlap between the unperturbed and the fully dressed states: $Z_B^{1/2} = \langle b_vJ;j0|B_v\rangle$.

To derive the heavy-quark expansion without ambiguities stemming from the normalization of states, one should set up perturbation theory so that Z_B does not appear. For example, the energy of a fully dressed state can be written [26]

$$E = \frac{\langle b_vJ;j0|H|B_v\rangle}{\langle b_vJ;j0|B_v\rangle} = \frac{\langle b_vJ;j0|H(0)U(0,-T)|b_vJ;j0\rangle}{\langle b_vJ;j0|U(0,-T)|b_vJ;j0\rangle}, \tag{5.3}$$

in which the normalization of states clearly cancels. Similarly, matrix elements for flavor-changing transitions can be expressed

$$\frac{\langle D_{v'}|T\,O_1\cdots O_n|B_v\rangle}{\langle D_{v'}|D_{v'}\rangle^{1/2}\;\langle B_v|B_v\rangle^{1/2}} = \frac{\langle c_{v'}J';j0|T\,O_1\cdots O_n e^{\int d^4x\,\mathcal{L}_I}|b_vJ;j0\rangle}{\langle c_{v'}J';j0|T\,e^{\int d^4x\,\mathcal{L}_I}|c_{v'}J';j0\rangle^{1/2}\,\langle b_vJ;j0|T\,e^{\int d^4x\,\mathcal{L}_I}|b_vJ;j0\rangle^{1/2}}, \tag{5.4}$$

where the upper (lower) sign of Eq. (5.1) is used for the initial (final) state, and products of $U(t_1,t_2)$ have been coalesced according to its well-known properties [27]. The factors $Z_B^{1/2}$ and $Z_D^{1/2}$ are eliminated in favor of the denominators by taking the modulus of each side of Eq. (5.1).

The operators O_j in Eq. (5.4) are operators of HQET. In general an operator from the underlying theory is described by a sum of operators in HQET, *cf.* Secs. VII and VIII. On the left-hand side the operators have the time dependence

$$O_j(t) = e^{iHt}O_je^{-iHt} \tag{5.5}$$

of the Heisenberg picture, whereas on the right-hand side they have the time dependence

$$O_j(t) = e^{iH_0t}O_je^{-iH_0t} \tag{5.6}$$

of the interaction picture. They are related by $O^{(H)}(t) = U^\dagger(t,0)O^{(I)}(t)U(t,0)$. When O contains explicit time derivatives, as in some cases in Appendix A, the time dependence of the Us generates additional contact terms in the T-product in the interaction picture.

This setup of time-ordered perturbation theory is equivalent to Rayleigh-Schrödinger perturbation theory [25]. The denominators on the right-hand side of Eq. (5.4) are rarely made explicit in the literature on HQET, but they are necessary. Indeed, in tracing the equivalence to Rayleigh-Schrödinger perturbation theory, one sees that the denominators generate wave-function renormalization and remove $|b_v J; j0\rangle$ and $|c_{v'} J'; j0\rangle$ from sums over intermediate states. The procedure is analogous to taking connected vacuum correlation functions.[5] The infinitesimal in the limit $T \to \infty(1 - i0^+)$ is needed to dampen the integrals in Minkowski space, and it is unnecessary in Euclidean space. There is no issue of analytic continuation here: the symbol $U(t, t_0)$ is just an integral representation of the energy denominators in ordinary perturbation theory.

The principal advantage of Eqs. (5.3) and (5.4) is that they separate cleanly how each term in the heavy-quark expansion affects the matrix element on the left-hand side. As desired, the normalization conditions on the full and infinite-mass states cancel separately. Each operator in $\mathcal{L}_I$ can be treated one insertion after another, and the expansion leads to matrix elements in the mass-independent states $|b_v J; j0\rangle$ and $\langle c_{v'} J'; j0|$. On the other hand, a formalism that starts with *vacuum* expectation values of time-ordered products and proceeds to the left-hand side via the reduction formula leads to expressions with "in" and "out" states whose masses equal those of the fully dressed states.

When employing HQET to describe lattice QCD it is especially helpful to obtain a series in the mass-independent eigenstates of $\mathcal{L}^{(0)} + \mathcal{L}_{\text{light}}$. These states depend only mildly on the lattice spacing, through the light degrees of freedom. Thus, the discretization effects of the heavy quark are truly encapsulated into the short-distance coefficients of $\mathcal{L}_I$, and one can estimate their effect simply by comparing the heavy-quark expansions of continuum and lattice QCD. With the expansions derived in subsequent sections, the comparison is made easily by substituting the usual coefficients for the modified ones.

Although not strictly necessary, it is convenient to choose normalization conditions for the states. In the underlying theory we normalize plane-wave states so that

$$\langle B(\boldsymbol{p}')|B(\boldsymbol{p})\rangle = v^0 (2\pi)^3 \delta(\boldsymbol{p}' - \boldsymbol{p}), \tag{5.7}$$

where $v^0 = \sqrt{1 + \boldsymbol{v}^2}$. In continuum QCD $\boldsymbol{v} = \boldsymbol{p}/M$ is the physical velocity of the true hadron, and in lattice QCD the relation between $\boldsymbol{v}$ and $\boldsymbol{p}$ should tend to the same as $\boldsymbol{p}a \to 0$. Equation (5.7) is convenient because it is relativistically invariant and its infinite mass limit is well behaved. We also normalize full HQET states so that

$$\langle B_v(\boldsymbol{k}')|B_v(\boldsymbol{k})\rangle = v^0 (2\pi)^3 \delta(\boldsymbol{k}' - \boldsymbol{k}), \tag{5.8}$$

where $\boldsymbol{k}^{(\prime)}$ is a small residual momentum [23], and likewise for the infinite-mass states. Note that in Eq. (5.8) the factor of v^0 does not introduce mass dependence; in HQET the velocity is an ingredient in the construction of the effective Lagrangian, not a property of the states.

To regulate δ functions one should smear plane-wave states into wave packets before expanding out Eq. (5.3) or (5.4). With the same normalization condition (5.8) for fully dressed and infinite-mass HQET states, the factors of v^0 and the smearing functions cancel completely after expanding. One can thus re-write Eq. (5.4)

[5]Reference [28] notes both the significance of the subtractions and the analogy with connected vacuum amplitudes, but prefers not to use HQET.

$$\langle D_{v'}|T\,O_1\cdots O_n|B_v\rangle = \langle c_{v'}J';j0|T\,O_1\cdots O_n e^{\int d^4x\,\mathcal{L}_I}|b_vJ;j0\rangle^\star \tag{5.9}$$

where the star on the right-hand side is a reminder to include the extra terms generated by expanding out the denominator of Eq. (5.4). In Sec. VII and Appendix A this notation is used for T-products $\langle c_{v'}J'|T\,O\mathcal{O}^b_X|b_vJ\rangle^\star$, $\langle c_{v'}J'|T\,\mathcal{O}^c_X O\mathcal{O}^b_Y|b_vJ\rangle^\star$, etc., where the operators $\mathcal{O}^h_X$ are those appearing in $\mathcal{L}_I$ for flavor h. The star means to collect all terms from the expansion with the specified insertions.

B. Trace formalism

To evaluate the right-hand side of Eq. (5.9) there is a powerful formalism, called the trace formalism, which takes full advantage of heavy-quark symmetry [29]. The objective is to calculate transition amplitudes of the form

$$\mathcal{T}^{A_1\cdots A_N}_{b_v\to c_{v'}} = \langle c_{v'}J';j0|T\,\bar{h}_{v'}\Gamma_1 G^{A_1}_1 h_{v'}\cdots\bar{h}_{v'}\Gamma_n G^{A_n}_n h_v\cdots\bar{h}_v\Gamma_N G^{A_N}_N h_v|b_vJ;j0\rangle^\star \tag{5.10}$$

and

$$\mathcal{T}^{A_1\cdots A_N}_{b_v\to 0} = \langle 0|T\,\bar{q}\Gamma_1 G^{A_1}_1 h_v\,\bar{h}_v\Gamma_2 G^{A_2}_2 h_v\cdots\bar{h}_v\Gamma_N G^{A_N}_N h_v|b_vJ;j0\rangle^\star, \tag{5.11}$$

where the $G^{A_k}_k$ is a combination of covariant derivatives $\mathcal{D}$ (including field strengths $F^{\mu\nu}$) and light-quark bilinears $\bar{q}q$ with Lorentz indices abbreviated by the superscript A_k.

The color and spin dependence of each static propagator $Th_v(x)\bar{h}_v(y)$ [or $Th_{v'}(x)\bar{h}_{v'}(y)$] factors into a Wilson line and a projector $P_+(v) =: P_+$ [or $P_+(v') =: P'_+$]. That means that the amplitudes can be written (for $j=\frac{1}{2}$ mesons)

$$\mathcal{T}^{A_1\cdots A_N}_{b_v\to c_{v'}} = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}(v')\Gamma_1 P'_+\cdots P'_+\Gamma_n P_+\cdots P_+\Gamma_N\mathcal{M}_J(v)\Xi^{A_1\cdots A_N}\}, \tag{5.12}$$

and

$$\mathcal{T}^{A_1\cdots A_N}_{b_v\to 0} = -\operatorname{tr}\{\Gamma_1 P_+\Gamma_2 P_+\cdots P_+\Gamma_N\mathcal{M}_J(v)\Xi^{A_1\cdots A_N}\}, \tag{5.13}$$

where $\mathcal{M}_J$ and $\bar{\mathcal{M}}_{J'}$ are spin wave-functions and $\Xi^{A_1\cdots A_N}$ parametrizes the spatial wave-functions and a trace over color of the Wilson lines, punctuated by the G^{A_k}, with the light quark propagator. There is only one trace over heavy-quark spin, because products of traces correspond to disconnected terms, which are subtracted when expanding Eq. (5.9). The minus sign arises because the trace over spin is obtained after anti-commuting the left-most quark field all the way to the right.

Spin wave-functions such as $\mathcal{M}_J(v)$ and $\bar{\mathcal{M}}_{J'}(v')$ are determined by spin symmetry alone. For $j=\frac{1}{2}$ they are

$$\mathcal{M}_0(v) = i2^{-1/2}P_+(v)\gamma_5, \tag{5.14}$$

$$\mathcal{M}_1(v) = i2^{-1/2}P_+(v)\not{\epsilon}. \tag{5.15}$$

Charge conjugates are $\bar{\mathcal{M}} = \gamma_4\mathcal{M}^\dagger\gamma_4$,

$$\bar{\mathcal{M}}_0(v) = i2^{-1/2}\gamma_5 P_+(v), \tag{5.16}$$

$$\bar{\mathcal{M}}_1(v) = i2^{-1/2}\bar{\not\epsilon} P_+(v), \tag{5.17}$$

where $\bar{\epsilon} = \epsilon^*$ in Minkowski space-time and $\bar{\epsilon} = (\boldsymbol{\epsilon}^*, -\epsilon_4^*)$ in Euclidean space-time. Note that $\mathcal{M} = P_+\mathcal{M}P_-$ and $\bar{\mathcal{M}} = P_-\bar{\mathcal{M}}P_+$. Generalizations to $j=0$ and $j=1$ baryons [30,31] and to higher angular momentum [32] are available in the literature.

The functions $\Xi^{A_1\cdots A_N}$ cannot be obtained from symmetry considerations alone. They depend on the velocities v' and v and the quantum numbers of the light degrees of freedom. They parametrize the long-distance dynamics of $\mathcal{L}^{(0)}+\mathcal{L}_{\text{light}}$, so they do not depend on flavor, and they suffer from lattice artifacts only of the light degrees of freedom. As explained above, cutoff effects of the heavy quark are captured in the coefficients of the modified HQET, which multiply matrix elements (5.10) and (5.11).

VI. HADRON MASSES

The simplest application of the HQET formalism is to generate an expansion for the rest mass of a heavy-light hadron. In numerical lattice calculations the energy of a state of momentum $\boldsymbol{p}$ is computed by looking at the (imaginary) time evolution of a correlation function

$$\langle \Phi_{\boldsymbol{p}'}(x_4)\Phi^\dagger_{\boldsymbol{p}}(0)\rangle = \delta_{\boldsymbol{p}'\boldsymbol{p}}\left[\theta(x_4)\sum_n e^{-x_4E_n(\boldsymbol{p})}|\langle B_n|\Phi^\dagger_{\boldsymbol{p}}|0\rangle|^2 + \theta(-x_4)\sum_{n'} e^{x_4E_{n'}(\boldsymbol{p})}|\langle \bar{B}_{n'}|\Phi_{\boldsymbol{p}}|0\rangle|^2\right], \tag{6.1}$$

where $|B_n\rangle$ ($|\bar{B}_{n'}\rangle$) are full lattice-QCD states connected to the vacuum by $\Phi^\dagger_{\boldsymbol{p}}$ ($\Phi_{\boldsymbol{p}}$). By a combination of judicious choices of $\Phi^\dagger_{\boldsymbol{p}}$ and taking x_4 large enough, one can isolate the lower-lying states. At small momentum, the relation between energy and momentum is

$$E(\boldsymbol{p}) = M_1 + \frac{\boldsymbol{p}^2}{2M_2}, \tag{6.2}$$

which defines the hadron's rest mass M_1 and kinetic mass M_2. (Some authors call M_1 the "pole" mass, but M_1 and M_2 are both properties of the particle's pole.) In this paper upper-case is used to denote hadron masses, and lower-case to denote quark masses.

These energies can be thought of as eigenvalues of a Hamiltonian, defined via the transfer matrix, which HQET models with Eq. (4.4). In Eq. (5.3) H is always to the left of $U(0,-T)$, so one can make the split $H = H_{\text{light}} + H^{(0)} - L_I$ and act the first two terms on the bra $\langle b_vJ;j0|$. Setting $\boldsymbol{p}=\boldsymbol{0}$ and calling the leading eigenvalue

$$m_1 + \bar{\Lambda} = \frac{\langle b_vJ;j0|[H_{\text{light}} + H^{(0)}]|b_vJ;j0\rangle}{\langle b_vJ;j0|b_vJ;j0\rangle}, \tag{6.3}$$

the heavy-quark expansion of the hadron mass is generated by

$$M_1 = m_1 + \bar{\Lambda} - \langle b_vJ;j0|L_I\,T\,e^{\int d^4x\,\mathcal{L}_I}|b_vJ;j0\rangle^\star, \tag{6.4}$$

where L_I is at time 0, the time integration is from $-\infty$ to 0, and the star is a reminder not to neglect the denominator in Eq. (5.3). The quark's rest mass enters solely additively because its term in the Hamiltonian commutes with all others.

