

Chapter 7

BEAM LOADING AND ROBINSON'S STABILITY

A klystron coupled to a rf cavity generates electromagnetic fields. The electric field across the gap of the cavity gives the required acceleration to the particle beam. However, the particle beam will also excite electromagnetic fields inside the cavity in the same way as the klystron or the rf source. This excitation of the cavity by the particle beam is called *beam loading*. Beam loading has two effects on the rf system. First, the electric field from beam loading generates a potential, called the beam loading voltage, across the cavity gap and opposes the accelerating voltage delivered by the klystron. Thus more power has to be supplied to the rf cavity in order to overcome the effect of beam loading. Second, to optimize the power of the klystron, the cavity has to be detuned. The detuning has to be performed correctly. If not, the power delivered by the klystron will not be efficient. Worst of all, an incorrect detuning will excite an instability. We first study the steady-state beam loading and later the transient beam loading. Most of the material in this chapter comes from lecture notes of Wilson [1], Wiedemann [2], and Boussard [3].

7.1 EQUIVALENT CIRCUIT

The rf cavity can be represented by a parallel resonant circuit with an inductance L , a capacitance C , and a resistor R_s as shown in Fig. 7.1. The resistor R_s is also called the shunt impedance of the resonator because it is the impedance of the circuit at the

resonant frequency $\omega_r/(2\pi)$, which is given by

$$\omega_r = \frac{1}{\sqrt{LC}} . \quad (7.1)$$

The image current of the particle beam is represented by a current source i_{im} . This

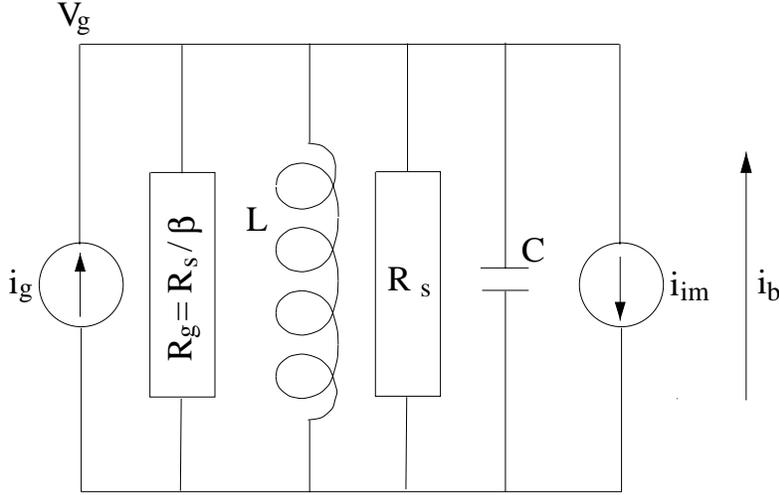


Figure 7.1: Circuit model representing an rf generator current source i_g driving a rf cavity with a beam loading current i_{im} .

is a valid representation from the rigid-bunch approximation, because the velocities and therefore the current of the beam particles are assumed roughly constant when the beam passes through the cavity gap. We reference image current here instead of the beam current i_b , because it is the image current that flows across the cavity gap and also into the cavity. The image current is in opposite direction to the beam current.

On the other hand, the situation is different for the klystron. The velocities of the electrons as they pass through the the gap of the output cavity of the klystron can change in response to the cavity fields of the klystron. As a consequence, the rf source is represented by a current source i_g in parallel to loading admittance Y_g or impedance $R_g = 1/Y_g$. The latter is written in terms of the shunt admittance Y_s or shunt impedance R_s of the rf cavity as

$$Y_g = \beta Y_s = \frac{\beta}{R_s} , \quad (7.2)$$

where β is the *coupling coefficient* still to be defined. The power generated by the klystron or rf generator consists of two parts: the power dissipated to the generator admittance Y_g

and the power that is available to the cavity and the particle beam. The latter, which is usually referred to loosely as the *generator power*, is

$$P_g = \frac{1}{2} Y_L V_g^2 , \quad (7.3)$$

where V_g is the generator voltage as indicated in Fig. 7.1, which is also the voltage across the gap of the rf cavity. Here, all currents and voltages referenced are the magnitudes of sinusoidally varying currents and voltages at or near the cavity resonant frequency, and the factor $\frac{1}{2}$ in Eq. (7.3) indicates the rms value of P_g has been taken. For example, i_{im} is the magnitude of the Fourier component of the image current at or near the cavity resonant frequency. Thus, for a *short bunch*, we have (Exercise 7.1),

$$i_{im} = 2I_0 , \quad (7.4)$$

with I_0 being the *dc current* of the beam. As phasors, however, they are in the opposite direction. The admittance Y_L is called the *load cavity admittance*, which includes the admittance of the cavity Y_s and also all the contribution from the particle beam. In other words, for a weak beam, when $i_b \rightarrow 0$, $Y_L \rightarrow Y_s$. At the resonant frequency ω_r , the generator voltage and the generator current are in phase and are related by

$$V_g = \frac{i_g}{Y_g + Y_L} , \quad (7.5)$$

and the generator power becomes

$$P_g = \frac{1}{2} \frac{Y_L}{(Y_g + Y_L)^2} i_g^2 . \quad (7.6)$$

The generator power is minimized by equating its derivative with respect to Y_L to zero, giving a matching between the source and the load,

$$Y_L = Y_g = \beta Y_s . \quad (7.7)$$

This is just the usual matching of the input impedance to the output impedance. The maximized generator power is then

$$P_g = \frac{i_g^2}{8\beta Y_s} = \frac{R_s i_g^2}{8\beta} \quad (7.8)$$

Notice that in the situation of an extremely weak beam, this matched condition is just $Y_g = Y_s$ with the coupling coefficient $\beta = 1$. Equation (7.8) will be used repeatedly below

and whenever the generator power P_g is referenced, we always imply the matched quantity satisfying Eq. (7.7).

In high energy electron linacs, bunches are usually accelerated at the peak or crest of the rf voltage wave in order to achieve maximum possible energy gain. As a result, the klystron is operated at exactly the same frequency as the resonant frequency of the rf cavities. Without the rf generator, the beam or image current sees the *unloaded shunt impedance* R_s in the cavity and the *unloaded quality factor* Q_0 , which can easily be found to be

$$Q_0 = \omega_r C R_s . \quad (7.9)$$

With the rf generator attached, however, the beam image current source sees an effective shunt impedance R_L in the cavity, which is the parallel combination of the generator shunt impedance and the cavity shunt impedance. This is called the cavity *loaded shunt impedance* in contrast with the cavity unloaded shunt impedance R_s . We therefore have

$$R_L = (Y_s + Y_g)^{-1} = \frac{R_s}{1 + \beta} . \quad (7.10)$$

Correspondingly, the beam image current sees a *loaded quality factor* in the cavity, which is

$$Q_L = \omega_r C R_L = \frac{Q_0}{1 + \beta} . \quad (7.11)$$

The beam loading voltage is the voltage generated by the image current, and is given by

$$V_{br} = \frac{i_{im}}{Y_g + Y_s} = \frac{i_{im}}{Y_s(1 + \beta)} , \quad (7.12)$$

while the voltage produced by the generator is

$$V_{gr} = \frac{i_g}{Y_g + Y_s} = \frac{i_g}{Y_s(1 + \beta)} , \quad (7.13)$$

where the subscript “ r ” implies that the operation is at resonant frequency, so that the currents and voltages are in phase, although they may have sign difference. In terms of the generator power P_g in Eq. (7.8), the generator voltage at resonance becomes

$$V_{gr} = \frac{\sqrt{8\beta}}{1 + \beta} \sqrt{R_s P_g} . \quad (7.14)$$

It is clear that the beam loading voltage is in the opposite direction of the generator voltage. Thus, the net accelerating voltage is

$$V_{\text{rf}} = V_{gr} - V_{br} = \sqrt{R_s P_g} \left[\frac{\sqrt{8\beta}}{1 + \beta} \left(1 - \frac{K}{2\sqrt{\beta}} \right) \right], \quad (7.15)$$

where

$$K^2 = \frac{i_{im}^2 R_s}{2P_g} \quad (7.16)$$

plays the role of the ratio of the beam loading power to the generator power. The fraction of generator power delivered to the beam is

$$\eta = \frac{i_{im} V_{\text{rf}}}{2P_g} = \frac{2\sqrt{\beta}}{1 + \beta} K \left(1 - \frac{K}{2\sqrt{\beta}} \right). \quad (7.17)$$

The power dissipated in the cavity is

$$P_c = \frac{V_{\text{rf}}^2}{2R_s} = P_g \left(\frac{2\sqrt{\beta}}{1 + \beta} \right)^2 \left(1 - \frac{K}{2\sqrt{\beta}} \right)^2. \quad (7.18)$$

From the conservation of energy, we must have

$$P_g = \eta P_g + P_c + P_r, \quad (7.19)$$

where P_r is the power reflected back to the generator and is given by

$$\frac{P_r}{P_g} = \left(\frac{\beta - 1 - K\sqrt{\beta}}{1 + \beta} \right)^2. \quad (7.20)$$

So far we have not said anything about the coupling coefficient β . Now we can choose β so that the generator power is delivered to the cavity and the beam without any reflection, or from Eq. (7.20), the optimum coupling constant is

$$K = \frac{\beta_{\text{op}} - 1}{\sqrt{\beta_{\text{op}}}}. \quad (7.21)$$

Notice that this optimization is also a maximization of the accelerating voltage $V_a = V_{\text{rf}}$, as can be verified by its differentiation with respect to β .