The expansion of Eq. (6.4) leads to reduced matrix elements that depend on the spin j of the light degrees of freedom ($j = 0$ for the Λ_b baryons, $j = 1/2$ for the B and B^* mesons, etc.), but not on the heavy quark's spin. Through order $1/m_Q^2$ one defines

$$\langle b_v J; j0|\mathcal{O}_2|b_v J; j0\rangle = \lambda_1, \tag{6.5}$$

$$\langle b_v J; j0|\mathcal{O}_B|b_v J; j0\rangle = d_J\lambda_2, \tag{6.6}$$

$$\langle b_v J; j0|\mathcal{O}_D|b_v J; j0\rangle = -2\rho_1, \tag{6.7}$$

$$\langle b_v J; j0|\mathcal{O}_E|b_v J; j0\rangle = -2d_J\rho_2, \tag{6.8}$$

and, in the notation of Ref. [26],

$$\int d^4x\, \langle b_v J; j0|\mathcal{O}_2(0)\mathcal{O}_2(x)|b_v J; j0\rangle^\star = \mathcal{T}_1, \tag{6.9}$$

$$\int d^4x\, \langle b_v J; j0|\mathcal{O}_B(0)\mathcal{O}_B(x)|b_v J; j0\rangle^\star = \mathcal{T}_3 + d_J(\mathcal{T}_4 - \mathcal{T}_2), \tag{6.10}$$

$$\int d^4x\, \langle b_v J; j0|\mathcal{O}_2(0)\mathcal{O}_B(x)|b_v J; j0\rangle^\star = \int d^4x\, \langle b_v J; j0|\mathcal{O}_B(0)\mathcal{O}_2(x)|b_v J; j0\rangle^\star = d_J\mathcal{T}_2. \tag{6.11}$$

The J-dependence in Eqs. (6.5)–(6.11) is $d_0 = 3$ (for the B meson) and $d_1 = -1$ (for the B^* meson). For the Λ_b baryon there are fewer non-vanishing matrix elements; the above formulae hold if one sets $d_{1/2} = 0$. The parameters $\bar{\Lambda}$, λ_n, ρ_n, and $\mathcal{T}_n$ are the same as in continuum QCD, apart from lattice artifacts of the light degrees of freedom. Combining Eqs. (6.4)–(6.11) the rest mass becomes

$$M_1 = m_1 + \bar{\Lambda} - \frac{\lambda_1}{2m_2} - \frac{d_J\lambda_2}{2m_B} + \frac{\rho_1}{4m_D^2} + \frac{d_J\rho_2}{4m_E^2} - \frac{\mathcal{T}_1}{4m_2^2} - \frac{2d_J\mathcal{T}_2}{2m_2\, 2m_B} - \frac{\mathcal{T}_3 + d_J(\mathcal{T}_4 - \mathcal{T}_2)}{4m_B^2}. \tag{6.12}$$

The result (6.12) is simple enough that it could have been written down upon inspection of Eqs. (3.14) and (3.17) and comparing to the continuum papers [33,34,28,26].

This result is the first example of the expansion for which Eq. (1.1) is a prototype. Short-distance effects of the heavy quark, including lattice-spacing effects, are contained in the "masses" m_1, m_2, m_B, m_D, and m_E. If the bare mass is adjusted so that $m_2 = m_Q$, then the mass formula (6.12) shows that the spin-averaged splittings, such as $m_{\Lambda_b} - \frac{1}{4}(m_B + 3m_{B^*})$, are reproduced correctly to order $1/m_Q$. The Sheikholeslami-Wohlert action has a second parameter, with which $1/m_B$ can be adjusted (via a short-distance calculation) to reproduce correctly the spin splittings, such as $m_{B^*} - m_B$, to order $1/m_Q$. These adjustments are essential, because in matrix elements the rest mass plays no role whatsoever.

In the usual HQET with $m_1 = 0$, the quark mass is added to $\bar{\Lambda}$ and the higher-order terms. Ambiguities of the HQET renormalization scheme, including those of infrared renormalons in the on-shell scheme, cancel in the sum. Similarly, the difference $m_2 - m_1$ can be added to Eq. (6.12): $M = M_1 + m_2 - m_1$. Adding the residual mass in this way has the virtue that $m_2 - m_1$ does not suffer from infrared ambiguities, even in the on-shell scheme.

VII. SEMILEPTONIC FORM FACTORS

Another interesting application of HQET is the heavy-quark expansion of form factors in the exclusive semileptonic decays $B \to D^* l\nu$ and $B \to Dl\nu$. These decays offer the most promising way to decrease the uncertainty in the CKM element $|V_{cb}|$, provided the hadronic matrix elements can be calculated reliably. Recent work [35,36] shows that calculations of the form factors at zero recoil with statistical errors of a few percent are feasible. The aim of this section is to describe the $1/m_Q$ and $1/m_Q^2$ contributions to the lattice observables calculated in Refs. [35,36], and compare them to the description of the form factors in the usual HQET. The technical details are in Appendix A, mostly following Refs. [33,34].

The transitions are mediated by the charged weak currents

$$\mathcal{V}^\mu = \bar{c} i\gamma^\mu b, \quad \mathcal{A}^\mu = \bar{c} i\gamma^\mu \gamma_5 b, \tag{7.1}$$

where $\bar{c}$ and b are conventionally normalized continuum quark fields. Currents in lattice gauge theory and in HQET are introduced below, but the symbols $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ are reserved for the physical currents. The hadronic part of the transitions involves the matrix elements $\langle D^{(*)}|\mathcal{V}^\mu|B\rangle$ and $\langle D^*|\mathcal{A}^\mu|B\rangle$. For $B \to Dl\nu$ there are two form factors h_+ and h_-. With the normalization (5.8) they are related to the matrix element by

$$\langle D(\boldsymbol{v}')|\mathcal{V}^\mu|B(\boldsymbol{v})\rangle = \tfrac{1}{2}(v'+v)^\mu h_+(w) - \tfrac{1}{2}(v'-v)^\mu h_-(w), \tag{7.2}$$

where $w = -v' \cdot v$. Zero recoil corresponds to $w = 1$. In Eq. (7.2) the final velocity is kept distinct from the initial velocity to be able to obtain $h_-(1)$. For $B \to D^* l\nu$ there are three axial form factors, defined by

$$\langle D^*(\boldsymbol{v}', \epsilon')|\mathcal{A}^\mu|B(\boldsymbol{v})\rangle = \tfrac{1}{2}(w+1) i\overline{\epsilon'}^\mu h_{A_1}(w) + \tfrac{1}{2} i\overline{\epsilon'}\cdot v\, v^\mu h_{A_2}(w) + \tfrac{1}{2} i\overline{\epsilon'}\cdot v\, v'^\mu h_{A_3}(w), \tag{7.3}$$

and a vector form factor, but at zero recoil the decay rate depends only on $h_{A_1}(1)$. For reasons that will become clear below, the zero-recoil matrix element

$$\langle D^*(\boldsymbol{v}, \epsilon')|\mathcal{V}^\mu|B^*(\boldsymbol{v}, \epsilon)\rangle = \overline{\epsilon'}\cdot \epsilon\, v^\mu h_1(1) \tag{7.4}$$

and its form factor $h_1(1)$ are also of interest.

Note that continuum QCD currents define the form factors. To generate the heavy-quark expansion of these form factors, one replaces the currents $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ with effective currents built from the heavy-quark fields and the fields of the light degrees of freedom. The effective currents and the heavy-quark Lagrangian are treated to the desired order in $1/m_Q$, and Eq. (5.4) should be used to generate the expansion, consistent to that order.

The zeroth order is simple and worth reviewing briefly. The QCD currents are related to HQET currents via

$$\mathcal{V}^\mu \doteq \eta_V \bar{c}_{v'} i\gamma^\mu b_v - \tfrac{1}{2}\beta_V (v'-v)^\mu \bar{c}_{v'} b_v - \tfrac{1}{2}\gamma_V (v'-v)_\nu \bar{c}_{v'} i\sigma^{\mu\nu} b_v, \tag{7.5}$$

$$\mathcal{A}^\mu \doteq \eta_A \bar{c}_{v'} i\gamma^\mu \gamma_5 b_v - \tfrac{1}{2}\beta_A (v'-v)^\mu \bar{c}_{v'} \gamma_5 b_v - \tfrac{1}{2}\gamma_A (v'-v)_\nu \bar{c}_{v'} i\sigma^{\mu\nu}\gamma_5 b_v, \tag{7.6}$$

where the symbol $\doteq$ means that the operators, though defined in different field theories, have the same matrix elements. The short-distance coefficients depend on the two masses; η_j and

γ_j are symmetric upon interchanging the masses ($j \in \{V, A\}$); β_j is anti-symmetric; at the tree level they satisfy $\eta_j = 1$, $\beta_j = \gamma_j = 0$. To obtain the leading heavy-quark expansion, one simply takes matrix elements of the effective currents in the states of the infinite-mass theory. From the trace formalism one finds

$$h_+(w) = \left[\eta_V + \tfrac{1}{2}(w-1)\gamma_V\right]\xi(w) + O(1/m_Q), \tag{7.7}$$

$$h_-(w) = \tfrac{1}{2}(w+1)\beta_V\xi(w) + O(1/m_Q), \tag{7.8}$$

$$h_1(w) = \left[\eta_V + \tfrac{1}{2}(w-1)\gamma_V\right]\xi(w) + O(1/m_Q), \tag{7.9}$$

$$h_{A_1}(w) = \eta_A\xi(w) + O(1/m_Q), \tag{7.10}$$

with a single HQET form factor $\xi(w)$, called the Isgur-Wise function. At zero recoil it is normalized by heavy-quark symmetry [8], so $\xi(1) = 1$. Therefore, the leading term in heavy-quark expansion is $h_+(1) = h_1(1) = \eta_V$, $h_-(1) = \beta_V$, and $h_{A_1}(1) = \eta_A$.

The $1/m_Q$ [29] and $1/m_Q^2$ [33,34] corrections to Eqs. (7.7)–(7.10) have been worked out with HQET. This section repeats the analysis through order $1/m_Q^2$ for the lattice approximants to the form factors introduced in Refs. [35,36]. The only crucial difference is that the short-distance coefficients are tracked carefully and their contributions are kept separate in the final results.

A. Lattice and HQET currents

To compute the form factors in Eqs. (7.2)–(7.4) with lattice gauge theory one introduces combinations of lattice fields with the same quantum numbers as $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$. The lattice currents are given by a series of dimension-three, -four, -five, etc., operators, with coefficients chosen to attain the right normalization and to reduce lattice artifacts. Several choices have been made in the literature, but with Wilson fermions they can all be described by HQET: the different choices simply have different short-distance coefficients.

Let $Z_{V^{cb}}V^\mu_{\text{lat}}$ ($Z_{A^{cb}}A^\mu_{\text{lat}}$) denote the lattice approximant to the charged $b \to c$ vector (axial-vector) current. To conform with much of the literature on lattice gauge theory, the current's normalization factor $Z_{j^{cb}}$ in shown explicitly. Then, suppressing the space-time index, the lattice currents are related to HQET currents via

$$Z_{V^{cb}}V_{\text{lat}} \doteq V^{(0)} + \sum_{s=1}\sum_{r=0}^{s} V^{(r,s-r)} \tag{7.11}$$

$$\doteq V^{(0)} + V^{(0,1)} + V^{(1,0)} + V^{(0,2)} + V^{(1,1)} + V^{(2,0)} + \cdots \tag{7.12}$$

and similarly for A^μ_{lat}. The HQET operator $V^{(r,s)}$ carries dimension $3 + r + s$. To make contact with the usual HQET, it is helpful to think of the dimensions being balanced by r powers of $1/m_c$ and s powers of $1/m_b$. The dimension-three vector current is

$$V^{(0)}_\mu = (\eta_V + \delta\eta^{\text{lat}}_{V_\mu})\bar{c}_{v'}i\gamma_\mu b_v - \tfrac{1}{2}\beta^{\text{lat}}_{V_\mu}(v'-v)_\mu\bar{c}_{v'}b_v - \tfrac{1}{2}\gamma^{\text{lat}}_{V_{\mu\nu}}(v'-v)^\nu\bar{c}_{v'}i\sigma_{\mu\nu}b_v. \tag{7.13}$$

In general the coefficients depend on the directional indices, because the lattice singles out the time direction. The overall factor Z_V is conventionally chosen so that $\delta\eta^{\text{lat}}_{V_0} = 0$. Then

$\delta\eta^{\rm lat}_{V_i}$ vanishes at the tree level, but not in general. As with the usual HQET $\beta^{\rm lat}_{V_\mu}$ and $\gamma^{\rm lat}_{V_{\mu\nu}}$ are, respectively, antisymmetric and symmetric upon interchange of heavy quark masses and both vanish at the tree level. The operator multiplying $\beta^{\rm lat}_{V_\mu}$ ($\gamma^{\rm lat}_{V_{\mu\nu}}$) makes a contribution at first (second) order in $v'-v$.

At dimension four and higher many operators arise, and a complete catalog requires a voluminous notation. Only the η-like terms are listed here. β-like terms are not needed until Sec. VII D, and γ-like terms are not needed at all. With this restriction, the dimension-four currents are

$$V_\mu^{(0,1)} = -\eta_V^{(0,1)} \frac{\bar{c}_{v'} i\gamma_\mu \not{D}_\perp b_v}{2m_{3b}}, \tag{7.14}$$

$$V_\mu^{(1,0)} = +\eta_V^{(1,0)} \frac{\bar{c}_{v'} \overleftarrow{\not{D}}_{\perp'} i\gamma_\mu b_v}{2m_{3c}}, \tag{7.15}$$

where $D_{\perp'} = D + v' \cdot D v'$. The notation $\eta_V^{(1,0)}/m_{3c}$ and $\eta_V^{(0,1)}/m_{3b}$ for the short-distance coefficients follows a helpful convention: for degenerate quarks the coefficient is merely $1/m_3$, which thus depends only on the indicated flavor; $\eta_V^{(r,s)}$ then describes the additional radiative corrections for non-degenerate masses. The dimension-five currents are

$$V_\mu^{(0,2)} = \eta_{VD_\perp^2}^{(0,2)} \frac{\bar{c}_{v'} i\gamma_\mu D_\perp^2 b_v}{8m_{D_\perp^2 b}^2} + \eta_{VsB}^{(0,2)} \frac{\bar{c}_{v'} i\gamma_\mu s^{\alpha\beta} B_{\alpha\beta} b_v}{8m_{sBb}^2} - \eta_{V\alpha E}^{(0,2)} \frac{\bar{c}_{v'} i\gamma_\mu i\not{E} b_v}{4m_{\alpha Eb}^2}, \tag{7.16}$$

$$V_\mu^{(2,0)} = \eta_{VD_\perp^2}^{(2,0)} \frac{\bar{c}_{v'} \overleftarrow{D}_{\perp'}^2 i\gamma_\mu b_v}{8m_{D_\perp^2 c}^2} + \eta_{VsB}^{(2,0)} \frac{\bar{c}_{v'} s^{\alpha\beta} B'_{\alpha\beta} i\gamma_\mu b_v}{8m_{sBc}^2} + \eta_{V\alpha E}^{(2,0)} \frac{\bar{c}_{v'} i\not{E}' i\gamma_\mu b_v}{4m_{\alpha Ec}^2}, \tag{7.17}$$

$$V_\mu^{(1,1)} = -z_{V1}^{(1,1)} \frac{\bar{c}_{v'} (\overleftarrow{\not{D}}_{\perp'} i\gamma_\mu \not{D}_\perp)_1 b_v}{2m_{3c}\, 2m_{3b}} - z_{Vs}^{(1,1)} \frac{\bar{c}_{v'} (\overleftarrow{\not{D}}_{\perp'} i\gamma_\mu \not{D}_\perp)_s b_v}{2m_{3c}\, 2m_{3b}}, \tag{7.18}$$

where again $1/m_{Xh}^2$ depends only on the indicated flavor and $\eta_V^{(r,s)}$ depends on both masses. The two coefficients $z_{V1}^{(1,1)}$ and $z_{Vs}^{(1,1)}$ multiply the spin-independent and spin-dependent part of the Dirac matrix structure. They do not reduce to 1 for equal masses, because $1/m_3$ is defined through the dimension-four currents, but for most choices of the lattice current they do equal 1 at the tree level.

B. At zero recoil: $h_+(1)$ and $h_1(1)$

The matrix elements that are to be described are

$$\langle D|Z_j j_{\rm lat}|B\rangle = \langle D_{v'}|j^{(0)}|B_v\rangle + \langle D_{v'}|j^{(1)}|B_v\rangle + \langle D_{v'}|j^{(2)}|B_v\rangle, \tag{7.19}$$

where j is V or A, and $j^{(1)} = j^{(0,1)} + j^{(1,0)}$, $j^{(2)} = j^{(0,2)} + j^{(1,1)} + j^{(2,0)}$. The first two matrix elements on the right-hand side of Eq. (7.19) must be expanded via Eq. (5.4) to second and first order in $\mathcal{L}_I$, respectively. There are, consequently, many HQET matrix elements to introduce. The matrix elements and their abbreviations, analogous to those in Sec. VI, are listed in Table I. The notation mostly follows previous work [33,34].

TABLE I. Notation for HQET matrix elements in Refs. [33,34] and this work.

contribution	Ref. [33]	Ref. [34]	this work
$\langle j^{(0)}\rangle$	$\xi(w)$	1	$\xi(w)$
$\langle j^{(1)}\rangle$	$\xi_{\pm,3}(w)$		$\xi_{\pm,3}(w)$
$\langle Tj^{(0)}\mathcal{O}_2\rangle^\star$	$A_1(w)$	χ_1	$A_1(w)$
$\langle Tj^{(0)}\mathcal{O}_B\rangle^\star$	$A_{2,3}(w)$	χ_3	$A_{2,3}(w)$
$\langle j^{(2)}\rangle$	$\phi_{0,\ldots,3}(w)$	$\lambda_{1,2}$	$\lambda_{1,\ldots,4}(w)$
$\langle Tj^{(1)}\mathcal{O}_2\rangle^\star$	$E_{1,2,3}(w)$; $E'_{1,2,3}(w)$		$\Xi_{2,3}(w)$; $F_{1,3}(w)$
$\langle Tj^{(1)}\mathcal{O}_B\rangle^\star$	$E_{4,\ldots,11}(w)$; $E'_{4,\ldots,11}(w)$		$\Xi_{4,\ldots,11}(w)$; $F_{4,\ldots,11}(w)$
$\langle Tj^{(0)}\mathcal{Q}_D\rangle^\star$	$2B_1(w)$	$2\Xi_1$	λ_1
$\langle Tj^{(0)}\mathcal{Q}_E\rangle^\star$	$2B_{2,3}(w)$	$2\Xi_3$	λ_2
$\langle T\mathcal{O}_2j^{(0)}\mathcal{O}_2\rangle^\star$	$D_1(w)$	D	D
$\langle T\mathcal{O}_2j^{(0)}\mathcal{O}_B\rangle^\star$	$D_{2,3}(w)$	E	E
$\langle T\mathcal{O}_Bj^{(0)}\mathcal{O}_B\rangle^\star$	$D_{4,\ldots,10}(w)$	$R_{1,2}$	$R_{1,2}$
$\langle Tj^{(0)}\mathcal{O}_2\mathcal{O}_2\rangle^\star$	$C_1(w)$	A	$A=-\frac{1}{2}D$
$\langle Tj^{(0)}\mathcal{O}_2\mathcal{O}_B\rangle^\star$	$C_{2,3}(w)$	B	$B=-E$
$\langle Tj^{(0)}\mathcal{O}_B\mathcal{O}_B\rangle^\star$	$C_{4,\ldots,12}(w)$	$C_{1,3}$	$C_{1,3}=-\frac{1}{2}R_{1,2}$

One can work out the matrix elements using the trace formalism. At zero recoil $\langle D_v|j^{(1)}|B_v\rangle$ vanishes. For the vector-current transitions $B\to D$ and $B^*\to D^*$ with $\boldsymbol{v}=\boldsymbol{v}'=\boldsymbol{0}$ one finds

$$\langle D^{(*)}|Z_{V^{cb}}V^0_{\mathrm{lat}}|B^{(*)}\rangle=\eta_V W^{(0)}_{JJ}+W^{(2)}_{JJ}, \tag{7.20}$$

in which $\langle D_v|V^{(0)}|B_v\rangle$ yields

$$\begin{aligned} W^{(0)}_{JJ}=1&+\left(\frac{1}{4m_{2c}^2}+\frac{1}{4m_{2b}^2}\right)A+\left(\frac{1}{2m_{2c}}\frac{1}{2m_{Bc}}+\frac{1}{2m_{2b}}\frac{1}{2m_{Bb}}\right)d_JB\\ &+\left(\frac{1}{4m_{Bc}^2}+\frac{1}{4m_{Bb}^2}\right)[C_1+d_JC_3]+\frac{1}{2m_{2c}}\frac{1}{2m_{2b}}D\\ &+\left(\frac{1}{2m_{Bc}}\frac{1}{2m_{2b}}+\frac{1}{2m_{2c}}\frac{1}{2m_{Bb}}\right)d_JE+\frac{1}{2m_{Bc}}\frac{1}{2m_{Bb}}[R_1+d_JR_2], \end{aligned} \tag{7.21}$$

and $\langle D_v|V^{(2)}|B_v\rangle$ yields

$$W^{(2)}_{JJ}=\left(\frac{\eta^{(2,0)}_{VD_\perp^2}}{8m^2_{D_\perp^2c}}+\frac{\eta^{(0,2)}_{VD_\perp^2}}{8m^2_{D_\perp^2b}}-\frac{z^{(1,1)}_{V1}}{2m_{3c}\,2m_{3b}}\right)\lambda_1+\left(\frac{\eta^{(2,0)}_{VsB}}{8m^2_{sBc}}+\frac{\eta^{(0,2)}_{VsB}}{8m^2_{sBb}}-\frac{z^{(1,1)}_{Vs}}{2m_{3c}\,2m_{3b}}\right)d_J\lambda_2. \tag{7.22}$$

The subscript JJ' denotes the initial and final spins, although here $J'=J$. The spin factor $d_0=3$ for $B\to D$ and $d_1=-1$ for $B^*\to D^*$. The coefficient factors reveal the origin of the contribution. By heavy-quark symmetry λ_1 and λ_2 are exactly the same as in Sec. VI, and A, B, C_1, C_3, D, E, R_1, and R_2 are new constants parametrizing the light degrees of freedom, introduced in Appendix A and the last six rows of Table I.

Equation (7.20) gives lattice approximants to the form factors $h_+(1)$ and $h_1(1)$. One striking feature of Eqs. (7.20)–(7.22) is that there are no contributions of order $1/m_Q$. For continuum QCD, this is known as Luke's theorem [29]. Matrix elements of $j^{(1)}$ in the infinite-mass states contribute only when $v' \neq v$, so a single power of $1/m_3$ does not appear in Eq. (7.20). As shown in Appendix A 1 b, terms with a single power of $1/m_2$ and $1/m_B$ are absent as a consequence of heavy-quark symmetry and Eq. (5.4). Thus, Luke's theorem holds for lattice QCD also.

At order $1/m_Q^2$, matrix elements that might have multiplied $1/m_D^2$ and $1/m_E^2$ also vanish; so do matrix elements involving four-quark operators. Furthermore, the parameters A, B, C_1, and C_3 can be eliminated, as indicated in the right-most column of the last three rows of Table I. As shown in Appendix A 1 e, this is another consequence of heavy-quark symmetry and Eq. (5.4). Taking these relations into account

$$W_{JJ}^{(0)} = 1 - \tfrac{1}{2}\Delta_2^2 D - \Delta_2\Delta_B d_J E - \tfrac{1}{2}\Delta_B^2[R_1 + d_J R_2], \tag{7.23}$$

where

$$\Delta_X = \frac{1}{2m_{Xc}} - \frac{1}{2m_{Xb}}. \tag{7.24}$$

So $W_{JJ}^{(0)}$ is correctly reproduced if $1/m_2$ and $1/m_B$ are adjusted to their continuum values, in particular if the analysis identifies m_2 with the heavy quark mass.

Another lattice approximant to $h_+(1)$ and $h_1(1)$ is given by double ratios introduced in Refs. [35,36]

$$R_+ = \frac{\langle D|V_{\rm lat}^0|B\rangle\langle B|V_{\rm lat}^0|D\rangle}{\langle D|V_{\rm lat}^0|D\rangle\langle B|V_{\rm lat}^0|B\rangle}, \tag{7.25}$$

$$R_1 = \frac{\langle D^*|V_{\rm lat}^0|B^*\rangle\langle B^*|V_{\rm lat}^0|D^*\rangle}{\langle D^*|V_{\rm lat}^0|D^*\rangle\langle B^*|V_{\rm lat}^0|B^*\rangle}. \tag{7.26}$$

Then $|h_{+,1}(1)|^2$ are approximated by $\rho_{V_0}^2 R_{+,1}$, where $\rho_{V_0}^2 = Z_{V^{cb}}Z_{V^{bc}}/Z_{V^{cc}}Z_{V^{bb}}$. To see the advantage of the double ratios, let us rewrite $W_{JJ}^{(2)} = \bar{W}_{JJ}^{(2)} + \delta W_{JJ}^{(2)}$,

$$\bar{W}_{JJ}^{(2)} = \tfrac{1}{2}\Delta_3^2\left[z_{V1}^{(1,1)}\lambda_1 + z_{Vs}^{(1,1)}d_J\lambda_2\right], \tag{7.27}$$

$$\begin{aligned}\delta W_{JJ}^{(2)} = {}& \left(\frac{\eta_{VD_\perp^2}^{(2,0)}}{8m_{D_\perp^2 c}^2} + \frac{\eta_{VD_\perp^2}^{(0,2)}}{8m_{D_\perp^2 b}^2} - \frac{z_{V1}^{(1,1)}}{8m_{3c}^2} - \frac{z_{V1}^{(1,1)}}{8m_{3b}^2}\right)\lambda_1 \\ & + \left(\frac{\eta_{VsB}^{(2,0)}}{8m_{sBc}^2} + \frac{\eta_{VsB}^{(0,2)}}{8m_{sBb}^2} - \frac{z_{Vs}^{(1,1)}}{8m_{3c}^2} - \frac{z_{Vs}^{(1,1)}}{8m_{3b}^2}\right) d_J\lambda_2.\end{aligned} \tag{7.28}$$

From Eq. (7.20) and the definitions one finds

$$\rho_{V_0}\sqrt{R_{+,1}} = \eta_V W_{JJ}^{(0)} + \bar{W}_{JJ}^{(2)} + O((\eta_V^{(r,s)} - 1)/m_Q^2). \tag{7.29}$$

The contribution of $\delta W_{JJ}^{(2)}$, which stems from the dimension-five currents, largely cancels. Hence, the double ratios depend most strongly on $1/m_2$, $1/m_B$, and $1/m_3$, namely the coefficients in $\mathcal{L}^{(1)}$ and $V_\mu^{(1)}$.

Equation (7.29) is an important practical result. If one tolerates errors of order α_s/m_Q^2 and $1/m_Q^3$, then $\rho_{V_0}\sqrt{R_{+,1}}$ only requires $m_2 = m_B = m_3$ and $z_V^{(1,1)} = 1$ at the tree level, and details of the currents $V^{(0,2)}$ and $V^{(2,0)}$ do not matter at all. With the widely used Sheikholeslami-Wohlert action [5], this accuracy is easy to arrange [1,35]. In practice an error comes also from η_V and ρ_{V_0}, which are available only to two loops [37] and one loop [38], respectively. So the recent result [35] for $h_+(1)$ has a heavy-quark discretization effect of order α_s^2, which could be reduced by calculating ρ_{V_0} to two loops.

C. At zero recoil: $h_{A_1}(1)$

To obtain lattice approximants to $h_{A_1}(1)$ one must work out Eq. (7.19) for a $B \to D^*$ transition mediated by the axial current. In HQET the currents are as in Eqs. (7.13)–(7.18) with a factor γ_5 inserted in the obvious places. In this case, the overall factor $Z_{A^{cb}}$ is conventionally chosen so that $\delta\eta^{\text{lat}}_{A_i^{cb}} = 0$.

A useful matrix element has the D^* spin is aligned along the i direction and $\boldsymbol{v} = \boldsymbol{v}' = \mathbf{0}$. One finds

$$\langle D^*|Z_{A^{cb}}A^i_{\text{lat}}|B\rangle = \eta_{A^{cb}}W_{01}^{(0)} + \bar{W}_{01}^{(2)} + \delta W_{01}^{(2)} \tag{7.30}$$

in which $\langle D_v^*|A^{(0)}|B_v\rangle$ yields—after eliminating A, B, C_1, and C_3—

$$W_{01}^{(0)} = 1 - \tfrac{1}{2}\Delta_2(\Delta_2 D - 2\Theta_B E) - \tfrac{1}{2}\Delta_B(\Delta_B R_1 - \Theta_B R_2) - \frac{1}{2m_{Bc}\,2m_{Bb}}\left(\tfrac{4}{3}R_1 + 2R_2\right) \tag{7.31}$$

with

$$\Theta_X = \frac{1}{2m_{Xc}} + \frac{3}{2m_{Xb}}. \tag{7.32}$$

As before, the zero-recoil matrix element does not depend on the dimension-six Lagrangian. The matrix element $\langle D_v^*|A^{(2)}|B_v\rangle$ of the dimension-five current yields

$$\bar{W}_{01}^{(2)} = \left(\tfrac{1}{2}\Delta_3^2 + \tfrac{4}{3}\frac{1}{2m_{3c}2m_{3b}}\right) z^{(1,1)}_{A^{cb}1}\lambda_1 - \left(\tfrac{1}{2}\Delta_3\Theta_3 - 2\frac{1}{2m_{3c}2m_{3b}}\right) z^{(1,1)}_{A^{cb}s}\lambda_2, \tag{7.33}$$

$$\begin{aligned}\delta W_{01}^{(2)} = {} & \left(\frac{\eta^{(2,0)}_{A^{cb}D_\perp^2}}{8m^2_{D_\perp^2 c}} + \frac{\eta^{(0,2)}_{A^{cb}D_\perp^2}}{8m^2_{D_\perp^2 b}} - \frac{z^{(1,1)}_{A^{cb}1}}{8m^2_{3c}} - \frac{z^{(1,1)}_{A^{cb}1}}{8m^2_{3b}}\right)\lambda_1 \\ & - \left(\frac{\eta^{(2,0)}_{A^{cb}sB}}{8m^2_{sBc}} - \frac{3\eta^{(0,2)}_{A^{cb}sB}}{8m^2_{sBb}} - \frac{z^{(1,1)}_{A^{cb}s}}{8m^2_{3c}} + \frac{3z^{(1,1)}_{A^{cb}s}}{8m^2_{3b}}\right)\lambda_2,\end{aligned} \tag{7.34}$$

after grouping terms as in Eqs. (7.27) and (7.28).