The description so far has been the steady-state. This means that the rf generator and the beam current have been turned on for a time long compared to the filling time of the cavity, which is given by

$$T_f = \frac{2Q_L}{\omega_r}. \quad (7.22)$$

7.2 BEAM LOADING IN AN ACCELERATOR RING

In a synchrotron ring or storage ring, it is necessary to operate the rf system off the crest of the accelerating voltage wave form in order to have a sufficient large bucket area to hold the bunched beam and to insure stability of phase oscillation. The klystron or rf generator is operating at the *rf frequency* $\omega_{\text{rf}}/(2\pi) = h\omega_0/(2\pi)$, where h is an integer called the *rf harmonic*, and $\omega_0/(2\pi)$ is the revolution frequency of the *synchronized* beam particles. Notice that this rf frequency will be the frequency the beam particles experience at the cavity gap and is different from the intrinsic resonant frequency of the cavity ω_r given by Eq. (7.1). According to the circuit diagram of Fig. 7.1, the impedance of the cavity seen by the particle at rf frequency $\omega_{\text{rf}}/(2\pi)$ can be written as

$$Z_{\text{cav}} = \frac{R_L}{1 - jQ_L \left(\frac{\omega_r}{\omega_{\text{rf}}} - \frac{\omega_{\text{rf}}}{\omega_r} \right)} = R_L \cos \psi e^{j\psi} , \quad (7.23)$$

where ψ is called the *rf detuning angle* or just detuning. It is important to point out that loaded values have been used here, because those are what the image current sees. When the deviation of ω_{rf} from ω_r is small, the detuning angle can be approximated by

$$\tan \psi = 2Q_L \frac{\omega_r - \omega_{\text{rf}}}{\omega_r} . \quad (7.24)$$

Note that in this section we have used j instead of $-i$, because phasor diagrams are customarily drawn using this convention. Phasors, as illustrated in Fig. 7.2, are represented by overhead tildes rotating counter-clockwise with angular frequency ω_{rf} if there is only one bunch in the ring. If there are N_b equal bunches in the ring separated equally by $h_b = h/N_b$ rf buckets, where h is the rf harmonic, we can also imagine the phasors to be rotating at angular frequency ω_{rf}/h_b . They are therefore the Fourier components at the rf frequency or ω_{rf}/h_b . This implies that we are going to see the same phasor plot for each passage of a bunch through the rf cavity. In order to be so, the beam loading voltage should have negligible decay during the time interval $T_b = 2\pi h_b/\omega_{\text{rf}}$. In other words, we require $T_b \ll T_f$ in this discussion, where T_f is the fill time of the cavity.

In general, the image current phasor \tilde{i}_{im} has the same magnitude as that of the beam current phasor \tilde{i}_b , although in the opposite direction. When the image current \tilde{i}_{im} interacts with the loaded cavity, according to Eq. (7.23), a beam loading voltage \tilde{V}_b will

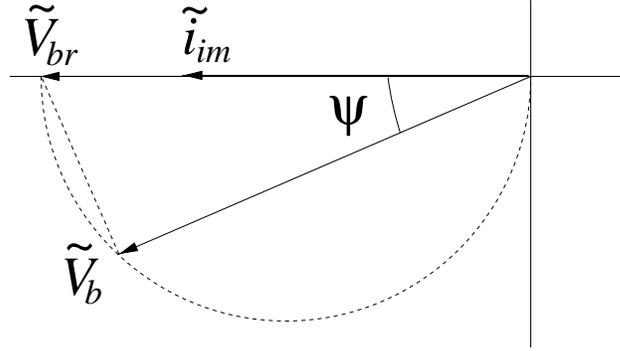


Figure 7.2: Phasor plot showing the beam loading voltage phasor \tilde{V}_b induced in the rf cavity by the image current phasor \tilde{i}_{im} , which lags \tilde{V}_b by the detuning angle ψ . Also plotted is the beam loading voltage phasor \tilde{V}_{br} , with $V_b = V_{br} \cos \psi$ when the beam current is at the crest of the rf wave with no detuning.

be produced and is given by

$$\tilde{V}_b = \tilde{i}_{im} R_L \cos \psi e^{j\psi} , \quad (7.25)$$

and

$$V_b = V_{br} \cos \psi . \quad (7.26)$$

Thus the voltage phasor always leads the current phasor by the detuning phase ψ and the magnitude of the phasor \tilde{V}_b is less than its value at the cavity resonant frequency V_{br} by the factor $\cos \psi$. If one likes, one can also introduce the phasor \tilde{V}_{br} which is in phase with the current phasor \tilde{i}_{im} and has the magnitude given by Eq. (7.26). This is illustrated in Fig. 7.2.

The phasor plot showing the contribution of both the beam loading voltage phasors \tilde{V}_b and the generator voltage phasor \tilde{V}_g is shown in Fig. 7.3. We see that both the beam loading voltage phasor \tilde{V}_b and the generator voltage phasor \tilde{V}_g are at a phase ψ ahead of their respective current phasors \tilde{i}_{im} and \tilde{i}_g . Since these two voltage phasors add up to give the gap voltage phasor \tilde{V}_{rf} which has a synchronous angle ϕ_s , we must have after dividing by $R_s \cos \psi$,

$$i_g \sin \psi = i_{im} \sin\left(\frac{\pi}{2} - \phi_s + \psi\right) . \quad (7.27)$$

Resolving the current contributions along \tilde{i}_g , we have

$$i_g = i_0 + i_{im} \sin \phi_s , \quad (7.28)$$

where $i_0 = V_{rf}/R_L = (1 + \beta)V_{rf}/R_s$ is the total current in phase with the cavity gap

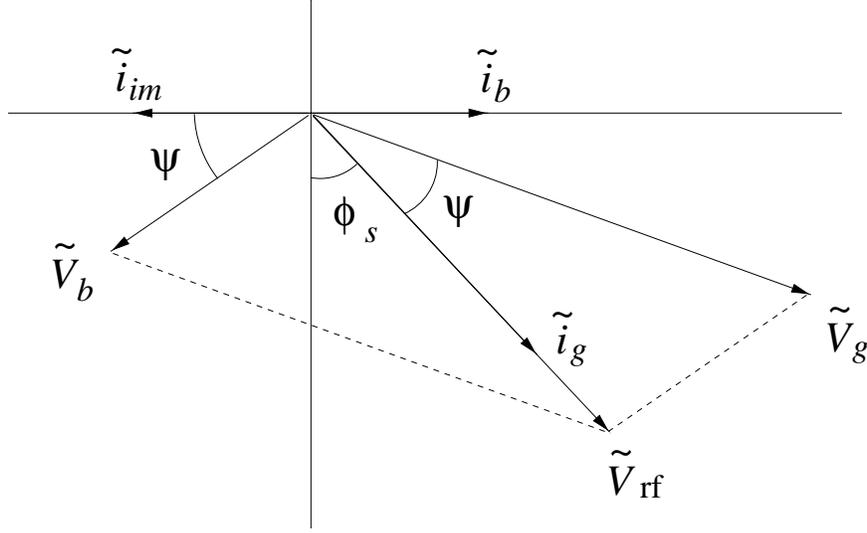


Figure 7.3: Phasor plot showing the vector addition of the generator voltage phasor \tilde{V}_g and the beam loading voltage phasor \tilde{V}_b to give the gap voltage phasor \tilde{V}_{rf} in a rf cavity. Note the detuning angle ψ which puts the gap current phasor \tilde{i}_g in phase with the gap voltage phasor.

voltage. Eliminating i_g , we arrive at

$$\tan \psi = \frac{i_{im} \cos \phi_s}{i_0} . \quad (7.29)$$

Now the generator power V_g can be computed with the aid of Eq. (7.14), namely,

$$P_g = \frac{(1 + \beta)^2 V_{gr}^2}{8\beta R_s} , \quad (7.30)$$

where V_{gr} is the generator voltage at the cavity resonant frequency, and is related to the generator voltage V_g at the rf frequency by $V_g = V_{gr} \cos \psi$. Using the cosine law for the triangle made up from \tilde{V}_g , \tilde{V}_b , and \tilde{V}_{rf} , it is easy to obtain

$$V_g^2 = V_b^2 + V_{\text{rf}}^2 - 2V_b V_{\text{rf}} \sin(\psi - \phi_s) , \quad (7.31)$$

or

$$V_{gr}^2 = V_{br}^2 + V_{\text{rf}}^2(1 + \tan^2 \psi) - 2V_{br} V_{\text{rf}}(\tan \psi \cos \phi_s - \sin \phi_s) , \quad (7.32)$$

where $V_{br} = V_b / \cos \psi$ is the beam loading voltage at the cavity resonant frequency. With the correct detuning, Eq. (7.29) and the definition of i_0 in Eq. (7.28), it is then easy to

show that

$$P_g = \frac{(1 + \beta)^2}{8\beta} \frac{(V_{\text{rf}} + V_{\text{br}} \sin \phi_s)^2}{R_s}, \quad (7.33)$$

where

$$V_{\text{br}} = \frac{i_{\text{im}} R_s}{1 + \beta} \quad (7.34)$$

is the beam loading voltage at the cavity resonant frequency. Again we can optimize the generator power by choosing the best coupling constant β , which turns out to be

$$\beta_{\text{op}} = 1 + \frac{i_{\text{im}} R_s \sin \phi_s}{V_{\text{rf}}} = 1 + \frac{P_b}{P_c}, \quad (7.35)$$

where

$$P_c = \frac{V_{\text{rf}}^2}{2R_s} \quad (7.36)$$

is the power dissipated in the walls of the cavity and

$$P_b = \frac{1}{2} i_{\text{im}} V_{\text{rf}} \sin \phi_s = I_0 V_{\text{rf}} \sin \phi_s. \quad (7.37)$$

Here, we have used Eq. (7.4), the fact that the Fourier component image current at the rf frequency (or at ω_{rf}/h_b) is nearly twice the dc beam current I_0 when the bunch is short. Obviously, $V_a = V_{\text{rf}} \sin \phi_s$ is the accelerating voltage. At the optimized coupling constant, the generator power becomes

$$P_{g\text{op}} = \frac{V_{\text{rf}}^2}{2R_g} = \frac{V_{\text{rf}}^2}{2R_s} \beta_{\text{op}} = P_b + P_c, \quad (7.38)$$

which just states that the power is transmitted to the cavity completely without any power reflected.

7.3 ROBINSON'S STABILITY CRITERIA

We are now in the position to discuss the conditions for phase stability. Suppose that center of the bunch has the same energy as the synchronous particle, but is at a small phase advance $\phi_{\text{rf}} = \epsilon > 0$, as depicted by Point 1 in the synchrotron oscillation and the phasor \tilde{i}_b in the phasor plot in Fig. 7.4. The phasor \tilde{i}_b arrives earlier by being ahead of the x -axis at a small angle $\epsilon > 0$. Then the accelerating voltage it sees will be $V_{\text{rf}} \sin(\phi_s - \epsilon)$ instead of $V_{\text{rf}} \sin \phi_s$, or an extra decelerating voltage of $\epsilon V_{\text{rf}} \cos \phi_s$ if $0 < \phi_s < \frac{1}{2}\pi$. Receiving less

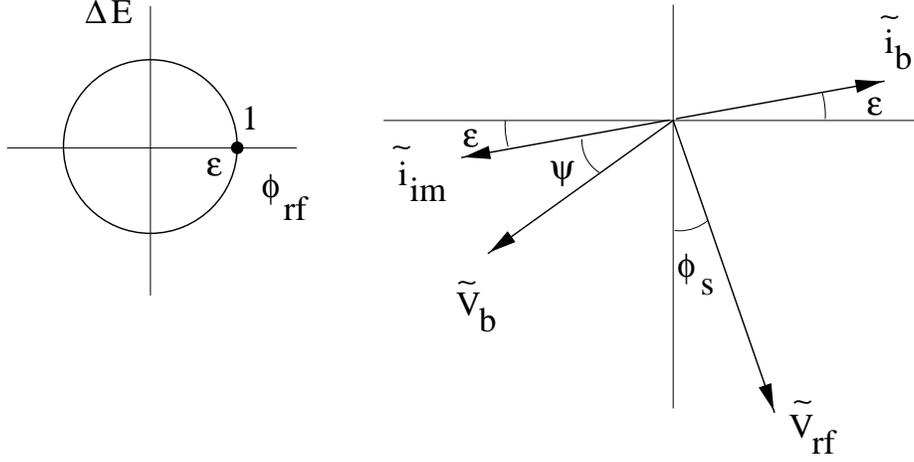


Figure 7.4: With the bunch center at Point 1 in the synchrotron oscillation, the beam current phasor \tilde{i}_b arrives earlier by being ahead of the x -axis at a small angle $\epsilon > 0$ in the phasor plot. The bunch sees a smaller rf voltage $V_{\text{rf}} \sin(\phi_s - \epsilon)$ if the synchronous phase $0 < \phi_s < \frac{1}{2}\pi$. It is decelerated. Below transition, it will arrive not so early in the next turn and phase stability is therefore established.

energy from the rf voltage than the synchronous particle will slow the bunch. If the beam is below transition, this implies the reduction of its revolution frequency, so that after the next h rf periods its arrival ahead of the synchronous particle will be smaller or ϵ will become smaller. The motion is therefore stable. Therefore to establish stable phase oscillation when beam loading can be neglected, one requires

$$\begin{cases} 0 < \phi_s < \frac{\pi}{2} & \text{below transition,} \\ \frac{\pi}{2} < \phi_s < \pi & \text{above transition.} \end{cases} \quad (7.39)$$

This is just the condition of phase stability and there is no damping at all. There is usually a loop that monitors the beam loading and feeds back onto the generator current so as to maintain the required rf gap voltage and synchronous phase. This explains why we have considered the phasor \tilde{V}_{rf} unperturbed.

Next, we consider the interaction of the beam with the impedance of the rf system. During half of a synchrotron period, the center of the bunch is at a higher energy than the synchronous particle. For the sake of convenience, choose the particular moment when the phase of bunch center is just in phase with the synchronous particle, so that the phasor \tilde{i}_b is exactly along the x -axis. This is illustrated by Point 2 in the synchrotron oscillation and the beam current phasor being in phase with the x -axis in the phasor plot

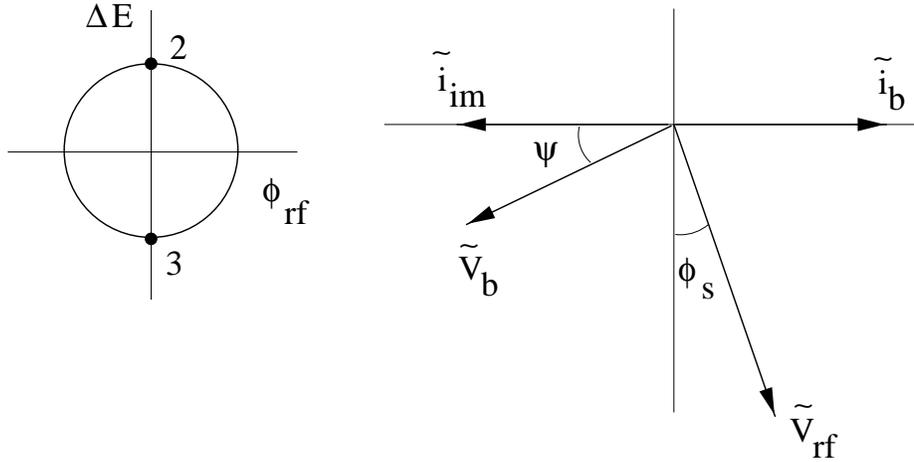


Figure 7.5: With bunch center at Point 2 in the synchrotron oscillation, the beam current phasor \tilde{i}_b is in phase with the x -axis in the phasor plot. The bunch sees a smaller rf voltage. Below transition, higher energy implies higher effective rf frequency ω_{rf} . The bunch center sees a smaller effective detuning angle and loses more energy per turn than if the energy of the bunch is larger at Point 3. The synchrotron oscillation amplitude is therefore damped.

in Fig. 7.5. Below transition, however, higher energy implies higher revolution frequency ω_0 . The detuning ψ which is defined by

$$\tan \psi = 2Q_L \frac{\omega_r - \omega_{rf}}{\omega_r} \quad (7.40)$$

will therefore be effectively smaller from the view of the bunch center, when we consider the effective rf frequency as $\omega_{rf} = h\omega_0$. The energy loss per turn, which is $i_{im}|Z_{cav}|\cos \psi$, will be larger than if the bunch center is synchronous. For the other half of the synchrotron period, for example, at Point 3, the beam particle has an energy smaller than the synchronous particle and revolves with a lower frequency, and therefore sees a larger effective detuning. Again we choose the moment when the phase of the bunch center is just in phase with synchronous particle. The bunch will lose less energy than if it is synchronous. The result is a gradual decrease in the energy offset oscillation after oscillation. This reduction of synchrotron oscillation amplitude is called *Robinson damping*. Notice that if the detuning is in the other direction, $\psi < 0$, the beam particle will lose less energy when its energy is higher than synchronous and lose more energy when its energy is less. The beam will therefore be Robinson unstable. The opposite is true if the beam is above

transition. We therefore have the criterion of Robinson stability:

$$\begin{cases} \psi > 0 & \text{or } \omega_r > \omega_{\text{rf}} & \text{below transition,} \\ \psi < 0 & \text{or } \omega_r < \omega_{\text{rf}} & \text{above transition.} \end{cases} \quad (7.41)$$

Notice that so far we have not imposed any optimization condition on the rf system. If the cavity tuning is adjusted so that the generator current \tilde{i}_g is in the same direction as the rf voltage \tilde{V}_{rf} , so that the beam-cavity impedance appears to be real as demonstrated in Fig. 7.3, the beam will always be Robinson stable, because the detuning will always satisfy Eq. (7.41) according to Eq. (7.29).

When the beam current is very intense, the phase loop may not be able to maintain the proper \tilde{V}_{rf} . Thus the condition of phase stability in Eq. (7.39) will be modified, because the effect of beam loading must be included. Now, go back to Fig. 7.4 when the beam current phasor arrives at an angle $\epsilon > 0$ ahead of the x -axis but is at the same energy as the synchronous particle, the image current phasor \tilde{i}_{im} will also advance by the same angle ϵ after h rf periods. Therefore, there will be an extra beam loading voltage phasor $\epsilon \tilde{i}_{\text{im}} R_L \cos \psi e^{j(\psi+3\pi/2)}$. If $\psi < 0$, this phasor will point into the 3rd quadrant and decelerate the particle in concert with $\epsilon V_{\text{rf}} \cos \phi_s$ in slowing the beam, thus causing no instability below transition. On the other hand, if $\psi > 0$, this phasor will point into the 4th quadrant and accelerate the particle instead. To be stable, the extra accelerating voltage on the beam must be less than the amount of decelerating voltage $\epsilon V_{\text{rf}} \cos \phi_s$, or

$$[V_{\text{rf}} \sin(\phi_s - \epsilon) - V_{\text{rf}} \sin \phi_s] + \epsilon \tilde{i}_{\text{im}} R_L \cos \psi \sin \psi \approx -\epsilon V_{\text{rf}} \cos \phi_s + V_{br} \cos \psi \sin \psi < 0. \quad (7.42)$$

Thus for phase stability, we require

$$\frac{V_{br}}{V_{\text{rf}}} < \frac{\cos \phi_s}{\sin \psi \cos \psi} \quad \begin{cases} \psi > 0 & \text{below transition,} \\ \psi < 0 & \text{above transition,} \end{cases} \quad (7.43)$$

which is called Robinson's *high-intensity criterion* of stability. In above, $V_{br} = \tilde{i}_{\text{im}} R_L$ is the *in-phase* beam loading voltage when the beam is in phase with the loaded cavity impedance.