Reference [36] introduces a third double ratio

$$R_{A_1} = \frac{\langle D^*|A^i_{\text{lat}}|B\rangle\langle B^*|A^i_{\text{lat}}|D\rangle}{\langle D^*|A^i_{\text{lat}}|D\rangle\langle B^*|A^i_{\text{lat}}|B\rangle}. \tag{7.35}$$

After substituting for each matrix element the foregoing expressions one finds

$$\rho_A\sqrt{R_{A_1}} = \check{\eta}_{A^{cb}}\check{W}_{01}^{(0)} + \check{W}_{01}^{(2)} + O((\check{\eta}_A^{(r,s)}-1)/m_Q^2), \tag{7.36}$$

where $\rho_A^2 = Z_{A^{cb}}Z_{A^{bc}}/Z_{A^{cc}}Z_{A^{bb}}$, $\check{\eta}_{A^{cb}}^2 = \eta_{A^{cb}}\eta_{A^{bc}}/\eta_{A^{cc}}\eta_{A^{bb}}$, and

$$\begin{aligned} \check{W}_{01}^{(0)} &= 1 - \tfrac{1}{2}\Delta_2^2 D - \Delta_2\Delta_B E + \tfrac{1}{6}\Delta_B^2(R_1+3R_2) \\ \check{W}_{01}^{(0)} &= -\tfrac{1}{6}\Delta_3^2(\check{z}_{A^{cb}1}^{(1,1)}\lambda_1 + 3\check{z}_{A^{cb}s}^{(1,1)}\lambda_2), \end{aligned} \tag{7.37}$$

where $\check{z}_{A^{cb}} = \check{\eta}_{A^{cb}}z_{A^{cb}}/\eta_{A^{cb}}$. As before the contribution $\delta W_{01}^{(2)}$ of the dimension-five currents largely cancels and the double ratio depends most strongly on $1/m_2$, $1/m_B$, and $1/m_3$, namely the coefficients in $\mathcal{L}^{(1)}$ and $A_\mu^{(1)}$.

Note, however, that $\rho_A\sqrt{R_{A_1}}$ does not yield $h_{A_1}(1)$ but

$$\rho_A^2 R_{A_1} = \frac{h_{A_1}^{B\to D^*}(1)h_{A_1}^{D\to B^*}(1)}{h_{A_1}^{D\to D^*}(1)h_{A_1}^{B\to B^*}(1)}. \tag{7.38}$$

Nevertheless, if the action and currents are tuned so that $1/m_2$, $1/m_B$, $1/m_3$, and $z_j^{(1,1)}$ match the usual HQET (to a desired accuracy), the three double ratios R_+, R_1, and R_{A_1} can be combined to yield the $1/m_Q^2$ contribution to $h_{A_1}(1)$. For example, if one tolerates errors of order α_s/m_Q^2, as well as $1/m_Q^3$, one only requires $m_2 = m_B = m_3$ at the tree level, and one may set $z_V^{(1,1)} = \check{z}_A^{(1,1)} = 1$. Then, dropping the distinction between m_2, m_B, and m_3, the double ratios are

$$\bar{\rho}_V^2 R_+ = 1 - 2\Delta^2\ell_P, \tag{7.39}$$

$$\bar{\rho}_V^2 R_1 = 1 - 2\Delta^2\ell_V, \tag{7.40}$$

$$\bar{\rho}_A^2 R_{A_1} = 1 - \Delta^2(\ell_P + \ell_V + \ell_A), \tag{7.41}$$

where $\bar{\rho}_V = \rho_V/\eta_V$, $\bar{\rho}_A = \rho_A/\check{\eta}_A$, and

$$2\ell_P = D + R_1 - \lambda_1 + 3(2E + R_2 - \lambda_2) \tag{7.42}$$

$$2\ell_V = D + R_1 - \lambda_1 - (2E + R_2 - \lambda_2) \tag{7.43}$$

$$\ell_A = \tfrac{4}{3}(\lambda_1 - R_1) + 2(\lambda_2 - R_2). \tag{7.44}$$

In the approximation being considered the desired form factor is

$$\frac{h_{A_1}(1)}{\eta_A} = 1 - \Delta\left(\frac{\ell_V}{2m_c} - \frac{\ell_P}{2m_b}\right) + \frac{\ell_A}{2m_c\,2m_b}. \tag{7.45}$$

By fitting Eqs. (7.39)–(7.41) one can extract ℓ_P, ℓ_V, and $\ell_P+\ell_V+\ell_A$, and then one has the information necessary to reconstitute $h_{A_1}(1)$. As with $h_+(1)$ there are, in practice, further errors because η_A and ρ_A are available only at finite-loop order [37,38].

D. Near zero recoil: $h_-(1)$

To extract $h_-(1)$ matrix elements with non-zero velocity transfer are needed, and some new features appear in the analysis. For example, lattice approximants to h_- receive a contribution from the term in $V^{(0)}$ proportional to $\beta^{\rm lat}_{V_\mu}$. Similar "β-like" terms omitted from Eqs. (7.14)–(7.18) also make contributions. We shall not write out all these terms but indicate instead how they contribute to matrix elements.

Suppose one extracts the form factor from a matrix element with $\boldsymbol{v}' = -\boldsymbol{v}$, pointing in the i direction. Then

$$\langle D(-\boldsymbol{v})|V_i|B(\boldsymbol{v})\rangle = v_i \left\{\beta^{\rm lat}_{V_i} W^{(0)}_{00} + Y^{(2)}_{00} + X^{(1)}_{00} + X^{(2)}_{00}\right\}, \tag{7.46}$$

where $W^{(0)}_{00}$ is given in Eq. (7.23), and $Y^{(2)}_{00}$ is like $W^{(2)}_{00}$ but with β-like coefficients replacing $\eta^{(2,0)}_V$, $\eta^{(0,2)}_V$, and $z^{(1,1)}_V$. The expression in braces is a lattice approximant to $h_-(w)$. For infinitesimal v_i the matrix element $\langle D_{v'}|V^{(1)}|B_v\rangle$ yields

$$\begin{aligned} X^{(1)}_{00} = {} & \left(\frac{\eta^{(1,0)}_V}{2m_{3c}} - \frac{\eta^{(0,1)}_V}{2m_{3b}}\right)\left[2\xi_3(1) - \bar\Lambda + 2\Sigma_2\Xi_3(1) + 2\Sigma_B\Xi_-(1)\right] \\ & - \left(\frac{\eta^{(1,0)}_V}{2m_{3c}} + \frac{\eta^{(0,1)}_V}{2m_{3b}}\right)\left[\Delta_2\tilde\phi_0(1) + \Delta_B\tilde\phi_-(1)\right], \end{aligned} \tag{7.47}$$

with Δ_2, Δ_B as in Eq. (7.24) and

$$\Sigma_X = \frac{1}{2m_{Xc}} + \frac{1}{2m_{Xb}}. \tag{7.48}$$

The chromoelectric part of $\langle D_{v'}|V^{(2)}|B_v\rangle$ yields

$$X^{(2)}_{00} = \left(\frac{\eta^{(2,0)}_{V\alpha E}}{4m^2_{\alpha Ec}} - \frac{\eta^{(0,2)}_{V\alpha E}}{4m^2_{\alpha Eb}}\right)\frac{2}{3}\left[\lambda_1 + 3\lambda_2\right]. \tag{7.49}$$

The coefficient factors, together with Table I, make clear the origin of each term. The infinite-mass matrix elements $\bar\Lambda$, λ_1, and λ_2 are exactly those introduced earlier, and the new ones $\xi_3(1)$, $\Xi_3(1)$, $\Xi_-(1)$, $\tilde\phi_0(1)$, and $\tilde\phi_-(1)$ are introduced in Appendix A 2. As before, the dimension-six effective Lagrangian drops out, but the dimension-five currents contribute in several places: in $W^{(0)}_{00}$, $Y^{(2)}_{00}$, and $X^{(2)}_{00}$. Further operators $-i\bar c_{v'} D^\mu_\perp b_v$, $i\bar c_{v'} \overleftarrow{D}^\mu_{\perp'} b_v$, $\bar c_{v'} E^\mu b_v$, and $\bar c_{v'} E'^\mu b_v$, whose coefficients vanish at the tree level, modify the short-distance coefficients of $\bar\Lambda$ in $X^{(1)}_{00}$ and of λ_1 in $X^{(2)}_{00}$. Thus, many short-distance coefficients influence the accuracy of Eq. (7.46).

The main drawback of Eq. (7.46) is, however, the requirement $\boldsymbol{v}' = -\boldsymbol{v}$ for hadrons of unequal mass. Numerical calculations employ a finite volume and, hence, discrete momentum. Moreover, with the many "masses" the relation between momentum and velocity is not plain. To remove these ambiguities Ref. [35] introduced another double ratio

$$R_- = \frac{\langle D|V^i_{\rm lat}|B\rangle}{\langle D|V^0_{\rm lat}|B\rangle}\frac{\langle D|V^0_{\rm lat}|D\rangle}{\langle D|V^i_{\rm lat}|D\rangle}. \tag{7.50}$$

In the spatial matrix elements, the initial state is at rest and the final state has a small velocity in the i direction; in the temporal matrix elements, initial and final states both are at rest. In continuum QCD, the analogous first ratio is [*cf.* Eq. (7.2)]

$$\frac{\langle D|\mathcal{V}^i|B\rangle}{\langle D|\mathcal{V}^0|B\rangle} = \tfrac{1}{2}v'_i\left[1 - \frac{h_-(1)}{h_+(1)}\right], \tag{7.51}$$

to first order in v'_i, and the second is

$$\frac{\langle D|\mathcal{V}^i|D\rangle}{\langle D|\mathcal{V}^0|D\rangle} = \tfrac{1}{2}v'_i, \tag{7.52}$$

because in the elastic case $h_+(1) = 1$ and $h_-(w) = 0$. Thus, with a suitable adjustment of the lattice currents, one can use R_- to obtain a lattice approximant to $h_-(1)/h_+(1)$.

In the double ratio of lattice currents the (mass-dependent) factors Z_V cancel. With the results of Appendix A 2, and noting that $\eta_{V^{cc}} = 1$ and $\beta^{\text{lat}}_{V_i^{cc}} = 0$,

$$R_- = \frac{\eta_{V^{cb}} + \delta\eta^{\text{lat}}_{V_i^{cb}} - [\beta^{\text{lat}}_{V_i^{cb}} + Y^{(2)}_{00} + X^{(1)}_{00} + X^{(2)}_{00}] + \delta\bar{\eta}^{(2)}_{V^{cb}}W^{(2)}_{00}}{\eta_{V^{cb}}(1 + \delta\eta^{\text{lat}}_{V_i^{cc}} + \delta\bar{\eta}^{(2)}_{V^{cc}}W^{(2)}_{00})} + O(1/m_Q^3), \tag{7.53}$$

where $\delta\bar{\eta}^{(2)}_{V^{cb}}$ is a combination of $\delta\eta^{(r,s)}_{V_i}/\eta^{(r,s)}_V$ and $(\delta\eta^{\text{lat}}_{V_i} - \beta^{\text{lat}}_{V_i})/\eta_V$. Here $W^{(2)}_{00}$, $X^{(s)}_{00}$, and $Y^{(2)}_{00}$ are precisely as above, though in the denominator $W^{(2)}_{00}$ is evaluated with flavor c in both final and initial states. As with the other double ratios, one would like to extract the long-distance information from R_-. To do so one must have a way to calculate the short-distance coefficients, either to adjust them so $\beta^{\text{lat}}_{V_i} = \beta_V$ and $\delta\eta^{\text{lat}}_{V_i} = \delta\bar{\eta}^{(2)}_V = 0$, or to constrain a fit.

A simple version of the latter strategy is available if one tolerates errors in $h_-(1)$ of order α_s/m_Q and α_s/m_Q^2. Then it is enough to adjust $m_3 = m_2 = m_B$ at the tree level, one may set $\eta^{(r,0)}_V = \eta^{(0,s)}_V = 1$, and one may neglect $Y^{(2)}_{00}$ and $\delta\bar{\eta}^{(2)}_V W^{(2)}_{00}$. In the approximation and hand, Eq. (7.53) can be rearranged to yield

$$\eta_{V^{cb}}[1 - (1 + \delta\eta^{\text{lat}}_{V_i^{cc}})R_-] + \delta\eta^{\text{lat}}_{V_i^{cb}} - \beta^{\text{lat}}_{V_i^{cb}} = X^{(1)}_{00} + X^{(2)}_{00}. \tag{7.54}$$

Setting $m_3 = m_2 = m_B$, but keeping $m_{\alpha E}$ distinct, Eqs. (7.47) and. (7.49) yield

$$X^{(1)}_{00} + X^{(2)}_{00} = \Delta\ell^{(1)}_- + \Delta\Sigma\ell^{(2)}_- + \tfrac{2}{3}\Delta_{\alpha E}\Sigma_{\alpha E}[\lambda_1 + 3\lambda_2], \tag{7.55}$$

where

$$\ell^{(1)}_- = 2\xi_3(1) - \bar{\Lambda}, \tag{7.56}$$

$$\ell^{(2)}_- = 2\Xi_3(1) + 2\Xi_-(1) - \tilde{\phi}_0(1) - \tilde{\phi}_-(1). \tag{7.57}$$

One may fit the left-hand side of Eq. (7.54) to the right-hand side of Eq. (7.55) with $1/m^2_{\alpha E}$ at the easily obtained tree level. After the fit one may reconstitute $h_-(1)$ from

$$\frac{h_-(1)}{h_+(1)} = \beta_V + \Delta\ell^{(1)}_- + \Delta\Sigma\left[\ell^{(2)}_- + \tfrac{2}{3}(\lambda_1 + 3\lambda_2)\right]. \tag{7.58}$$

In practice, there are also errors of order α_s^n because the coefficients η_V, β_V, $\delta\eta^{\text{lat}}_V$, and β^{lat}_V are available only to a finite loop order. Note that the matrix element $\lambda_1 + 3\lambda_2$ appears also as the $1/m_Q$ correction to the pseudoscalar meson mass, *cf.* Eq. (6.12), so a simultaneous fit may turn out to be useful.

VIII. LEPTONIC DECAYS

A straightforward application of the trace formalism gives the first-order heavy-quark expansion of the matrix element in leptonic decays. The result for lattice QCD is in Ref. [39], but for completeness the derivation is given here.

With the states normalized as in Eq. (5.8), the QCD amplitudes appearing in leptonic decays of heavy-light pseudoscalar and vector mesons can be written

$$\langle 0|\mathcal{A}^\mu|H(v)\rangle = iv^\mu \phi_H/\sqrt{2}, \tag{8.1}$$

$$\langle 0|\mathcal{V}^\mu|H^*(v,\epsilon)\rangle = \epsilon^\mu \phi_{H^*}/\sqrt{2}, \tag{8.2}$$

where $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ are now the vector and axial vector currents with a light and a heavy quark, and H (H^*) is the pseudoscalar (vector) meson with heavy flavor h. The relation between the parameter ϕ_H and the conventional pseudoscalar meson decay constant is

$$\phi_H = f_H\sqrt{M_H}. \tag{8.3}$$

There are several conventions for defining the vector meson decay constant, but only ϕ_{H^*} is considered here.