Now let us impose the condition that the generator current \tilde{i}_g is in phase with the rf voltage \tilde{V}_{rf} . First, we have $i_0 = V_{\text{rf}}/R_L$, so that Eq. (7.43) can be rewritten as

$$\frac{\tilde{i}_{\text{im}}}{i_0} < \frac{\cos \phi_s}{\sin \psi \cos \psi} \quad \begin{cases} \psi > 0 & \text{below transition,} \\ \psi < 0 & \text{above transition.} \end{cases} \quad (7.44)$$

Second, the in-phase condition implies Eq. (7.29), which simplifies the above to

$$\frac{i_{im}}{i_0} < \frac{1}{\sin \phi_s}, \quad (7.45)$$

after eliminating the detuning. If we further optimize the generator power by choosing the coupling constant β_{op} given by Eq. (7.35), it is easy to show that

$$\frac{i_{im} \sin \phi_s}{i_0} = \frac{\beta_{op} - 1}{\beta_{op} + 1} < 1. \quad (7.46)$$

In other words, this phase stability criterion will always be satisfied.

Notice that this Robinson's high-intensity criterion of stability is only a criterion of phase stability similar to the phase stability condition of Eq. (7.39). Satisfying this criterion just enables stable oscillating like sitting inside a stable potential well. Violating this criterion will place the particle in an unstable potential well so that phase oscillation will not be possible. To include damping or antidamping due to the interaction of the beam with the cavity impedance, the first criterion of Robinson stability, Eq. (7.41) must be satisfied also.

7.4 TRANSIENT BEAM LOADING

By transient we mean that the fill time of the cavity T_f is not necessarily much longer than the time interval T_b for successive bunches to pass through the cavity. In other words, the beam loading voltage from the first bunch will have significant decay before the successive bunch arrives.

First, let us understand how the transient beam loading occurs. As the bunch of charge $q > 0$ passes through the cavity gap, a negative charge equal to that carried by the bunch will be left by the image current at the upstream end of the cavity gap. Since the negative image current will resume from the downstream end of the cavity gap following the bunch, an equal amount of positive charge will accumulate there. Thus, a voltage will be created at the gap opposing the beam current and this is the transient beam loading voltage as illustrated in Fig. 7.6. For an infinitesimally short bunch, this transient voltage is

$$V_{b0} \sim \frac{q}{C} = \frac{q\omega_r R_s}{Q_0}, \quad (7.47)$$

where C is the equivalent capacitance across the gap of the cavity. Notice that we will arrive at the same value if the loaded shunt impedance R_L and the loaded quality factor

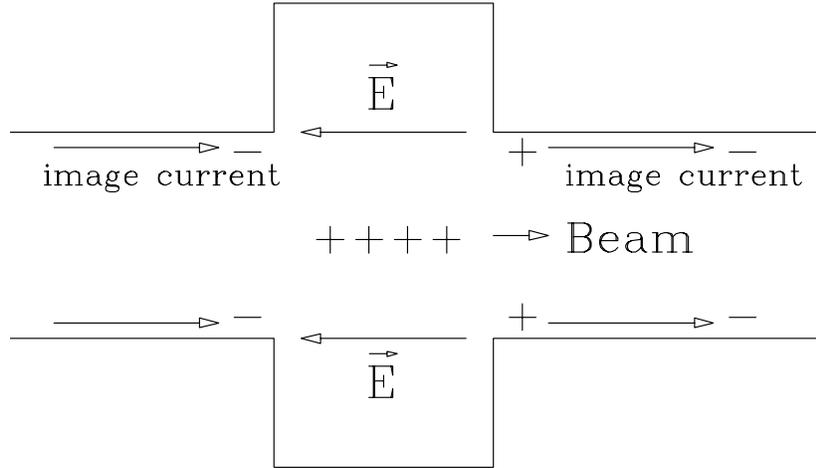


Figure 7.6: As a positively charged bunch passes through a cavity, the image current leaves a negative charge at the upstream end of the cavity gap. As the image current resumes at the downstream side of the cavity, a positive charge is created at the downstream end of the gap because of charge conservation, thus setting up an electric field \vec{E} and therefore the induced beamloading voltage.

Q_L are used instead. Due to the finite quality factor Q_L , this induced voltage across the gap starts to decay immediately, hence the name transient beam loading. We will give concrete example about the size of the voltage later. The next question is how much of this beam loading voltage will be seen by the bunch. This question is answered by the fundamental theorem of beam loading first derived by P. Wilson.

7.4.1 FUNDAMENTAL THEOREM OF BEAM LOADING

When a particle of charge q passes through a cavity that is lossless (infinite R_s and infinite Q_0), it induces a voltage V_{b0} which will start to oscillate with the resonant frequency of the cavity. Suppose that the particle sees a fraction f of V_{b0} , which opposes its motion. After half an oscillation of the cavity, a second particle of charge q passes through the cavity. The first induced voltage left by the first is now in the direction of the motion of the second particle and accelerates the particle. At the same time, this second particle will induce another retarding voltage \tilde{V}_{b0} which it will see as a fraction f . This second retarding voltage will cancel exactly the first one inside the cavity, since the cavity is assumed to be lossless. In other words, no field will be left inside the cavity after the

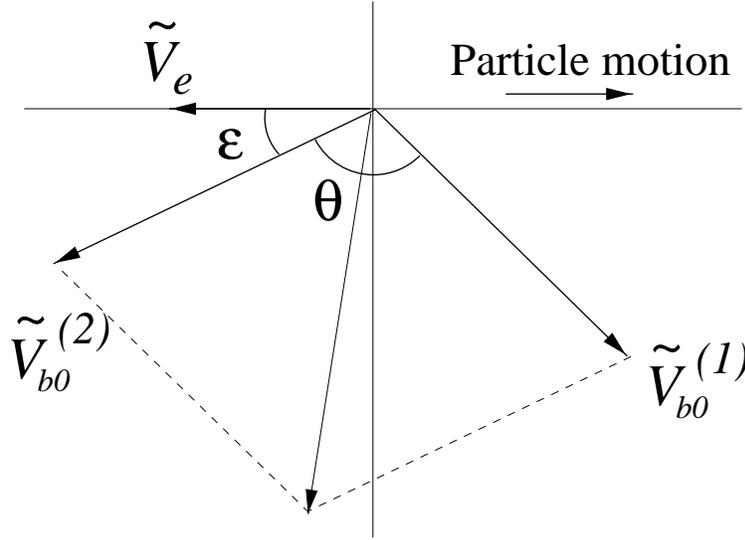


Figure 7.7: Phasor plot showing the the instant just after the second passage of the charged particle through the lossless cavity. The induced beam loading voltage phasors for the two passages are labeled as $\tilde{V}_{b0}^{(1)}$ and $\tilde{V}_{b0}^{(2)}$, respectively.

passage of the two particles. The net energy gained by the second particle is

$$\Delta E_2 = qV_{b0} - fqV_{b0} , \quad (7.48)$$

while the first particle gains

$$\Delta E_1 = -fqV_{b0} . \quad (7.49)$$

Conservation of energy requires that the total energy gained by the two particles must be zero. This implies $f = \frac{1}{2}$. In other words, the particle sees one half of its transient beam loading voltage, which is the fundamental theorem of beam loading.

The following is a more general proof by Wilson. The first particle induces a voltage phasor $\tilde{V}_{b0}^{(1)}$ in the cavity which may lie at an angle ϵ with respect to the voltage \tilde{V}_e seen by that particle. As before, we suppose $V_e = fV_{b0}$, where V_e and V_{b0} are the magnitudes of, respectively, \tilde{V}_e and $\tilde{V}_{b0}^{(1)}$. Some time later when the cavity phase changes by θ , the same particle returns via bending magnets or whatever and passes through the cavity again. It induces a second beam loading voltage phasor $\tilde{V}_{b0}^{(2)}$. At this moment, the phasor $\tilde{V}_{b0}^{(1)}$ rotates to a new position as illustrated in Fig. 7.7. The net energy lost by the particle on the two passes is

$$\Delta E = 2fqV_{b0} + qV_{b0} \cos(\epsilon + \theta) . \quad (7.50)$$

The cavity, however, gains energy because of the beam loading fields left behind. The energy inside a cavity is proportional to the square of the gap voltage. If the cavity is free of any field to start with, the final energy stored there becomes

$$\Delta E_c = \alpha \left(2V_{b0} \cos \frac{\theta}{2} \right)^2 = 2\alpha V_{b0}^2 (1 + \cos \theta) , \quad (7.51)$$

where α is a proportionality constant. From the conservation of energy, we get

$$2fqV_b + qV_{b0}(\cos \epsilon \cos \theta - \sin \epsilon \sin \theta) - 2\alpha V_{b0}^2 (1 + \cos \theta) = 0 . \quad (7.52)$$

Since θ is an arbitrary angle, we first obtain

$$\begin{aligned} qV_{b0} \sin \epsilon &= 0 , \\ qV_{b0} \cos \epsilon &= 2\alpha V_{b0}^2 , \\ 2fqV_{b0} &= 2\alpha V_{b0}^2 . \end{aligned} \quad (7.53)$$

The first equation gives $\epsilon = 0$ implying that the transient beam loading voltage must have a phase such as to maximally oppose the motion of the inducing charge. Clearly $\epsilon = \pi$ will not be allowed because this leads to the unphysical situation of the particle gaining energy from nowhere. Finally, we obtain $f = \frac{1}{2}$.

7.4.2 MULTI-BUNCH PASSAGE

Let the bunch spacing be h_b rf buckets or T_b in time. The cavity time constant or filling time is $T_f = 2Q_L/\omega_r$ and the e-folding voltage decay decrement between two successive bunch passages is $\delta_L = T_b/T_f$. During this time period, the phase of the rf fields changes by $\omega_r T_b$ and the rf phase by $\omega_{rf} T_b = 2\pi h_b$. The phasors therefore rotate by the angle $\Psi = \omega_r T_b - 2\pi h_b$, which can also be written in terms of the detuning angle,

$$\Psi = (\omega_r - \omega_{rf})T_b = \delta_L \tan \psi , \quad (7.54)$$

where Eq. (7.24) has been used. The transient beam loading voltage left by the first passage of a short bunch carrying charge q is $V_{b0} = q/C = q\omega_r R_L/Q_L$. The total beam loading voltage V_b seen by a short bunch is obtained by adding up vectorially the beam loading voltage phasors for all previous bunch passages. The result is

$$V_b = \frac{1}{2}V_{b0} + V_{b0}(e^{-\delta_L} e^{j\Psi} + e^{-2\delta_L} e^{j2\Psi} + \dots) , \quad (7.55)$$

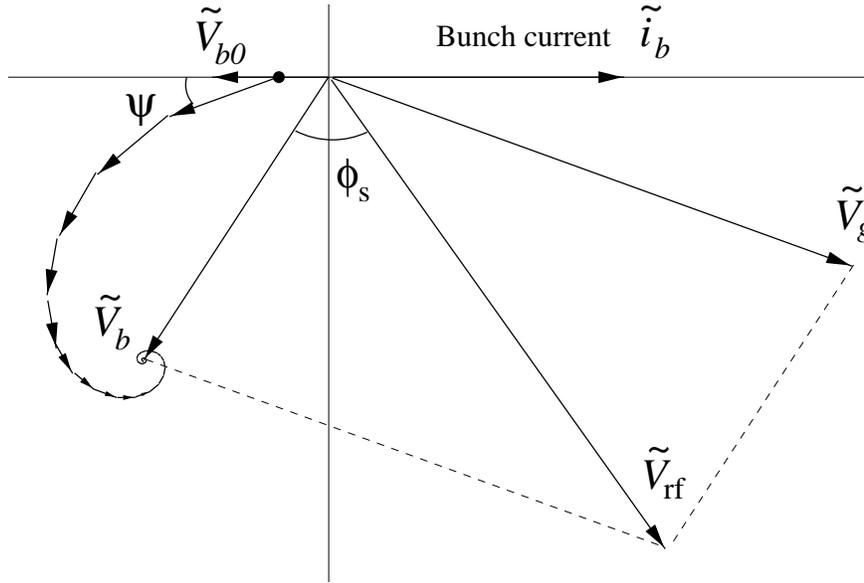


Figure 7.8: Transient beam loading voltages from equally spaced bunches. Each preceding voltage phasor has a phase advance of ψ because of detuning and a decay of $e^{-\delta_L}$. Note that the bunch that is just passing by sees only half of its induced voltage \tilde{V}_{b0} . These voltage phasors add up to the total beam loading voltage phasor \tilde{V}_b . Together with the generator voltage \tilde{V}_g , the cavity gap voltage results at the synchronous angle ϕ_s .

where the $\frac{1}{2}$ in the first term on the right side is the result of Wilson's fundamental theorem of beam loading, which states that a particle sees only one-half of its own induced voltage. It is worth pointing out that these voltages are excitations of the cavity and are therefore oscillating at the cavity resonant frequency (all higher order modes of the cavity are neglected). These infinite series of induced voltage phasors are illustrated in Fig. (7.8). The summation can be performed exactly giving the result

$$V_b = V_{b0} \left[F_1(\delta_L, \psi) + jF_2(\delta_L, \psi) \right], \quad (7.56)$$

with

$$F_1 = \frac{1 - e^{-\delta_L}}{2D}, \quad F_2 = \frac{e^{-\delta_L} \sin(\delta_L \tan \psi)}{D}, \quad (7.57)$$

$$D = 1 - 2e^{-\delta_L} \cos(\delta_L \tan \psi) + e^{-2\delta_L}. \quad (7.58)$$

In terms of the coupling constant β and detuning angle ψ , we have

$$\begin{aligned}\tan \psi &= 2Q_L \frac{\omega_r - \omega_{rf}}{\omega_r}, \\ Q_L &= \frac{Q_0}{1 + \beta}, \\ \delta_L &= \delta_0(1 + \beta),\end{aligned}\tag{7.59}$$

where we have defined $\delta_0 = T_b/T_{f0}$ with T_{f0} being the filling time of the unloaded cavity. Then the single bunch induced beam loading voltage becomes

$$V_{b0} = 2I_0 R_s \delta_0, \tag{7.60}$$

use has been made of the approximation for short bunches, so that the Fourier component of the current of a bunch at frequency ω_{rf}/h_b is equal to twice its dc value or $i_b = 2I_0$ and $I_0 = q/T_b$. Putting things together, we get

$$V_b = 2I_0 R_s \delta_0 \left[F_1(\beta, \phi) + jF_2(\beta, \phi) \right], \tag{7.61}$$

with

$$F_1(\beta, \phi) = \frac{1 - e^{-\delta_0(1+\beta)}}{2D}, \tag{7.62}$$

$$F_2(\beta, \phi) = \frac{e^{-\delta_0(1+\beta)} \sin[\delta_0(1 + \beta) \tan \psi]}{D}, \tag{7.63}$$

$$D = 1 - 2e^{-\delta_0(1+\beta)} \cos[\delta_0(1 + \beta) \tan \psi] + e^{-2\delta_0(1+\beta)}. \tag{7.64}$$

Some comments are in order. In Fig. 7.8, if we consider the beam loading voltage phasors that rotate by the angle Ψ and have its magnitude diminished by the factor $e^{-\delta_L}$ for each successive time period T_b to come from the passage of *one* short bunch, the plot shows the transient nature of beam loading. However, what we consider is in fact the diminishing beamloading voltage phasors coming from successive bunches that pass through the cavity at successive time period T_b earlier. For this reason, what Fig. 7.8 shows is actually the steady-state situation of the beam loading voltages, because for each time interval T_b later, we will see exactly the spiraling beam loading phasor plot and the same total beam loading voltage phasor \tilde{V}_b . For this reason, we can add into the plot the generator voltage phasor \tilde{V}_g in the same way as the plot in Fig. 7.3. In fact, the plot in Fig. 7.3 provides only an approximate steady-state plot, because the beam loading voltage phasor there does attenuate a little bit after a 2π rotation of the phasors,

although a high Q_L has been assumed. However, such attenuation has already been taken care of in Fig. 7.8, resulting in the plotting of an exact steady state. Although the total beam loading voltage phasor \tilde{V}_b seen by the passing bunch in Fig. 7.8 has the period of T_b , nevertheless, it is not sinusoidal. On the other hand, the beam loading voltage phasor \tilde{V}_b seen by the bunch in Fig. 7.3 is sinusoidal because it is induced by a sinusoidal component of the beam.

Using Eq. (7.14), the generator power V_g can now be computed:

$$P_g = \frac{(1 + \beta)^2 V_{\text{rf}}^2}{8\beta R_s \cos^2 \psi} \left\{ \left[\sin \phi_s - \frac{i_b R_s \delta_0}{V_{\text{rf}}} F_1(\delta_0, \beta) \right]^2 + \left[\cos \phi_s + \frac{i_b R_s \delta_0}{V_{\text{rf}}} F_2(\delta_0, \beta) \right]^2 \right\}. \quad (7.65)$$

In the situation when the generator current \tilde{i}_g is in phase with the rf voltage \tilde{V}_{rf} , the generator power can be minimized so that there will not be any reflection. Similarly, the generator power can also be optimized by choosing a suitable coupling coefficient β . Unfortunately, these optimized powers cannot be written as simple analytic expressions.

A. LIMITING CASE WITH $\delta_0 \rightarrow 0$

When the bunch spacing T_b is short compared to the unloaded cavity filling time T_{f0} , simplified expressions can be written for the total beam loading voltage V_b . One gets

$$F_1(\delta_0, \beta) = \frac{1}{\delta_0(1 + \beta)(1 + \tan^2 \psi)}, \quad (7.66)$$

$$F_2(\delta_0, \beta) = \frac{\tan \psi}{\delta_0(1 + \beta)(1 + \tan^2 \psi)}, \quad (7.67)$$

so that

$$V_b = \frac{i_b R_s}{1 + \beta} \frac{1}{1 - j \tan \psi}. \quad (7.68)$$

Notice that this is exactly the same expression in Eq. (7.25). In fact, this is to be expected, because we are in the situation of $T_b \ll T_f$, or the case of a high Q_L resonating cavity.