In lattice gauge theory the decay constants are approximated with matrix elements of lattice currents $Z_{V^{qh}}V^{qh}$ and $Z_{A^{qh}}A^{qh}$ with the same quantum numbers as $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$. As before, they are not made explicit, to allow for a variety of choices. The underlying currents are described by HQET currents,

$$Z_{V^{qh}}V^{qh}_\mu \doteq \eta_{V^{qh}}\bar{q}i\gamma_\mu h_v + \zeta_{V^{qh}}v_\mu\bar{q}h_v - \frac{\eta^{(0,1)}_{V^{qh}}}{2m_3}\bar{q}i\gamma_\mu \not{D}_\perp h_v + \cdots \tag{8.4}$$

$$Z_{A^{qh}}A^{qh}_\mu \doteq \eta_{A^{qh}}\bar{q}i\gamma_\mu\gamma_5 h_v + \zeta_{A^{qh}}v^\nu\bar{q}i\sigma_{\mu\nu}\gamma_5 h_v - \frac{\eta^{(0,1)}_{A^{qh}}}{2m_3}\bar{q}i\gamma_\mu\gamma_5 \not{D}_\perp h_v + \cdots \tag{8.5}$$

where $\bar{q}$ is a light anti-quark field. The coefficient $1/2m_3$ is defined through the degenerate-mass heavy-heavy vector current, and $\eta^{(0,1)}_j$ captures the remaining radiative corrections. At the tree level $\eta^{(0,1)}_j = 1$. The coefficients ζ_j vanish at the tree level, and the operators that they multiply do not affect $\phi_{H^{(*)}}$. Additional dimension-four operators, whose coefficients vanish at the tree level, are not written out.

The static limit is given by the matrix element of the first term of the HQET currents:

$$\begin{aligned}\langle 0|\bar{q}i\Gamma_\mu h_v|h_v J\rangle &= -\tfrac{1}{2}\phi_\infty \operatorname{tr}\left[i\Gamma_\mu \mathcal{M}_J\right] \\ &= i\omega_\mu\phi_\infty/\sqrt{2},\end{aligned} \tag{8.6}$$

where $\Gamma_\mu = \gamma_\mu\gamma_5$ or γ_μ and $\omega_\mu = v_\mu$ or $-i\epsilon_\mu$, for $J=0$ or 1. The constant $\phi_\infty/2$ is introduced to parametrize the light degrees of freedom; in the static limit, $\phi_H = \phi_{H^*} = \phi_\infty$. As with the quantities introduced in Secs. VI and VII, ϕ_∞ differs from its continuum limit, but the difference stems only from the light degrees of freedom.

At order $1/m_Q$ there are three contributions to $\phi_{H^{(*)}}$, from the kinetic and chromomagnetic energy, and from the correction to the current. They take the form

$$\langle 0|Z_j j_\mu|H^{(*)}\rangle = \eta_j \langle 0|\bar{q}i\Gamma_\mu h_v|h_v J\rangle + \frac{\eta_j}{2m_2}\int d^4x\,\langle 0|T\,\bar{q}i\Gamma_\mu h_v(0)\mathcal{O}_2(x)|h_v J\rangle^\star + \frac{\eta_j}{2m_B}\int d^4x\,\langle 0|T\,\bar{q}i\Gamma_\mu h_v(0)\mathcal{O}_B(x)|h_v J\rangle^\star - \frac{\eta_j^{(0,1)}}{2m_3}\langle 0|\bar{q}i\Gamma_\mu \not{D}_\perp h_v|h_v J\rangle. \tag{8.7}$$

Spin-dependent factors may be obtained with the trace formalism. One has

$$\int d^4x\,\langle 0|T\bar{q}i\Gamma_\mu h_v\,\mathcal{O}_2(x)|h_v J\rangle = -\tfrac{1}{2}(-\phi_\infty A_2)\,\mathrm{tr}\,[i\Gamma_\mu \mathcal{M}_J] = -\frac{\phi_\infty A_2}{\sqrt{2}} i\omega_\mu \tag{8.8}$$

$$\int d^4x\,\langle 0|T\bar{q}i\Gamma_\mu h_v\,\mathcal{O}_B(x)|h_v J\rangle = -\tfrac{1}{12}(-\phi_\infty A_B)\,\mathrm{tr}\,[i\Gamma_\mu \sigma^{\rho\sigma}\mathcal{M}_J\sigma_{\alpha\beta}]\,\eta^\alpha_\rho\eta^\beta_\sigma = -\frac{d_J\phi_\infty A_B}{3\sqrt{2}} i\omega_\mu \tag{8.9}$$

$$\langle 0|\bar{q}i\Gamma_\mu \not{D}_\perp h_v|h_v J\rangle = -\tfrac{1}{2}\phi_\infty A_3\,\mathrm{tr}\,[i\Gamma_\mu \gamma_\perp^\alpha \mathcal{M}_J\gamma_\alpha] = +\frac{d_J\phi_\infty A_3}{\sqrt{2}} i\omega_\mu \tag{8.10}$$

where A_2, A_B, and A_3 parametrize the light degrees of freedom, and $d_H = 3$, $d_{H^*} = -1$. Combining the equations of motion and heavy-quark symmetry,

$$A_3 = \tfrac{1}{3}(\bar{\Lambda} - m_q), \tag{8.11}$$

where m_q is the mass of the light quark [21]. Combining Eqs. (8.6)–(8.11)

$$\phi_{H^{(*)}} = \phi_\infty \left[\eta_j\left(1 - \frac{A_2}{2m_2} - \frac{d_J}{3}\frac{A_B}{2m_B}\right) - \eta_j^{(0,1)}\frac{d_J}{3}\frac{\bar{\Lambda} - m_q}{2m_3}\right]. \tag{8.12}$$

As expected on the general grounds outlined in Sec. IV, the rest mass does not appear. Previously this had been shown only by explicit calculation [21]. Like the mass formula (6.12), this result is simple enough that it could have been written down upon inspection of the corresponding continuum formula [40,21].

To obtain the correct static limit of the decay constants, one must adjust the normalization factors Z_j to yield η_j in the leading terms. This is known at the one-loop level for the Wilson [41] and Sheikholeslami-Wohlert actions [42]. Similarly, to obtain the $1/m_Q$ corrections, one must adjust the lattice action and currents so that $m_2 = m_B = m_3 = m_Q$, which is easy at the tree level. With these choices, Eq. (8.12) predicts that the heavy-light decay constants should depend mildly on the lattice spacing. Explicit calculation supports this prediction [43,44]. On the other hand, when not all these choices are made, the dependence on the lattice spacing could be more pronounced, because then $1/m_3$ or $1/m_B$ could vary rapidly with $m_Q a$. Explicit calculation supports this prediction too [45].

IX. DISCUSSION AND CONCLUSIONS

Two themes run through Symanzik's application of effective field theory to the study of cutoff effects. The first is descriptive [2]. The local effective Lagrangian organizes deviations from the continuum limit through a series of higher-dimension operators, multiplied with certain coefficients. When the higher-dimension terms are small, they can be treated as perturbations, and their influence can be propagated from the effective Lagrangian to physical quantities. The second theme turns the description into a weapon [16]. Details of the underlying lattice action alter the effective Lagrangian only via the short-distance coefficients. If a given action leads to a reduced (or vanishing) coefficient, then the process independence of the coefficient guarantees that its associated operator has a reduced (or vanishing) effect on all observables.

The two themes also run through the application of HQET to lattice QCD. The concrete results—the expansions given in Eq. (6.12), Eqs. (7.20)–(7.28), Eqs. (7.30)–(7.34), Eqs. (7.46)–(7.49), and Eq. (8.12)—describe the deviations from the static limit of the mass, semileptonic form factors, and decay constant of heavy-light mesons. These descriptions hold, as always in HQET, when momentum transfers are much smaller than the heavy quark mass(es). Details of the lattice alter the validity of the description superficially: they merely change the short-distance coefficients. On the other hand, the details alter the utility of the description greatly: if a coefficient is tuned correctly, to some accuracy, in one observable, then its associated operator contributes correctly, to that accuracy, in all observables. In all examples, one sees that the leading $1/m_Q$ dependence is reproduced correctly if the short-distance coefficients $1/m_2$, $1/m_B$, and $1/m_3$ are adjusted correctly. These conditions can be obtained, respectively, through suitable adjustments of the bare mass, of the "clover" coupling in the Sheikholeslami-Wohlert action, and of a tunable parameter in the current.

It may be worthwhile to contrast the formalism developed here with other methods for treating heavy quarks in lattice gauge theory. One approach is to derive HQET or NRQCD in the continuum and discretize the result. In fact, both effective theories were originally formulated with this idea in mind [9–12]. The resulting lattice theory has ultraviolet divergences that are more severe than QCD, so one must either keep $a^{-1} \sim m_Q$ and employ a highly improved lattice action [10,11] or restrict one's attention to the infinite-mass limit [46]. The approach developed here and in Ref. [1] examines the large-mass limit of Wilson fermions, and as $a \to 0$ the only ultraviolet divergences that are encountered are those of QCD.

Another approach is based on lattice actions that are asymmetric under interchange of the temporal and spatial axes [1]. With a suitable adjustment of the asymmetry couplings, the physics can be made relativistically covariant. For example, one can adjust the action so that $m_1 = m_2$. Cutoff effects can be analyzed either with Symanzik's effective Lagrangian, provided one retains the full dependence on $m_Q a$ in the coefficient functions, or with the HQET description developed here. Initial results [47] with the asymmetric action indicate that the Symanzik and HQET interpretations give the same physical results.

Many papers have followed an *ad hoc* combination of Symanzik and heavy-quark effective theories. Numerical data are generated with artificially small heavy-quark masses, to reduce $m_Q a$. Then these data are extrapolated up in mass guided by the (continuum) $1/m_Q$ expansion. In practice, however, it is hard to find a region with $m_Q a \ll 1$, for Symanzik's analysis genuinely to apply to cutoff effects, and $\Lambda_{\mathrm{QCD}}/m_Q \ll 1$, for HQET genuinely to apply to the mass dependence. Often neither asymptotic condition realistically describes the numerical data. The description developed in this paper naturally applies to the subset of such data where HQET is indeed valid, so these data could be reanalyzed in light of the expansions given above.

One might also imagine reducing the lattice spacing a by an order of magnitude or so. In this regime, the pictures painted by HQET and Symanzik's effective Lagrangian become indistinguishable from each other, even for the bottom quark [1]. The brute-force approach is costly, however. Processor requirements grow as a^{-5} (if not faster) and memory as a^{-4}. For B physics it makes more sense to invest steady improvements in computers into removing the quenched approximation, rather than into a radical reduction of the lattice spacing.

A gap left by this paper is the calculation of the short-distance coefficients, which depend on the lattice action. They can be obtained with some accuracy through perturbation theory in the gauge coupling. There are, for example, general formulae, valid to every order in perturbation theory, relating the self energy of the underlying lattice theory to the first two coefficients of the effective Lagrangian, m_1 and $1/m_2$ [22]. Similarly, radiative corrections to the currents are related to the (on-shell) vertex function [38]. Beyond the one-loop level the calculations will not be easy, but at least they are well defined.

An even better strategy would be to devise nonperturbative methods for tuning, if not explicitly calculating, the short-distance physics. For example, heavy-quark expansions of a hadron's kinetic mass, chromomagnetic mass, *etc.*, would be useful, because with them one could remove HQET scheme dependence. Other possibilities might mimic strategies invented for light quarks, such as imposing—at finite lattice spacing—identities of the continuum limit. For heavy quarks, reparametrization invariance [17], which is closely related to Lorentz invariance and heavy-quark symmetry, may be helpful.

The heavy-quark expansions in this paper are just the beginning. A wide variety of physically interesting observables have been studied with the usual HQET, and matrix elements of the infinite-mass limit are almost always needed. One can re-analyze each observable with the modified coefficients appropriate to the HQET description of lattice gauge theory, to find out how a direct lattice calculation compares to the continuum. Furthermore, it might be possible to extract parameters such as $\bar{\Lambda}$ and λ_1 by calculating the short-distance coefficients (in a suitable scheme) and fitting lattice data. The idea is similar to a proposal [48] for extracting kaon matrix elements from current-current correlation functions $\langle J(x)J(0)\rangle$. (A significant difference is that here the ratio m_Qa of short distances is treated exactly, whereas in Ref. [48] the analogous ratio a/x is presumed small.) Determinations of $\bar{\Lambda}$ and λ_1 are intriguing, because they also appear in heavy-quark expansions of inclusive processes [49,28,24].

ACKNOWLEDGMENTS

I thank Shoji Hashimoto, Aida El-Khadra, and Paul Mackenzie for collaboration on related work [1,38], Laurent Lellouch for probing questions at Lattice '99, and Zoltan Ligeti and Dan Pirjol for helpful remarks. Fermilab is operated by Universities Research Association Inc., under contract with the U.S. Department of Energy.

APPENDIX A: TRACES FOR SEMI-LEPTONIC FORM FACTORS

This appendix gives the traces needed to express the semi-leptonic form factors, at zero recoil. Matrix elements with $v' = v$ are considered first, in Appendix A 1. They enter into $h_+(1)$, $h_1(1)$, and $h_{A_1}(1)$. To extract $h_-(1)$ one must take v' different from v, focus on terms multiplying $\frac{1}{2}(v'-v)^\mu$, and then set $w=1$; *cf.* Appendix A 2.

1. At zero recoil

The traces needed to express matrix elements used to obtain $h_+(1)$, $h_1(1)$, and $h_{A_1}(1)$ are worked out here. One finds no contribution of the types $\langle j^{(1)}\rangle$ and $\langle j^{(1)}\mathcal{L}^{(1)}\rangle$ when $w=1$.

a. Contributions from $\langle j^{(0)}\rangle$

At leading order in the heavy-quark expansion, all matrix elements are written

$$\langle c_{v'}J'|\bar{c}_{v'}\Gamma b_v|b_v J\rangle = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma\mathcal{M}_J\}\xi(w) \tag{A1}$$

where $w = -v'\cdot v$. The spin dependence factors completely; there is only one function $\xi(w)$ to parametrize the light degrees of freedom. At zero recoil the current $iv^\mu\bar{c}_v b_v$ is the Noether current of heavy-quark flavor symmetry. The associated charge changes nothing but the heavy-quark flavor, namely

$$\int d^3y\langle c_v J|iv^0\bar{c}_v b_v(y) = \langle b_v J|, \tag{A2}$$

and hence $\xi(1)=1$. Fortunately, this conclusion does *not* depend on the conservation of the current in the underlying theory, because for lattice QCD one usually computes the transition with a current that is not conserved. (That is why Z_V is written explicitly.) The violation of current conservation is a short-distance effect, however, so it can appear only in the short-distance coefficients.

The matrix elements of interest are

$$\langle c_{v'}0|\bar{c}_{v'}i\gamma^\mu b_v|b_v 0\rangle = \tfrac{1}{2}(v'+v)^\mu\xi(w), \tag{A3}$$

$$\langle c_{v'}1|\bar{c}_{v'}i\gamma^\mu b_v|b_v 1\rangle = \tfrac{1}{2}(v'+v)^\mu\overline{\epsilon'}\cdot\epsilon\,\xi(w), \tag{A4}$$

$$\langle c_{v'}1|\bar{c}_{v'}i\gamma^\mu\gamma_5 b_v|b_v 0\rangle = \tfrac{1}{2}[(1+w)i\overline{\epsilon'}^\mu + i\overline{\epsilon'}\cdot v v'^\mu]\xi(w). \tag{A5}$$

$$\langle c_{v'}0|\bar{c}_{v'}i\gamma^\mu\gamma_5 b_v|b_v 1\rangle = -\tfrac{1}{2}[(1+w)i\epsilon^\mu + i\epsilon\cdot v'\, v^\mu]\xi(w). \tag{A6}$$

In Eqs. (A5) and (A6) note that $\overline{\epsilon'}\cdot v = 0$ and $\epsilon\cdot v' = 0$ at zero recoil.

The Isgur-Wise function $\xi(w)$ is ubiquitous, reappearing, for example, in $h_-(w)$, which is considered in the next section. Here we are concerned with $v'=v$, and then

$$\langle c_v J'|\bar{c}_v i\Gamma^\mu b_v|b_v J\rangle = \omega^\mu\xi(1) = \omega^\mu, \tag{A7}$$

where $\Gamma^\mu = \gamma^\mu$ or $\gamma^\mu\gamma_5$ and $\omega^\mu = v^\mu$, $v^\mu\overline{\epsilon'}\cdot\epsilon$, $i\overline{\epsilon'}^\mu$, or $-i\epsilon^\mu$, as the case may be.

b. Contributions from $\langle j^{(0)}\mathcal{L}^{(1)}\rangle$

The dimension-five interactions in the HQET Lagrangian lead to time-ordered products of $j^{(0)}$ with $\mathcal{O}_2$ and $\mathcal{O}_B$. For unequal velocities the matrix elements are parametrized by three functions

$$\int_{-T}^{0} d^4x\, \langle c_{v'}J'|\bar{c}_{v'}\Gamma b_v(0)\mathcal{O}_2^b(x)|b_vJ\rangle^\star = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma\mathcal{M}_J\}A_1(w), \tag{A8}$$

$$\int_{-T}^{0} d^4x\, \langle c_{v'}J'|\bar{c}_{v'}\Gamma b_v(0)\mathcal{O}_B^b(x)|b_vJ\rangle^\star = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma s^{\alpha\beta}\mathcal{M}_J A_{\alpha\beta}(v,v')\}, \tag{A9}$$

where, like the chromomagnetic field $B_{\alpha\beta}$, the tensor $A_{\alpha\beta}(v,v')$ is anti-symmetric and $v^\alpha A_{\alpha\beta}(v,v')=0$. A general decomposition satisfying these constraints is

$$A_{\alpha\beta}(v,v') = (\eta i\sigma\eta)_{\alpha\beta}A_3(w) + (i\gamma_{\perp\alpha}v'_{\perp\beta} - iv'_{\perp\alpha}\gamma_{\perp\beta})A_2(w). \tag{A10}$$

The same functions appear for insertions of $\mathcal{O}_2^c$ and $\mathcal{O}_B^c$.

One can work out the traces to see how $A_1(w)$ and $A_3(w)$ contribute to $h_+(w)$, $h_1(w)$, and $h_{A_1}(w)$. [$A_2(w)$ contributes to $h_-(w)$.] We are, however, mainly interested in the zero-recoil point, $w=1$. Then the currents become Noether currents, and there are further constraints. With one insertion the starred time-ordered product is identical to the connected one:

$$\begin{aligned}\langle c_vJ'|\bar{c}_v\Gamma b_v(0)\mathcal{O}_X^b(x)|b_vJ\rangle^\star &= \langle c_vJ'|\bar{c}_v\Gamma b_v(0)\mathcal{O}_X^b(x)|b_vJ\rangle_c = \\ &\langle c_vJ'|\bar{c}_v\Gamma b_v(0)\mathcal{O}_X^b(x)|b_vJ\rangle - \langle c_vJ'|\bar{c}_v\Gamma b_v(0)|b_vJ\rangle(\ v^0\ )^{-1}\langle b_vJ|\mathcal{O}_X^b(x)|b_vJ\rangle,\end{aligned} \tag{A11}$$

for $x^0<0$, as in Eq. (A8). By translation invariance the left-hand side of Eq. (A8)

$$\int d^4x\, \langle c_vJ'|T\bar{c}_v\Gamma b_v(y)\mathcal{O}_X^b(x)|b_vJ\rangle^\star = i\int dx^0\, d^3y\, \langle c_vJ'|T\bar{c}_v\Gamma b_v(y)\mathcal{O}_X^b(x)|b_vJ\rangle^\star. \tag{A12}$$

Taking $\Gamma = iv^0$ and using Eq. (A2) one sees that the right-hand side of Eq. (A11) vanishes identically. Thus,

$$\int d^4y\, \langle c_vJ'|T\,\bar{c}_v\Gamma b_v(0)\mathcal{O}_2(y)|b_vJ\rangle^\star = 0, \tag{A13}$$

$$\int d^4y\, \langle c_vJ'|T\,\bar{c}_v\Gamma b_v(0)\mathcal{O}_B(y)|b_vJ\rangle^\star = 0, \tag{A14}$$

namely $A_1(1)=0$ and $A_3(1)=0$.

These results are properties of heavy-quark symmetry and not of the underlying theory. Usually it is argued that $A_1(1)=A_3(1)=0$ as a consequence of current conservation in QCD. This line of argument would not have been enough for our purposes, because for most choices of V^μ_{lat} current conservation fails. Fortunately, the foregoing argument does not rely on the underlying theory; indeed, it is equivalent to the derivation in Rayleigh-Schrödinger perturbation theory of the Ademollo-Gatto theorem.

c. Contributions from $\langle j^{(0)}\mathcal{L}^{(2)}\rangle$

By the same argument leading to Eqs. (A13) and (A14)

$$\int d^4y\,\langle c_vJ'|T\,\bar{c}_v\Gamma b_v(0)\mathcal{O}_D(y)|b_vJ\rangle^\star = 0, \tag{A15}$$

$$\int d^4y\,\langle c_vJ'|T\,\bar{c}_v\Gamma b_v(0)\mathcal{O}_E(y)|b_vJ\rangle^\star = 0. \tag{A16}$$

The same holds for insertions of the four-quark operators omitted from Eq. (3.17). Again, this is a property of heavy-quark symmetry and not of the underlying theory.

References [33,34] choose a basis with the operator

$$\mathcal{Q}_D = 2\bar{h}_v D_\perp^\mu(-iv\cdot\mathcal{D})D_{\perp\mu}h_v, \tag{A17}$$

and a similar, spin-dependent operator $\mathcal{Q}_E$, instead of $\mathcal{O}_D$ and $\mathcal{O}_E$. They are related by

$$\mathcal{Q}_D = \mathcal{O}_D + \bar{h}_v\overleftarrow{D}_\perp^2(-iv\cdot\mathcal{D})h_v + \bar{h}_v(iv\cdot\overleftarrow{\mathcal{D}})D_\perp^2h_v, \tag{A18}$$

up to total derivatives, and similarly for $\mathcal{Q}_E$. The additional terms, which superficially vanish by the equations of motion, generate contact terms. Thus,

$$\int d^4y\,\langle c_vJ'|T\,\bar{c}_v\Gamma b_v(0)\mathcal{Q}_D^b(y)|b_vJ\rangle^\star = \langle c_vJ'|\bar{c}_v\Gamma D_\perp^2b_v|b_vJ\rangle = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma\mathcal{M}_J\}\lambda_1, \tag{A19}$$

$$\int d^4y\,\langle c_vJ'|T\,\bar{c}_v\Gamma b_v(0)\mathcal{Q}_E^b(y)|b_vJ\rangle^\star = \langle c_vJ'|\bar{c}_v\Gamma\mathcal{B}b_v|b_vJ\rangle = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma s^{\alpha\beta}\mathcal{M}_Ji\sigma_{\alpha\beta}\}\lambda_2, \tag{A20}$$

where $\mathcal{B} = s^{\alpha\beta}B_{\alpha\beta}$, and λ_1 and λ_2 are the same constants (including any light-sector cut-off effects) as in Eqs. (6.5) and (6.6). The left-hand sides of Eqs. (A19) and (A20) were parametrized, respectively, with $2B_1(1)$ and $2B_3(1)$ in Ref. [33] and with $2\Xi_1$ and $2\Xi_3$ in Ref. [34], but the identification with λ_1 and λ_2 was not made.

In the basis employing $\mathcal{O}_D$ and $\mathcal{O}_E$, the counterpart of these contact terms are the contributions $\bar{c}_{v'}\Gamma D_\perp^2b_v$ and $\bar{c}_{v'}\Gamma\mathcal{B}b_v$ to the currents, *cf.* Eqs. (7.16) and (7.17). In the $\mathcal{Q}$-basis these currents have coefficients $(8m_{D_\perp^2}^2)^{-1} - (8m_D^2)^{-1}$ and $(8m_{sB}^2)^{-1} - (8m_E^2)^{-1}$.

d. Contributions from $\langle j^{(2)}\rangle$

There are two kinds of of second-order corrections: those which can be associated with a single leg and those which involve cross-talk between the legs. At zero recoil all can be expressed through the parameters λ_1 and λ_2, namely

$$\langle c_vJ'|\bar{c}_v\Gamma\mathcal{D}^\alpha\mathcal{D}^\beta b_v|b_vJ\rangle = -\operatorname{tr}\left\{\bar{\mathcal{M}}_{J'}\Gamma\mathcal{M}_J\left[\tfrac{1}{3}\lambda_1\eta^{\alpha\beta} + \tfrac{1}{2}\lambda_2 i\sigma^{\alpha\beta}\right]\right\}. \tag{A21}$$

By taking Γ to be the unit matrix or $s_{\alpha\beta}$ and contracting indices, it is easy to trace back to the definitions (6.5) and (6.6). By dimensional analysis, these are the only corrections that can arise, even beyond tree level.

The required matrix elements are

$$\langle c_v J'|\bar{c}_v i\Gamma_\mu D_\perp^2 b_v|b_v J\rangle = \langle c_v J'|\bar{c}_v \overleftarrow{D}_\perp^2 i\Gamma_\mu b_v|b_v J\rangle = \lambda_1\omega_\mu, \tag{A22}$$

$$\langle c_v J'|\bar{c}_v i\Gamma_\mu \mathcal{B} b_v|b_v J\rangle = d_J\lambda_2\omega_\mu, \tag{A23}$$
$$\langle c_v J'|\bar{c}_v \mathcal{B} i\Gamma_\mu b_v|b_v J\rangle = d_{J'}\lambda_2\omega_\mu. \tag{A24}$$

At zero recoil $\langle c_v J'|\bar{c}_v \overleftarrow{\mathcal{D}}^\alpha \Gamma \mathcal{D}^\beta b_v|b_v J\rangle = -\langle c_v J'|\bar{c}_v \Gamma \mathcal{D}^\alpha \mathcal{D}^\beta b_v|b_v J\rangle$, so $V^{(1,1)}$ and $A^{(1,1)}$ have matrix elements

$$\langle c_v J|\bar{c}_v \overleftarrow{\not{D}}_\perp i\gamma_\mu \not{D}_\perp b_v|b_v J\rangle = (\lambda_1 + d_J\lambda_2)\omega_\mu \tag{A25}$$
$$\begin{aligned}\langle c_v 1|\bar{c}_v \overleftarrow{\not{D}}_\perp i\gamma_\mu\gamma_5 \not{D}_\perp b_v|b_v 0\rangle &= \langle c_v 0|\bar{c}_v \overleftarrow{\not{D}}_\perp i\gamma_\mu\gamma_5 \not{D}_\perp b_v|b_v 1\rangle \\ &= -\tfrac{1}{3}(\lambda_1 + 3\lambda_2)\omega_\mu \end{aligned} \tag{A26}$$

Contributions with λ_1 (λ_2) are spin-independent (spin-dependent).

e. Contributions from $\langle \mathcal{L}^{(1)} j^{(0)} \mathcal{L}^{(1)}\rangle$

Several matrix elements are introduced for double insertions of $\mathcal{L}^{(1)}$. In the following the short-distance coefficients are stripped off, leading to insertions of $\int d^4z\, \mathcal{O}_X^h(z)$, where $X \in \{2, B\}$ and h labels the heavy flavor. When the operator comes from the numerator of Eq. (5.4) the time variable is integrated for $h = b$ over the interval $(-T, 0]$ and for $h = c$ over $[0, T)$; when the operator comes from the denominator the time variable is integrated over the interval $(-T, T)$. After generating all terms the limit $T \to \infty(1 - i0^+)$ is taken.

When two interactions occur on the incoming line

$$\tfrac{1}{2}\int d^4x\, d^4y\, \langle c_v J'|T\, \bar{c}_v i\Gamma_\mu b_v(0)\mathcal{O}_2^b(x)\mathcal{O}_2^b(y)|b_v J\rangle^\star = \omega_\mu A \tag{A27}$$
$$\int d^4x\, d^4y\, \langle c_v J'|T\, \bar{c}_v i\Gamma_\mu b_v(0)\mathcal{O}_2^b(x)\mathcal{O}_B^b(y)|b_v J\rangle^\star = \omega_\mu d_J B \tag{A28}$$
$$\tfrac{1}{2}\int d^4x\, d^4y\, \langle c_v J'|T\, \bar{c}_v i\Gamma_\mu b_v(0)\mathcal{O}_B^b(x)\mathcal{O}_B^b(y)|b_v J\rangle^\star = \omega_\mu (C_1 + d_J C_3) \tag{A29}$$

and, similarly, when two interactions occur on the outgoing line

$$\tfrac{1}{2}\int d^4x\, d^4y\, \langle c_v J'|T\, \mathcal{O}_2^c(x)\mathcal{O}_2^c(y)\bar{c}_v i\Gamma_\mu b_v(0)|b_v J\rangle^\star = \omega_\mu A \tag{A30}$$
$$\int d^4x\, d^4y\, \langle c_v J'|T\, \mathcal{O}_2^c(x)\mathcal{O}_B^c(y)\bar{c}_v i\Gamma_\mu b_v(0)|b_v J\rangle^\star = \omega_\mu d_{J'} B \tag{A31}$$
$$\tfrac{1}{2}\int d^4x\, d^4y\, \langle c_v J'|T\, \mathcal{O}_B^c(x)\mathcal{O}_B^c(y)\bar{c}_v i\Gamma_\mu b_v(0)|b_v J\rangle^\star = \omega_\mu (C_1 + d_{J'} C_3) \tag{A32}$$

where $\Gamma_\mu = \gamma_\mu$ or $\gamma_\mu\gamma_5$, as the case may be. When each line has one interaction

$$\int d^4x\, d^4y\, \langle c_v J'|T\, \mathcal{O}_2^c(x)\bar{c}_v i\Gamma_\mu b_v(0)\mathcal{O}_2^b(y)|b_v J\rangle^\star = \omega_\mu D \tag{A33}$$
$$\int d^4x\, d^4y\, \langle c_v J'|T\, \mathcal{O}_2^c(x)\bar{c}_v i\Gamma_\mu b_v(0)\mathcal{O}_B^b(y)|b_v J\rangle^\star = \omega_\mu d_J E \tag{A34}$$
$$\int d^4x\, d^4y\, \langle c_v J'|T\, \mathcal{O}_B^c(x)\bar{c}_v i\Gamma_\mu b_v(0)\mathcal{O}_2^b(y)|b_v J\rangle^\star = \omega_\mu d_{J'} E \tag{A35}$$

again where $\Gamma_\mu = \gamma_\mu$ or $\gamma_\mu\gamma_5$, as the case may be, and

$$\int d^4x\, d^4y\, \langle c_v J|T\, \mathcal{O}^c_B(x)\bar{c}_v i\gamma_\mu b_v(0)\mathcal{O}^b_B(y)|b_v J\rangle^\star = \omega_\mu(R_1 + d_J R_2), \tag{A36}$$

$$\int d^4x\, d^4y\, \langle c_v 1|T\, \mathcal{O}^c_B(x)\bar{c}_v i\gamma_\mu\gamma_5 b(0)\mathcal{O}^b_B(y)|b_v 0\rangle^\star = -\tfrac{1}{3}\omega_\mu(R_1 + 3R_2). \tag{A37}$$

Here we have used the notation of Ref. [34].