One may think that when $\delta_0 \rightarrow 0$, the phase angle $\Psi = \delta_0(1 + \beta) \tan \psi \rightarrow 0$. Thus, the transient beam loading voltage \tilde{V}_{b0} will not decay and will also line up for successive former bunch passages, leading to an infinite total beam loading voltage V_b seen by the bunch. However, $\delta_0 \rightarrow 0$ implies letting $Q_0 \rightarrow \infty$ while keeping the shunt impedance fixed. Thus, the instantaneous beam loading voltage $V_{b0} = q/C = q\omega_r R_s/Q_0 = 2i_b R_s \delta_0$ also goes to zero. Thus the summation has to be done with care. For successive V_{b0} to

wrap around in a circle, one needs approximately $2\pi/\Psi$ V_{b0} 's. The radius of this circle will be V_{b0}/Ψ . As $\delta_0 \rightarrow 0$, this radius becomes

$$\lim_{\delta_0 \rightarrow 0} \frac{V_{b0}}{\Psi} = \frac{2i_b R_s}{\tan \psi}, \quad (7.69)$$

which is finite. In fact, this is roughly the same as the total beam loading voltage V_b as $\delta_0 \rightarrow 0$.

B. LIMITING CASE WITH $T_b \gg T_f$

This is the situation when the instantaneous beam loading voltage decays to zero before a second bunch comes by. It is easy to see that $F_1(\delta_0, \beta) \rightarrow \frac{1}{2}$ and $F_2(\delta_0, \beta) \rightarrow 0$. From Eq. (7.65), it is clear that the generator power increases rapidly as the square of δ_0 . This is easy to understand, because the rf power that is supplied to the cavity gets dissipated rapidly. A pulse rf system will then be desirable. In such a system, the power is applied to the cavity for about a filling time preceding the arrival of the bunch. For most of the time interval between bunches, there is no stored energy in the cavity at all and hence no power dissipation.

7.5 AN EXAMPLE

Let us look into the design of a proposed future Fermilab pre-booster with has a circumference of 158.07 m. It accelerates 4 bunches each containing 0.25×10^{14} protons from kinetic energy 1 to 3 GeV. Because of the high intensity of the beam, the problems of space charge and beam loading must be addressed. We wish to examine the issues of beam loading and Robinson instabilities based on a preliminary rf system proposed by Griffin [5].

7.5.1 THE RAMP CURVE

Because of the high beam intensity, the longitudinal space-charge impedance per harmonic is $Z_{||}/n|_{\text{spch}} \sim -j100 \Omega$. But the beam pipe discontinuity will contribute only about $Z_{||}/n|_{\text{ind}} \sim j20 \Omega$ of inductive impedance. The space-charge force will be a large fraction of the rf-cavity gap voltage that intends to focus the bunch. A proposal is to insert ferrite rings into the vacuum chamber to counteract this space-charge force [6]. An experiment of ferrite insertion was performed at the Los Alamos Proton Storage Ring and the result has been promising [7]. Here we assume such an insertion will over-compensate

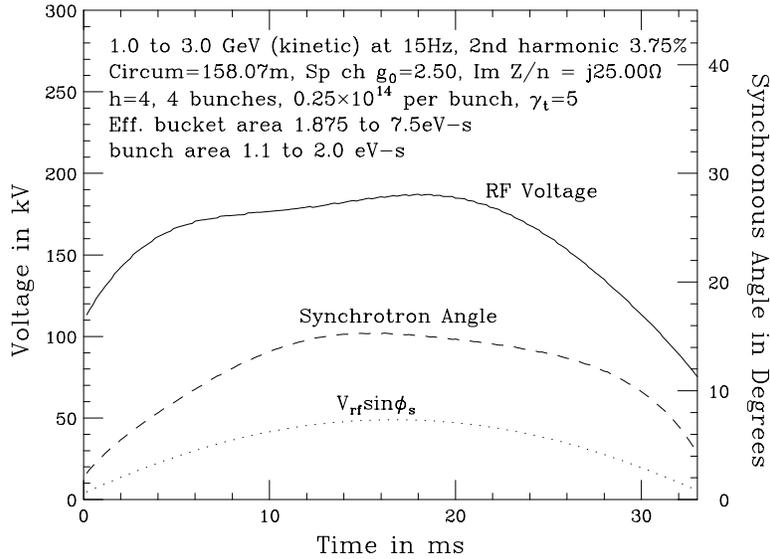


Figure 7.9: A typical ramp curve for the future Fermilab pre-booster.

all the space-charge force leaving behind about $Z_{\parallel}/n|_{\text{ind}} \approx j25 \Omega$ of inductive impedance. An over-compensation of the space charge will help bunching so that the required rf voltage needed will be smaller.

The acceleration from kinetic energy 1 to 3 GeV in 4 buckets at a repetition rate of 15 Hz is to be performed by resonant ramping. In order to reduce the maximum rf voltage required, about 3.75% of second harmonic is added. A typical ramp curve, with bucket area increasing quadratically with momentum, is shown in Fig. 7.9, which will be used as a reference for the analysis below. If the present choice of initial and final bucket areas and bunch areas is relaxed, the fraction of second harmonic can be increased. However, when the second harmonic is beyond $\sim 12.5\%$, it will only flatten the rf gap voltage in the ramp but will not decrease the maximum significantly.

7.5.2 THE RF SYSTEM

According to the ramp curve in Fig. 7.9, the peak voltage of the rf system is $V_{\text{rf}} \approx 185 \text{ kV}$. Griffin proposed 10 cavities [5], each delivering a maximum of 19.0 kV. Each cavity contains 26.8 cm of ferrite rings with inner and outer radii 20 and 35 cm, respectively. The ferrite has a relative magnetic permeability of $\mu_r = 21$. The inductance and capacitance of the cavity are $L \sim 0.630 \mu\text{H}$ and $C \sim 820 \text{ pF}$. Assuming an average ferrite loss of 134 kW/m^3 , the dissipation in the ferrite and wall of the cavity will be $P \sim 14.2 \text{ kW}$. The

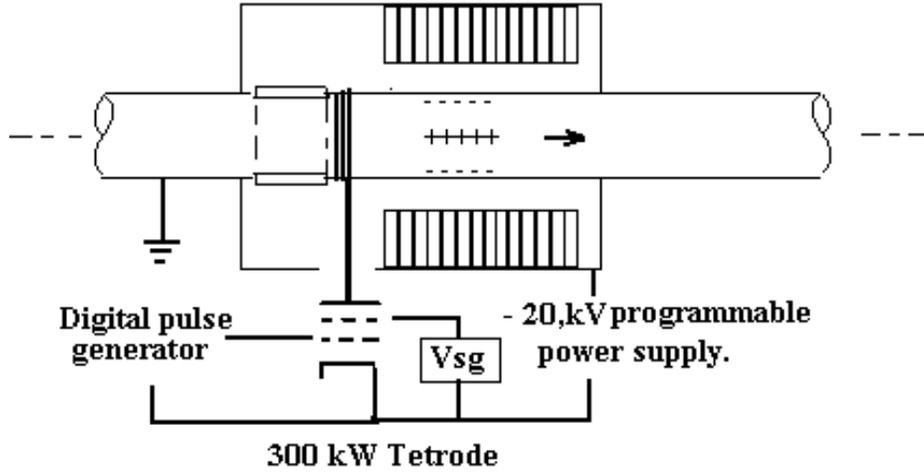


Figure 7.10: Transient beam-loading power tetrode connected directly to a rf cavity gap.

mean energy stored is $W \sim 0.15$ J. Therefore each cavity has a quality factor $Q \sim 459$ and a shunt impedance $R_s \sim 12.7$ k Ω .

Because each bunch contains $q = 4.005$ μC , the transient beam loading is large. For the passage of one bunch, 4.005 μC of positive charge will be left at downstream end of the cavity gap creating a transient beam loading voltage of $V_{b0} \sim q/C = 5.0$ kV, where $C = 820$ pF is the gap capacitance. We note from Fig. 7.9 that the accelerating gap voltages at both ends of the ramp are only about or less than 10 kV in each cavity. If the wakes due to the bunches ahead do not die out, we need to add up the contribution due to all previous bunch passages. Assuming a loaded quality factor of $Q_L = 45$, we find from Eq. (7.61) that the accumulated beam-loading voltage can reach a magnitude of $V_b = 36$ kV when the detuning angle is zero.

A feed-forward system is suggested which will deliver via a tetrode the same amount of negative charge to the downstream end of the gap so as to cancel the positive charge created there as the beam passes by. Without the excess positive charge, there will not be any more transient beam loading. This is illustrated in Fig. 7.10.

Here, we are in a situation where the image current i_{im} passing through the cavity gap is not equal to the beam current i_b . However, either at zero detuning or nonzero detuning, Eqs. (7.17) and (7.37) indicate that the portion of generator power transmitted to the acceleration of the beam is directly proportional to the magnitude of the image

current. If the image current goes to zero in this feed-forward scheme, this implies that the rf generator is not delivering any power to the particle beam at all, although the beam is seeing an accelerating gap voltage. Then, how can the particle beam be accelerated? The answer is simple, the power comes from the tetrode that is doing the feed-forward. This explains why the tetrode has to be of high power.