There are relations between these parameters, which follow solely from heavy-quark symmetry and properties of perturbation theory. Upon expanding Eq. (5.4) and sorting terms with like coefficients one finds

$$\int d^4x\, d^4y\, \langle c_v J'|T\, j(0)\mathcal{O}^b_X(x)\mathcal{O}^b_Y(y)|b_v J\rangle^\star = \int d^4x\, d^4y\, \langle c_v J'|T\, j(0)\mathcal{O}^b_X(x)\mathcal{O}^b_Y(y)|b_v J\rangle_c - \langle c_v J'|j(0)|b_v J\rangle Z^\star_{XY}, \tag{A38}$$

and

$$\int d^4x\, d^4y\, \langle c_v J'|\mathcal{O}^c_X(x)j(0)\mathcal{O}^b_Y(y)|b_v J\rangle^\star = \int d^4x\, d^4y\, \langle c_v J'|\mathcal{O}^c_X(x)j(0)\mathcal{O}^b_Y(y)|b_v J\rangle_c, \tag{A39}$$

with limits of integration on the time coordinates as given above. On the right-hand side of Eq. (A38) the second term is the contribution from state renormalization:

$$Z^\star_{XY} = \frac{1}{v^0}\int_0^T d^4x \int_{-T}^0 d^4y\, \langle b_v J|\mathcal{O}^b_X(x)\mathcal{O}^b_Y(y)|b_v J\rangle_c, \tag{A40}$$

which is flavor independent. In Eq. (A38) the operator j is left-most for all time orderings. When j is a Noether charge one can apply Eq. (A12) to show that the connected term vanishes, leaving only the term from state renormalization. In Eq. (A39) the operator j is in the middle for all time orderings. When j is a Noether charge, however, the right-hand side can be reduced to the same quantity as in the state renormalization. Inserting complete sets of states on both sides of j, and noting that Eq. (A12) applies equally well to excited states, one finds

$$\int d^4x\, d^4y\, \langle c_v J'|\mathcal{O}^c_X(x)j(0)\mathcal{O}^b_Y(y)|b_v J\rangle^\star = \langle c_v J'|j(0)|b_v J\rangle Z^\star_{XY}, \tag{A41}$$

making use of the flavor independence of $Z^\star_{XY}$. Apart from a sign, therefore, the two kinds of $\star$-products are the same, and

$$A = -\tfrac{1}{2}D, \tag{A42}$$

$$B = -E, \tag{A43}$$

$$C_1 = -\tfrac{1}{2}R_1, \tag{A44}$$

$$C_3 = -\tfrac{1}{2}R_2. \tag{A45}$$

These identities leave only four parameters. To my knowledge they have not been derived before. Since they do not depend on the underlying theory, they hold also for continuum QCD.

2. Near zero recoil: $h_-(1)$

At zero recoil several matrix elements vanish, but they are precisely of the type leading to $h_-(w)$ in Eq. (7.2). To extract $h_-(1)$ one must take $v' \neq v$ while evaluating matrix elements, read off the form factor, and then set w to 1. This subsection works out the relevant matrix elements, those of the dimension-four currents, the dimension-five currents $\bar{c}_{v'} i\gamma_\mu i\not{E} b_v$ and $\bar{c}_{v'} i\not{E}' i\gamma_\mu b_v$, and time-ordered products of dimension-four currents with $\mathcal{L}^{(1)}$.

a. Contributions from $\langle j^{(1)} \rangle$

For the matrix elements $\langle c_{v'}0|\bar{c}_{v'} i\gamma_\mu \not{D}_\perp b_v|b_v 0\rangle$ and $\langle c_{v'}0|\bar{c}_{v'} \overleftarrow{\not{D}}_{\perp'} i\gamma_\mu b_v|b_v 0\rangle$ one starts with the matrix element

$$\langle c_{v'}J'|\bar{c}_{v'}\Gamma\mathcal{D}^\alpha b_v|b_v J\rangle = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma\mathcal{M}_J i\xi^\alpha(v,v')\} \tag{A46}$$

where ξ^α parametrizes the light degrees of freedom. The equation of motion $(-iv\cdot\mathcal{D})b_v = 0$ implies that $v_\alpha\xi^\alpha(v,v') = 0$, leaving two independent form factors

$$\xi^\alpha(v,v') = v'^\alpha_\perp \xi_2(w) - i\gamma^\alpha_\perp \xi_3(w). \tag{A47}$$

A further constraint on $\xi^\alpha(v,v')$ comes from the "integration-by-parts" identity

$$\langle c_{v'}J'|\bar{c}_{v'}\overleftarrow{\mathcal{D}}^\alpha\Gamma b_v|b_v J\rangle + \langle c_{v'}J'|\bar{c}_{v'}\Gamma\mathcal{D}^\alpha b_v|b_v J\rangle = -i\bar{\Lambda}(v'-v)^\alpha\langle c_{v'}J'|\bar{c}_{v'}\Gamma b_v|b_v J\rangle, \tag{A48}$$

where $\mathcal{D}b_v = (D - im_{1b}v)b_v$ and $\bar{c}_{v'}\overleftarrow{\mathcal{D}} = \bar{c}_{v'}(\overleftarrow{D} + im_{1c}v')$. The first matrix element

$$\langle c_{v'}J'|\bar{c}_{v'}\overleftarrow{\mathcal{D}}^\alpha\Gamma b_v|b_v J\rangle = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma\mathcal{M}_J[-i\overline{\xi^\alpha(v',v)}]\}, \tag{A49}$$

where $\overline{\xi^\alpha(v',v)} = \gamma^4[\xi^\alpha(v',v)]^\dagger\gamma^4$. Substituting traces for matrix elements in Eq. (A48) yields the relation

$$(w+1)\xi_2(w) + \xi_3(w) = -\bar{\Lambda}\xi(w), \tag{A50}$$

which can be used to eliminate $\xi_2(w)$. In Eq. (A50) the constant $\bar{\Lambda}$ and the function $\xi(w)$ are the same—including lattice artifacts of the light degrees of freedom—as in Eqs. (6.3) and (A1), respectively. Evaluating the traces of interest and using Eq. (A50) one finds

$$\begin{aligned}\langle c_{v'}0|\bar{c}_{v'} i\gamma_\mu \not{D}_\perp b_v|b_v 0\rangle &= \langle c_{v'}0|\bar{c}_{v'}\overleftarrow{\not{D}}_{\perp'} i\gamma_\mu b_v|b_v 0\rangle \\ &= -\tfrac{1}{2}(v'-v)_\mu[2\xi_3(w) - \bar{\Lambda}\xi(w)],\end{aligned} \tag{A51}$$

There is no contribution to $h_+(w)$, and the vector-to-vector matrix elements make no contribution to $h_1(w)$, just to other form factors that are not considered in this paper. An equivalent analysis appears in Ref. [33]. The only significant addition is to extend to lattice QCD the identification of $\bar{\Lambda}\xi(w)$ in Eq. (A50) with the quantities in Eqs. (6.3) and (A1).

b. Contributions from $\langle j^{(2)} \rangle$

To obtain all of the second-order corrections to the current one can start with

$$\langle c_{v'} J' | \bar{c}_{v'} \overleftarrow{\mathcal{D}}^\alpha \Gamma \mathcal{D}^\beta b_v | b_v J \rangle = -\operatorname{tr}\{ \bar{\mathcal{M}}_{J'} \Gamma \mathcal{M}_J [-\lambda^{\alpha\beta}(v, v')] \}. \tag{A52}$$

The equations of motion $(-iv \cdot \mathcal{D}) b_v = 0$ and $\bar{c}_{v'}(iv' \cdot \overleftarrow{\mathcal{D}}) = 0$ imply that $\lambda^{\alpha\beta}(v, v') v_\beta = 0$ and $v'_\alpha \lambda^{\alpha\beta}(v, v') = 0$, and symmetry under exchanging final and initial states implies that $\overline{\lambda^{\beta\alpha}(v', v)} = \lambda^{\alpha\beta}(v, v')$, leaving four independent form factors,

$$\lambda^{\alpha\beta}(v, v') = \eta'^{\alpha}_{\ \gamma} [\tfrac{1}{3} g^{\gamma\delta} \lambda_1(w) + \tfrac{1}{2} i\sigma^{\gamma\delta} \lambda_2(w)] \eta^\beta_\delta + v^\alpha_{\perp'} v'^\beta_\perp \lambda_3(w) + [i\gamma^\alpha_{\perp'} v'^\beta_\perp + v^\alpha_{\perp'} i\gamma^\beta_\perp] \lambda_4(w). \tag{A53}$$

The pre-factors for the first two form factors are chosen so that $\lambda_1(1) = \lambda_1$ and $\lambda_2(1) = \lambda_2$ are the constants in Eq. (A21).

The matrix elements needed for $h_-(w)$ are $\langle c_{v'} 0 | \bar{c}_{v'} i\gamma_\mu i\not{E} b_v | b_v 0 \rangle$ and $\langle c_{v'} 0 | \bar{c}_{v'} i\not{E}' i\gamma_\mu b_v | b_v 0 \rangle$. They are related to Eq. (A52) by the identity

$$\langle c_{v'} J' | \bar{c}_{v'} \Gamma \mathcal{D}^\alpha \mathcal{D}^\beta b_v | b_v J \rangle = -i\bar{\Lambda}(v' - v)^\alpha \langle c_{v'} J' | \bar{c}_{v'} \Gamma \mathcal{D}^\beta b_v | b_v J \rangle - \langle c_{v'} J' | \bar{c}_{v'} \overleftarrow{\mathcal{D}}^\alpha \Gamma \mathcal{D}^\beta b_v | b_v J \rangle \tag{A54}$$

and the definitions $i\not{E} = -iv_\alpha [\mathcal{D}^\alpha, \mathcal{D}_\beta] \gamma^\beta_\perp$, $i\not{E}' = -iv'_\alpha [\mathcal{D}^\alpha, \mathcal{D}_\beta] \gamma^\beta_{\perp'}$. Evaluating the traces one finds

$$\begin{aligned} \langle c_{v'} 0 | \bar{c}_{v'} i\gamma^\mu i\not{E} b_v | b_v 0 \rangle &= \langle c_{v'} 0 | \bar{c}_{v'} i\not{E}' i\gamma^\mu b_v | b_v 0 \rangle \\ &= -\tfrac{1}{2}(v' - v)^\mu \left\{ \tfrac{1}{2}(w+1)\lambda(w) + (w-1)\bar{\Lambda} \left[2\xi_3(w) - \bar{\Lambda}\xi(w) \right] \right\} \end{aligned} \tag{A55}$$

where

$$\lambda(w) = \tfrac{2}{3} w \lambda_1(w) + (3 - w)\lambda_2(w) - 2(w^2 - 1)\lambda_3(w) + 8(w - 1)\lambda_4(w). \tag{A56}$$

At $w = 1$, $\lambda(1) = \frac{2}{3}(\lambda_1 + 3\lambda_2)$.

c. Contributions from $\langle j^{(1)} \mathcal{L}^{(1)} \rangle$

The time-ordered products of interest are $\int d^4y \, \langle c_{v'} J' | T \, \bar{c}_{v'} \Gamma \not{\mathcal{D}}_\perp b_v(x) \mathcal{O}^f_X(y) | b_v J \rangle^\star$ and $\int d^4y \, \langle c_{v'} J' | T \, \bar{c}_{v'} \overleftarrow{\not{\mathcal{D}}}_{\perp'} \Gamma b_v(x) \mathcal{O}^f_X(y) | b_v J \rangle^\star$, where $X \in \{2, B\}$ and $f \in \{c, b\}$. As before it is helpful to consider matrix elements with $\not{\mathcal{D}}_\perp$ replaced with $\mathcal{D}$ and derive constraints from the equations of motion and from "integrating by parts." This is a bit trickier now, with derivatives acting under the time-ordered product.

The equations of motion imply the identities

$$\int d^4y \, \langle c_{v'} J' | T \, \bar{c}_{v'} \Gamma (-iv \cdot \mathcal{D}) b_v(x) \mathcal{O}^b_2(y) | b_v J \rangle^\star = \langle c_{v'} J' | \bar{c}_{v'} \Gamma D^2_\perp b_v(x) | b_v J \rangle, \tag{A57}$$

$$\int d^4y \, \langle c_{v'} J' | T \, \bar{c}_{v'} \Gamma (-iv \cdot \mathcal{D}) b_v(x) \mathcal{O}^b_B(y) | b_v J \rangle^\star = \langle c_{v'} J' | \bar{c}_{v'} \Gamma \mathcal{B} b_v(x) | b_v J \rangle, \tag{A58}$$

$$\int d^4y \, \langle c_{v'} J' | T \, \mathcal{O}^c_2(y) \bar{c}_{v'} (iv' \cdot \overleftarrow{\mathcal{D}}) \Gamma b_v(x) | b_v J \rangle^\star = \langle c_{v'} J' | \bar{c}_{v'} \overleftarrow{D}^2_{\perp'} \Gamma b_v(x) | b_v J \rangle, \tag{A59}$$

$$\int d^4y \, \langle c_{v'} J' | T \, \mathcal{O}^c_B(y) \bar{c}_{v'} (iv' \cdot \overleftarrow{\mathcal{D}}) \Gamma b_v(x) | b_v J \rangle^\star = \langle c_{v'} J' | \bar{c}_{v'} \mathcal{B}' \Gamma b_v(x) | b_v J \rangle. \tag{A60}$$

The contact terms on the right-hand side were omitted from Eqs. (4.27) of Ref. [33] but do appear, for example, in Eq. (A21) of Ref. [49]. They arise from a careful definition of the T-product for operators containing time derivatives. A helpful mnemonic for checking them is to note that

$$(-iv\cdot\mathcal{D})T\,b_v(x)b_v(y)=\delta^{(4)}(x-y), \tag{A61}$$

$$T\,c_{v'}(y)\bar{c}_{v'}(x)(iv'\cdot\overleftarrow{\mathcal{D}})=\delta^{(4)}(y-x). \tag{A62}$$

Further identities come from taking the derivative $\partial^\rho\langle D_{v'}|\bar{c}_{v'}\Gamma b_v|B_v\rangle$ between fully dressed states, and generating the expansion. This leads to

$$\int d^4y\,\langle c_{v'}J'|T\,[\bar{c}_{v'}\Gamma\mathcal{D}^\rho b_v(x)+\bar{c}_{v'}\overleftarrow{\mathcal{D}}^\rho\Gamma b_v(x)]\mathcal{O}_2^b(y)|b_vJ\rangle^\star= \tag{A63}$$
$$-i\bar{\Lambda}(v'-v)^\rho\int d^4y\,\langle c_{v'}J'|T\,\bar{c}_{v'}\Gamma b_v(x)\mathcal{O}_2^b(y)|b_vJ\rangle^\star-i\lambda_1v^\rho\langle c_{v'}J'|\bar{c}_{v'}\Gamma b_v(x)|b_vJ\rangle,$$

$$\int d^4y\,\langle c_{v'}J'|T\,[\bar{c}_{v'}\Gamma\mathcal{D}^\rho b_v(x)+\bar{c}_{v'}\overleftarrow{\mathcal{D}}^\rho\Gamma b_v(x)]\mathcal{O}_B^b(y)|b_vJ\rangle^\star= \tag{A64}$$
$$-i\bar{\Lambda}(v'-v)^\rho\int d^4y\,\langle c_{v'}J'|T\,\bar{c}_{v'}\Gamma b_v(x)\mathcal{O}_B^b(y)|b_vJ\rangle^\star-id_J\lambda_2v^\rho\langle c_{v'}J'|\bar{c}_{v'}\Gamma b_v(x)|b_vJ\rangle,$$

$$\int d^4y\,\langle c_{v'}J'|T\,\mathcal{O}_2^c(y)[\bar{c}_{v'}\Gamma\mathcal{D}^\rho b_v(x)+\bar{c}_{v'}\overleftarrow{\mathcal{D}}^\rho\Gamma b_v(x)]|b_vJ\rangle^\star= \tag{A65}$$
$$-i\bar{\Lambda}(v'-v)^\rho\int d^4y\,\langle c_{v'}J'|T\,\mathcal{O}_2^c(y)\bar{c}_{v'}\Gamma b_v(x)|b_vJ\rangle^\star+i\lambda_1v'^\rho\langle c_{v'}J'|\bar{c}_{v'}\Gamma b_v(x)|b_vJ\rangle,$$

$$\int d^4y\,\langle c_{v'}J'|T\,\mathcal{O}_B^c(y)[\bar{c}_{v'}\Gamma\mathcal{D}^\rho b_v(x)+\bar{c}_{v'}\overleftarrow{\mathcal{D}}^\rho\Gamma b_v(x)]|b_vJ\rangle^\star= \tag{A66}$$
$$-i\bar{\Lambda}(v'-v)^\rho\int d^4y\,\langle c_{v'}J'|T\,\mathcal{O}_B^c(y)\bar{c}_{v'}\Gamma b_v(x)|b_vJ\rangle^\star+id_{J'}\lambda_2v'^\rho\langle c_{v'}J'|\bar{c}_{v'}\Gamma b_v(x)|b_vJ\rangle.$$

These identities do not agree with analogous ones from combining Eqs. (C4) and (C5) of Ref. [33]. Remarkably, Eq. (C5) of Ref. [33] contains the contact terms omitted from Eq. (4.27) of Ref. [33].

Once again the time-ordered products are parametrized by form factors. Consider first the case with the kinetic operator. It is enough to present the details for $\mathcal{O}_2^b$. One may write

$$\int d^4y\,\langle c_{v'}J'|T\,\bar{c}_{v'}\Gamma\mathcal{D}^\alpha b_v\mathcal{O}_2^b(y)|b_vJ\rangle^\star=-\,\mathrm{tr}\{\bar{\mathcal{M}}_{J'}\Gamma\mathcal{M}_Ji\Xi^\alpha(v,v')\}, \tag{A67}$$

$$\int d^4y\,\langle c_{v'}J'|T\,\bar{c}_{v'}\overleftarrow{\mathcal{D}}^\alpha\Gamma b_v\mathcal{O}_2^b(y)|b_vJ\rangle^\star=-\,\mathrm{tr}\{\bar{\mathcal{M}}_{J'}\Gamma\mathcal{M}_J[-iF^\alpha(v,v')]\}. \tag{A68}$$

In the first case, the equation of motion (A57) implies $v\cdot\Xi=\phi_0$, where

$$\phi_0(w)=\tfrac{1}{3}(2+w^2)\lambda_1(w)-(w-1)\{w[\tfrac{1}{2}\lambda_2(w)+(w+1)\lambda_3(w)-2\lambda_4(w)]+\bar{\Lambda}^2\xi(w)\} \tag{A69}$$

is obtained from Eqs. (A53) and (A54). Thus, Ξ^α has a decomposition

$$\Xi^\alpha(v,v')=-v^\alpha\phi_0(w)+v'^\alpha_\perp\Xi_2(w)-i\gamma^\alpha_\perp\Xi_3(w) \tag{A70}$$

similar to ξ^α but with $-\phi_0(w)$ multiplying v^α. On the other hand, the equation of motion still implies $v'\cdot F=0$, so F^α has the decomposition

$$F^{\alpha}(v,v') = v^{\alpha}_{\perp'}F_1(w) - i\gamma^{\alpha}_{\perp'}F_3(w). \tag{A71}$$

similar to $\overline{\xi^{\alpha}(v',v)}$. The form factors Ξ_2, F_1, and F_3 can be eliminated, because the identity (A63) implies

$$(w+1)\Xi_2 + \Xi_3 = -w\tilde{\phi}_0 - \bar{\Lambda}A_1, \tag{A72}$$

$$(w+1)F_1 + F_3 = \tilde{\phi}_0 - \bar{\Lambda}A_1, \tag{A73}$$

$$F_3 = \Xi_3, \tag{A74}$$

where

$$\tilde{\phi}_0(w) = \frac{\phi_0(w) - \lambda_1\xi(w)}{w-1}. \tag{A75}$$

At zero recoil the new constants that can arise are $\Xi_3(1)$ and, denoting differentiation with respect to w by a dot, $\tilde{\phi}_0(1) = \dot{\phi}_0(1) - \lambda_1\dot{\xi}(1)$. (Recall that $A_1(1) = 0$, as a consequence of heavy-quark flavor symmetry.)

Evaluating the traces for $h_-(w)$, one finds

$$\begin{aligned}\int d^4y\, \langle c_{v'}0|T\,\bar{c}_{v'}i\gamma^{\mu}\not{D}_{\perp}b_v(x)\mathcal{O}_2^b(y)|b_v0\rangle^{\star} &= \int d^4y\, \langle c_{v'}0|T\,\bar{c}_{v'}\mathcal{O}_2^c(y)\overleftarrow{\not{D}}_{\perp'}i\gamma^{\mu}b_v(x)|b_v0\rangle^{\star} \\ &= -\tfrac{1}{2}(v'-v)^{\mu}[2\Xi_3(w) - \bar{\Lambda}A_1(w) - w\tilde{\phi}_0(w)], \end{aligned} \tag{A76}$$

$$\begin{aligned}\int d^4y\, \langle c_{v'}0|T\,\bar{c}_{v'}\overleftarrow{\not{D}}_{\perp'}i\gamma^{\mu}b_v(x)\mathcal{O}_2^b(y)|b_v0\rangle^{\star} &= \int d^4y\, \langle c_{v'}0|T\,\bar{c}_{v'}\mathcal{O}_2^c(y)i\gamma^{\mu}\not{D}_{\perp}b_v(x)|b_v0\rangle^{\star} \\ &= -\tfrac{1}{2}(v'-v)^{\mu}[2\Xi_3(w) - \bar{\Lambda}A_1(w) + \tilde{\phi}_0(w)]. \end{aligned} \tag{A77}$$

Matrix elements of this kind make no contribution to $h_+(w)$, $h_1(w)$, or $h_{A_1}(w)$.

Finally there are the time-ordered products with the chromomagnetic energy. It is enough to show the details for $\mathcal{O}_B^b$. When the derivative acts on b_v,

$$\int d^4y\, \langle c_{v'}J'|T\,\bar{c}_{v'}\Gamma\mathcal{D}^{\rho}b_v(x)\mathcal{O}_B^b(y)|b_vJ\rangle^{\star} = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma s^{\alpha\beta}\mathcal{M}_J i\Xi^{\rho}{}_{\alpha\beta}(v,v')\}, \tag{A78}$$

and when the derivative acts on $\bar{c}_{v'}$,

$$\int d^4y\, \langle c_{v'}J'|T\,\bar{c}_{v'}\overleftarrow{\mathcal{D}}^{\rho}\Gamma b_v(x)\mathcal{O}_B^b(y)|b_vJ\rangle^{\star} = -\operatorname{tr}\{\bar{\mathcal{M}}_{J'}\Gamma s^{\alpha\beta}\mathcal{M}_J[-iF^{\rho}{}_{\alpha\beta}(v,v')]\}. \tag{A79}$$

The tensors $\Xi^{\rho}{}_{\alpha\beta}$ and $F^{\rho}{}_{\alpha\beta}$ inherit properties from the chromomagnetic field $B_{\alpha\beta}$: they are antisymmetric on the lower indices, and $\Xi^{\rho}{}_{\alpha\beta}v^{\beta} = F^{\rho}{}_{\alpha\beta}v^{\beta} = 0$. From the equations of motion $v'_{\rho}F^{\rho}{}_{\alpha\beta} = 0$ and $v_{\rho}\Xi^{\rho}{}_{\alpha\beta} = \phi_{\alpha\beta}$, where

$$\phi_{\alpha\beta}(v,v') = \eta_{\alpha\mu}[\lambda^{\mu\nu}(v,v') - \lambda^{\nu\mu}(v,v')]\eta_{\nu\beta} + \bar{\Lambda}[v'_{\perp\alpha}\xi_{\beta}(v,v') - v'_{\perp\beta}\xi_{\alpha}(v,v')]. \tag{A80}$$

Substituting Eq. (A53) into Eq. (A80)

$$\phi_{\alpha\beta}(v,v') = (\eta i\sigma\eta)_{\alpha\beta}\phi_3(w) - (i\gamma_{\perp\alpha}v'_{\perp\beta} - iv'_{\perp\alpha}\gamma_{\perp\beta})\phi_2(w), \tag{A81}$$

where $\phi_3(w) = \lambda_2(w)$ and $\phi_2(w) = -\frac{1}{2}\lambda_2(w) - (w+1)\lambda_4(w) - \bar{\Lambda}\xi_3(w)$. The constraints on $\Xi^{\rho}{}_{\alpha\beta}$ and $F^{\rho}{}_{\alpha\beta}$ lead to the decompositions

$$\Xi^{\rho}{}_{\alpha\beta}(v,v') = (\eta i\sigma\eta)_{\alpha\beta}\left[-v^{\rho}\phi_3 + v'^{\rho}_{\perp}\Xi_8 - i\gamma^{\rho}_{\perp}\Xi_9\right] - (i\gamma_{\perp\alpha}v'_{\perp\beta} - v'_{\perp\alpha}i\gamma_{\perp\beta})\left[-v^{\rho}\phi_2 + v'^{\rho}_{\perp}\Xi_5 + i\gamma^{\rho}_{\perp}\Xi_6\right] + (\eta^{\rho}_{\alpha}v'_{\perp\beta} - \eta^{\rho}_{\beta}v'_{\perp\alpha})\Xi_{10} + (\eta^{\rho}_{\alpha}i\gamma_{\perp\beta} - \eta^{\rho}_{\beta}i\gamma_{\perp\alpha})\Xi_{11}, \tag{A82}$$

and

$$F^{\rho}{}_{\alpha\beta}(v,v') = (\eta i\sigma\eta)_{\alpha\beta}\left[v^{\rho}_{\perp'}F_7 - i\gamma^{\rho}_{\perp'}F_9\right] - (i\gamma_{\perp\alpha}v'_{\perp\beta} - v'_{\perp\alpha}i\gamma_{\perp\beta})\left[v^{\rho}_{\perp'}F_4 + i\gamma^{\rho}_{\perp'}F_6\right] + \eta'^{\rho}_{\sigma}(\eta^{\sigma}_{\alpha}v'_{\perp\beta} - \eta^{\sigma}_{\beta}v'_{\perp\alpha})F_{10} + \eta'^{\rho}_{\sigma}(\eta^{\sigma}_{\alpha}i\gamma_{\perp\beta} - \eta^{\sigma}_{\beta}i\gamma_{\perp\alpha})F_{11}. \tag{A83}$$

The subscripts are chosen as in Ref. [33].

The identity (A64) can be applied to eliminate Ξ_5, Ξ_8, and all Fs:

$$F_k = \Xi_k, \qquad k \in \{6, 9, 10, 11\}, \tag{A84}$$
$$(w^2-1)\Xi_5 = -w\phi_2 - (w+1)\Xi_6 - \Xi_{11} + (w-1)\bar{\Lambda}A_2, \tag{A85}$$
$$(w^2-1)F_4 = \phi_2 + (w+1)\Xi_6 + w\Xi_{11} + (w-1)\bar{\Lambda}A_2, \tag{A86}$$
$$(w+1)\Xi_8 = -w\tilde{\phi}_3 - \Xi_9 - \bar{\Lambda}A_3, \tag{A87}$$
$$(w+1)F_7 = \tilde{\phi}_3 - \Xi_9 - \bar{\Lambda}A_3, \tag{A88}$$

where

$$\tilde{\phi}_3(w) = \frac{\phi_3(w) - \lambda_2\xi(w)}{w-1}. \tag{A89}$$

Each of Eqs. (A85) and (A86) implies $2\Xi_6(1) + \Xi_{11}(1) = -\phi_2(1)$.

Evaluating the traces for $h_-(w)$, one finds

$$\int d^4y\, \langle c_{v'}0|T\,\bar{c}_{v'}i\gamma^{\mu}\not{D}_{\perp}b_v(x)\mathcal{O}^b_B(y)|b_v0\rangle^{\star} = \int d^4y\, \langle c_{v'}0|T\,\bar{c}_{v'}\mathcal{O}^c_B(y)\overleftarrow{\not{D}}_{\perp'}i\gamma^{\mu}b_v(x)|b_v0\rangle^{\star} = -\tfrac{1}{2}(v'-v)^{\mu}[2\Xi_-(w) - w\tilde{\phi}_-(w)], \tag{A90}$$

$$\int d^4y\, \langle c_{v'}0|T\,\bar{c}_{v'}\overleftarrow{\not{D}}_{\perp'}i\gamma^{\mu}b_v(x)\mathcal{O}^b_B(y)|b_v0\rangle^{\star} = \int d^4y\, \langle c_{v'}0|T\,\bar{c}_{v'}\mathcal{O}^c_B(y)i\gamma^{\mu}\not{D}_{\perp}b_v(x)|b_v0\rangle^{\star} = -\tfrac{1}{2}(v'-v)^{\mu}[2\Xi_-(w) + \tilde{\phi}_-(w)], \tag{A91}$$

where

$$\Xi_- = 3\Xi_9 + (w+1)(2\Xi_6 + \Xi_{10}) - 2\Xi_{11} - \tfrac{3}{2}\bar{\Lambda}A_3 - (w-1)\bar{\Lambda}A_2. \tag{A92}$$
$$\tilde{\phi}_- = 3\tilde{\phi}_3 - 2\phi_2 \tag{A93}$$

At zero recoil the new constants that can arise are $\Xi_6(1)$, $\Xi_9(1)$, $\Xi_{10}(1)$, and $\tilde{\phi}_3(1)$. (Note that $A_2(1)$ drops out, and recall that $A_3(1) = 0$ as a consequence of heavy-quark flavor symmetry.) As in Eqs. (A76) and (A77), matrix elements of this kind make no contribution to $h_+(w)$, $h_1(w)$, or $h_{A_1}(w)$. In $h_-(1)$ they reduce to two constants

$$\Xi_-(1) = 3\Xi_9(1) + 8\Xi_6(1) + 2\Xi_{10}(1) + 2\phi_2(1), \tag{A94}$$
$$\tilde{\phi}_-(1) = 3[\dot{\phi}_3(1) - \lambda_2\dot{\xi}(1)] - 2\phi_2(1), \tag{A95}$$

which are needed in Eq. (7.47).

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