Actually, the feed-forward system is not perfect and we assume that the cancellation is 85 %. For a δ -function beam, the component at the fundamental rf frequency is 56.0 A. Therefore, the remaining image current across the gap is $i_{\text{im}} = 8.4$ A. To counter this remaining 15% of beam loading in the steady state, the cavity must be detuned according to Eq. (7.29) by the angle

$$\psi = \tan^{-1} \left(\frac{i_{\text{im}} \cos \phi_s}{i_0} \right), \quad (7.70)$$

where ϕ_s is the synchronous angle and $i_0 = V_{\text{rf}}/R_s$ is the cavity current *in phase* with the cavity gap voltage V_{rf} . For high quality factor of $Q = 459$ which is accompanied by a large shunt impedance, the detuning angle will be large. Corresponding to the ramp curve of Fig. 7.9, the detuning angle is plotted as dashes in Fig. 7.11 along with the synchronous angle and maximum cavity gap voltage. We see that the detuning angle is between 80° and 86° , which is too large. If a large driving tube is installed with anode (or cathode follower) dissipation at ~ 131 kW, the quality factor will be reduced to the loaded value of $Q_L \sim 45$ and the shunt impedance to the loaded value of $R_L \sim 1.38$ k Ω . The detuning angle then reduces to $\psi \sim 29^\circ$ at the center of the ramp and to $\sim 40^\circ$ or $\sim 56^\circ$ at either end. This angle is also plotted in Fig. 7.11 as a dot-dashed curve for comparison. Then, this rf system becomes workable.

7.5.3 FIXED-FREQUENCY RF CAVITIES

Now we want to raise the question whether it is possible to have a fixed resonant frequency for the cavity. A fixed-frequency cavity can be a very much simpler device because it may not need any biasing current at all. Thus the amount of cooling can be very much reduced and even unnecessary. It appears that the resonant frequency of the cavity should be chosen as the rf frequency at the *end* of the ramp, or $f_R = 7.37$ MHz so that the whole ramp will be immune to Robinson's phase-oscillation instability [4]. However, the detuning will be large. For example, at the beginning of the ramp where $f_{\text{rf}} = 6.64$ MHz, the detuning angle becomes $\psi = 85.2^\circ$. Since the beam-loading voltage V_{im} is small, the generator voltage phasor \tilde{V}_g will be very close to the gap voltage phasor

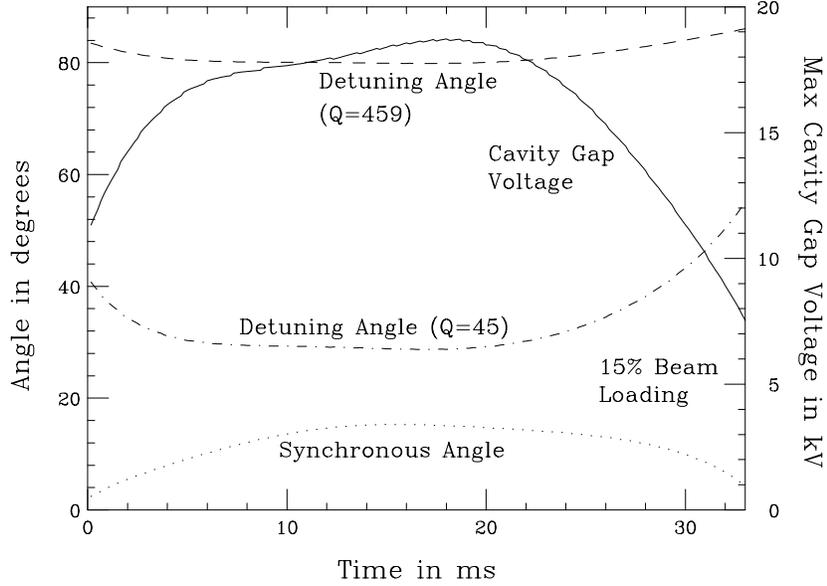


Figure 7.11: Detuning angle for the high $Q = 459$ and low $Q_L = 45$ situations.

\tilde{V}_{rf} . As a result, the angle θ between the gap voltage \tilde{V}_{rf} and the generator current phasor \tilde{i}_g will be close to the detuning angle, as demonstrated in Fig. 7.12. For example, Fig. 7.13 shows that, at the beginning of the ramp, the detuning angle is $\psi = 85.2^\circ$. Although the total power delivered by the generator

$$\frac{1}{2}\tilde{i}_g \cdot \tilde{V}_{\text{rf}} = \frac{V_{\text{rf}}^2}{2R_L} + \frac{1}{2}i_{\text{im}}V_{\text{rf}} \cos \phi_s \quad (7.71)$$

is independent of θ , the energy capacity of the driving tube has to be very large.

Another alternative is to choose the resonant frequency of the cavity to be the rf frequency near the *middle* of the ramp. Then the detuning angle ψ and therefore the angle θ between \tilde{V}_{rf} and \tilde{i}_g will be much smaller at the middle of the ramp when the gap voltage is large. Although θ will remain large at both ends of the ramp, however, this is not so important because the gap voltages are relatively smaller there. Figure 7.14 shows the scenario of setting the cavity resonating frequency f_R equal to f_{rf} at the ramp time of 13.33 ms.

There is a price to pay for this choice of f_R ; namely, there will be Robinson phase instability for the second half of the ramp when the rf frequency is larger than f_R . The instability comes from the fact that, below transition, the particles with larger energy have

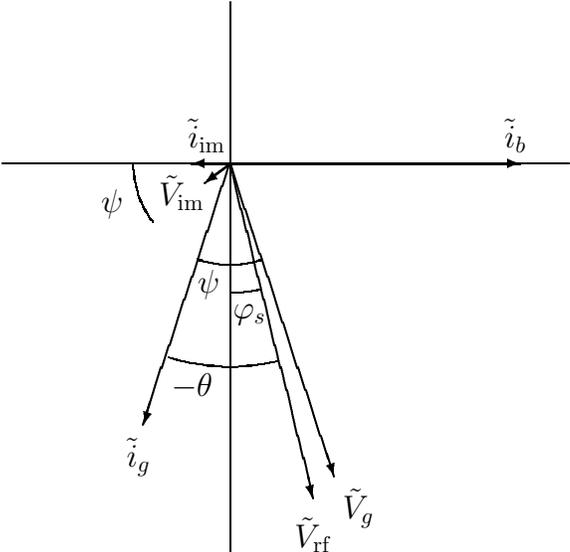


Figure 7.12: For a fixed cavity resonant frequency, the detuning angle ψ is fixed at each ramp time. When beam-loading is small, the angle θ between the gap voltage \tilde{V}_{rf} and the generator current \tilde{i}_g will be close to ψ and will be large.

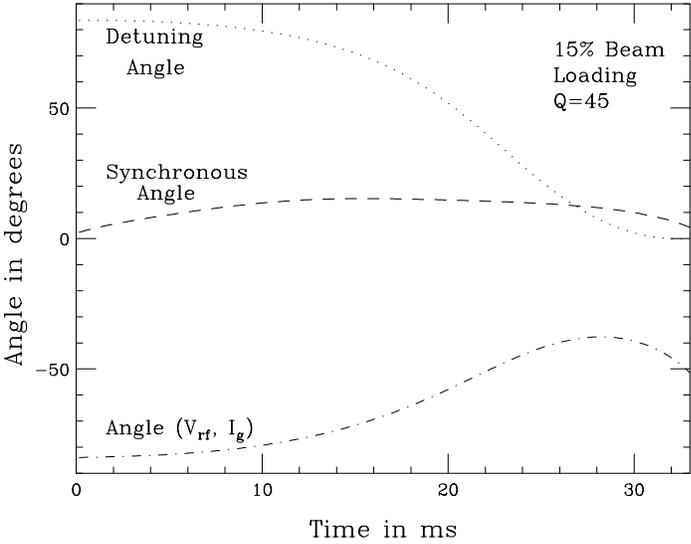


Figure 7.13: When the cavity resonant frequency is chosen as the rf frequency at the end of the ramp, both the detuning angle as well as the angle between the cavity gap voltage \tilde{V}_{rf} and the generator current \tilde{I}_g are large.

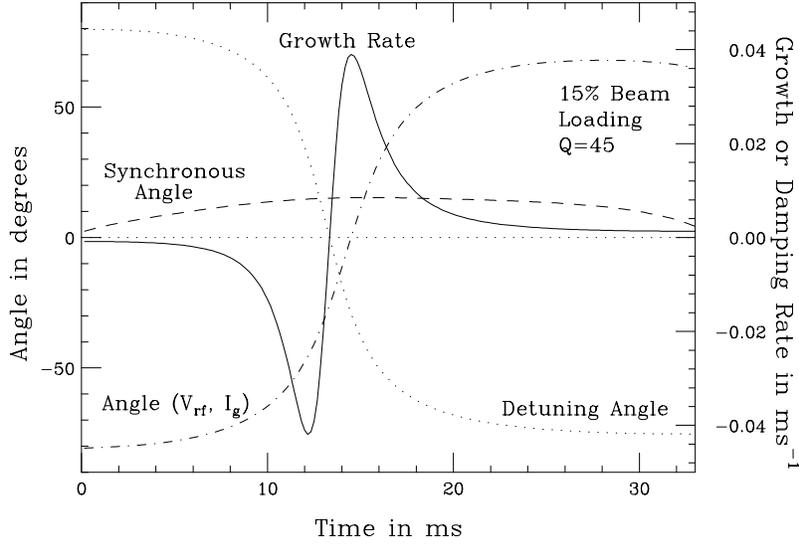


Figure 7.14: When the cavity resonant frequency is chosen as the rf frequency at the middle of the ramp at 13.33 ms, although the detuning angle as well as the angle between the cavity gap voltage \tilde{V}_{rf} and the generator current \tilde{I}_g are large at both ends of the ramp, they are relatively smaller at the middle of the ramp where the gap voltage is large.

higher revolution frequency and see a smaller real impedance of the cavity, thus losing less energy than particles with smaller energy. Therefore, the synchrotron amplitude will grow. In other words, the upper synchrotron sideband of the image current interacts with a smaller real impedance of the cavity resonant peak than the lower synchrotron sideband. However, since the loaded quality factor Q_L is not small, the difference in real impedance at the two sidebands is only significant when the rf frequency is very close to the cavity resonant frequency. Thus, we expect the instability will last for only a very short time during the second half of the ramp. The growth rate of the synchrotron oscillation amplitude has been computed and is equal to [2]

$$\frac{1}{\tau} = -\frac{i_{\text{im}}\beta\omega_s(R_+ - R_-)}{2V_{\text{rf}}\cos\phi_s}, \quad (7.72)$$

where

$$R_+ - R_- = \text{Re} \left[Z_{\text{cav}}(\omega_{\text{rf}} + \omega_s) - Z_{\text{cav}}(\omega_{\text{rf}} - \omega_s) \right], \quad (7.73)$$

i_{im} is the image current, β is the velocity with respect to light velocity, $\omega_s/(2\pi)$ is the synchrotron frequency, and Z_{cav} is the longitudinal impedance of the cavity. We see from

Fig. 7.14 that the growth occurs for only a few ms and the growth time is at least ~ 25 ms. The total integrated growth increment from ramp time 13.33 ms is $\Delta G = \int \tau^{-1} dt = 0.131$ and the total growth is $e^{\Delta G} - 1 = 14.0\%$ which is acceptable.

We also want to see whether Robinson's criterion for stable phase oscillation is satisfied for this rf consideration. For the second half of the ramp where the detuning angle $\psi < 0$, the phase is stable because we are below transition and the synchronous angle ϕ_s is between 0 and $\frac{1}{2}\pi$. For the first half of the ramp where $\psi > 0$, the sufficient condition for stability is, from Eq. (7.43), the high-intensity Robinson's criterion:

$$\frac{V_{\text{br}}}{V_{\text{rf}}} < \frac{\cos \phi_s}{\sin \psi \cos \psi}, \quad (7.74)$$

where $V_{\text{br}} = i_{\text{im}} R_L$ is the in-phase beam loading voltage. Figure 7.15 plots both sides of the criterion and shows that the criterion is well satisfied.

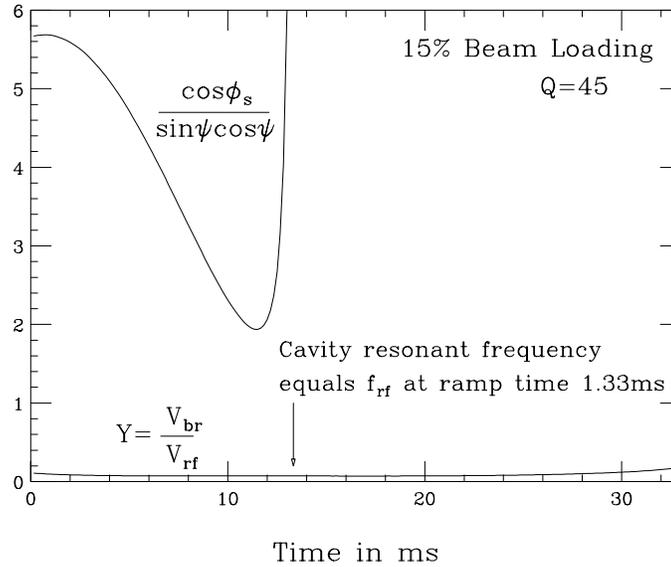


Figure 7.15: Plot showing the high-intensity Robinson's phase-stability criterion is satisfied.

Finally let us compute the beam loading voltage seen by a bunch including all the effects of the previous bunch passage. In this example, $\delta_L \approx \pi h_b / Q_L = 0.0698$ for $h_b = 1$ and $Q_L = 45$. When the detuning angle $\psi = 0$, $V_b \approx V_{b0} / (2\delta_L)$. The functions F_1 and F_2 are computed at some other values of ψ , which are listed in Table 7.1 and plotted in

Fig. 7.16. We see that the total transient beam loading V_t falls rapidly as the detuning angle ψ increases. It vanishes approximately $\sim 88.7^\circ$ and oscillates rapidly after that. However, the choice of a large ψ is not a method to eliminate beam loading, because the steady-state beam loading will not be reduced.

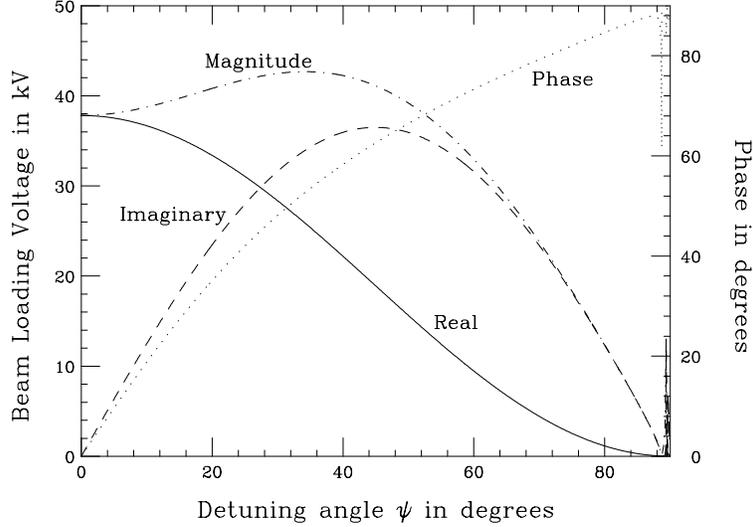


Figure 7.16: Plot of transient beam-loading voltage including all previous bunch passages, $\frac{q}{C}(F_1 + jF_2)$, versus detuning angle ψ .

Table 7.1: F_1 and F_2 for some values of the detuning angle ψ .

ψ	$\Psi = \delta_L \tan \psi$	F_1	F_2
0°	0°	$\sim \frac{1}{2\delta_L}$	0
84.9°	45°	0.061	1.197
87.5°	90°	$\sim \frac{\delta_L}{4}$	$\sim \frac{1}{2}$
88.7°	180°	$\sim \frac{\delta_L}{8}$	0

7.6 EXERCISES

- 7.1. For a Gaussian bunch with rms length σ_τ in a storage ring, find the Fourier component of the current at the rf frequency. Give the condition under which this component is equal to twice the dc current.
- 7.2. Prove the fundamental theorem of beam loading when there are electromagnetic fields inside before the passage of any charged particle.
- 7.3. In Section 7.2, rf-detuning and Robinson's stability condition have been worked out below transition. Show that above transition the detuning according the Fig. 7.3 leads to instability. Draw a new phasor diagram for the situation above transition with stable rf-detuning. Rederive Robinson's high-intensity stability criterion above transition.
- 7.4. Derive Eq. (7.65), the generator power delivered to the rf system with multi-passage of equally spaced bunches.
- 7.5. On passage through a cavity, the beam loading potential seen by a particle inside a bunch at a distance z behind the bunch center is

$$V(z) = \int_{-\infty}^z dz' \rho(z') W_0(z - z') , \quad (7.75)$$

where $\rho(z)$ is the charge distribution of the bunch, and W_0 is the wake function of the cavity. (a) Show that for a Gaussian charge distribution with rms length σ_ℓ the beam loading voltage seen by the bunch is

$$V(z) = -\frac{q\omega_r R_s}{2Q_0 \cos \phi_0} \mathcal{R}e e^{j\phi_0 - z^2/(2\sigma_\ell^2)} w \left[\frac{\sigma_\ell \omega_r e^{j\phi_0}}{c\sqrt{2}} - \frac{jz}{\sqrt{2}\sigma_\ell} \right] , \quad (7.76)$$

where q is the total charge in the bunch, $\sin \phi_0 = 1/(2Q_0)$, and w is the complex error function defined as

$$w(z) = e^{-z^2} \left(1 + \frac{2j}{\sqrt{\pi}} \int_0^z e^{t^2} dt \right) . \quad (7.77)$$

(b) Using the property of the complex error function,

$$\lim_{\sigma_\ell \rightarrow 0} w \left(\frac{-jz}{\sqrt{2}\sigma_\ell} \right) = \lim_{\sigma_\ell \rightarrow 0} \frac{2}{\sqrt{\pi}} e^{z^2/(2\sigma_\ell^2)} \int_{-\frac{z}{\sqrt{2}\sigma_\ell}}^{\infty} e^{-t^2} dt = \begin{cases} 0 & z < 0 , \\ 1 & z = 0 , \\ 2 & z > 0 , \end{cases} \quad (7.78)$$

show that as the bunch length shortens to zero, the head, center, and tail of the bunch are seeing the transient beamloading voltage

$$V(z) = \begin{cases} 0 & z < 0 \text{ (head) ,} \\ -\frac{q\omega_r R_s}{2Q_0 \cos \phi_0} & z = 0 \text{ (center) ,} \\ -\frac{q\omega_r R_s}{Q_0 \cos \phi_0} & z > 0 \text{ (tail) .} \end{cases} \quad (7.79)$$

- 7.6. Compute the transient beam loading voltage in the last problem by using a parabolic distribution

$$\rho(z) = \frac{3q}{4\ell} \left(1 - \frac{z^2}{\ell^2} \right) . \quad (7.80)$$

where ℓ is the half length of the bunch.

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