Chapter 13

BEAM BREAKUP

In a high-energy electron linac,* the longitudinal positions of the particles inside a bunch do not change. Thus, the tail particles are always affected by the head particles. We have shown that the longitudinal wake will cause the tail particles to lose energy. This loss accumulated throughout the whole length of the linac can be appreciable, leading to an undesirable spread in energy within the bunch. If the linac is the upstream part of a linear collider, this energy spread will have chromatic effect in the final focusing and eventually enlarging the spot size of the beam at the interaction point. We have also discussed how this energy spread can be corrected by placing the center of the bunch at a rf phase angle where the rf voltage gradient is equal and opposite to the energy gradient along the bunch.

Here, we would like to address the effect of the transverse wake potential. A small offset of the head particles will translate into a transverse force on the particles following. The deflections of the tail particles will accumulate along the linac. When the particles hit the vacuum chamber, they will be lost. Even if the aperture is large enough, the transverse emittance will be increased to an undesirable size. This phenomenon is called *beam breakup*. This is not a collective instability, however.

Recently, there is a lot of interest in isochronous or quasi-isochronous rings, where the spread in the slippage factor for all the particles in the bunch is very tiny, for example, $\Delta\eta \lesssim 10^{-6}$. In these rings, the head and tail particles hardly exchange longitudinal

*All proton linacs in existence are not ultra-relativistic. The highest energy is less than 1 GeV. Therefore synchrotron oscillations occur.

position, and we are having a situation very similar to linacs. Problems of beam breakup will also show up in these rings. The beam breakup discussed in this chapter does not allow particles to exchange longitudinal positions or change their longitudinal positions. We therefore assume that their velocities are equal to the velocity of light.

13.1 TWO-PARTICLE MODEL

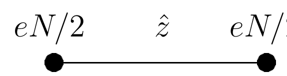

Figure 13.1: The two-particle model, where the bunch is represented by two macro-particles each carrying half the charge of the bunch separated by a distance $\hat{z}$.

Take the simple two-particle model in Fig. 13.1, by which the bunch is represented by two macro-particles of charge $\frac{1}{2}eN$ separated by a distance $\hat{z}$. The transverse displacements of the head, y_1, and the tail, y_2, satisfy

$$\frac{d^2y_1}{ds^2} + k_{\beta_1}^2 y_1 = 0 \ , \tag{13.1}$$

$$\frac{d^2y_2}{ds^2} + k_{\beta_2}^2 y_2 = -\frac{e^2NW_1(\hat{z})}{2LE}\, y_1 \ , \tag{13.2}$$

where s is the longitudinal distance measured along the designed particle path, W_1 is the transverse wake function for one linac cavity of length L, and k_β is the betatron wave number. For an isochronous ring, L will be taken as the ring circumference $C = 2\pi R$ and

$$k_\beta = \frac{\nu_\beta}{R} = \frac{1}{\langle\beta\rangle} \ , \tag{13.3}$$

where ν_β is the betatron tune and $\langle\beta\rangle$ is the average betatron function. This model has been giving a reasonably accurate description to the beam breakup mechanism for short electron bunches when $\hat{z}$ is taken as the rms bunch length. The head oscillates as $y_1(s) = y_{10}\cos k_\beta s$ and the tail is initially at $y_2 = y_{10}$ with $y_2' = 0$. The displacement of the tail can be readily solved and the result is

$$y_2(s) = y_{10}\cos\bar{k}_\beta s\cos\Delta k_\beta s - \left[\frac{e^2NW_1(\hat{z})}{4\pi E\bar{k}_\beta} + \frac{L\Delta k_\beta}{2\pi}\right]\left[y_{10}\sin\bar{k}_\beta s\right]\left[\frac{\sin\Delta k_\beta s}{\Delta k_\beta}\right] \ , \tag{13.4}$$

where $\bar{k}_\beta = \frac{1}{2}(k_{\beta_1} + k_{\beta_2})$ is the mean of the two betatron wave numbers of the head and tail. When the tune difference $\Delta k_\beta = k_{\beta_2} - k_{\beta_1}$ approaches zero, the tail is driven resonantly by the head and its displacement grows linearly with s:

$$y_2(s) = y_1(s) - \frac{e^2NW_1(\hat{z})}{4ELk_\beta}\left[y_{10}\sin k_{\beta_1}s\right]\ s\ . \tag{13.5}$$

In the length ℓ, the displacement of the tail will grow by Υ folds, where [2]

$$\Upsilon = -\frac{e^2NW_1(\hat{z})\ell}{4ELk_\beta} = -\frac{e^2NW_1(\hat{z})\langle\beta\rangle\ell}{4EL}\ , \tag{13.6}$$

where $W_1(\hat{z})$ is negative for small $\hat{z}$. In the above, we have written the growth in term of the average betatron function $\langle\beta\rangle$. This is because the transverse impedance initiates a kick y' of the beam and the size of the kicked displacement depends on the betatron function at the location of the impedance. This can be easily visualized from the transfer matrix.

For a broadband impedance, the transverse wake function at a distance z behind the source particle is, for $z > 0$,

$$W_1(z) = -\frac{\omega_r^2 Z_\perp}{Q\bar{\omega}}\, e^{-\alpha z/c}\sin\frac{\bar{\omega}z}{c}\ , \tag{13.7}$$

where $Z_\perp$ is the transverse impedance at the angular resonant frequency ω_r, which is shifted to $\bar{\omega} = \sqrt{\omega_r^2 - \alpha^2}$ by the decay rate $\alpha = \omega_r/(2Q)$ of the wake. Let us introduce the dimensionless variables

$$v = \frac{\omega_r\sigma_\ell}{c}\ ,\quad t = \frac{z}{\sigma_\ell}\ ,\quad \text{and}\quad \phi = vt\cos\phi_0 = \frac{\bar{\omega}z}{c}\ , \tag{13.8}$$

where the angle ϕ_0 is defined as

$$\cos\phi_0 = \sqrt{1 - \frac{1}{4Q^2}}\quad \text{or}\quad \sin\phi_0 = \frac{1}{2Q}\ , \tag{13.9}$$

assuming that $Q > \frac{1}{2}$. Then, the transverse wake in Eq. (13.7) can be rewritten as, for $\phi > 0$,

$$W_1(\phi) = -2\omega_r Z_\perp \tan\phi_0 \sin\phi\, e^{-\phi\tan\phi_0}\ , \tag{13.10}$$

The wake function decreases linearly from zero when $\phi = \bar{\omega}z/c \ll 1$ and reaches a minimum

$$W_1|_{\min} = -2\omega_r Z_\perp \tan\phi_0 \cos\phi_0\, e^{-\left(\frac{\pi}{2}-\phi_0\right)\tan\phi_0} \tag{13.11}$$

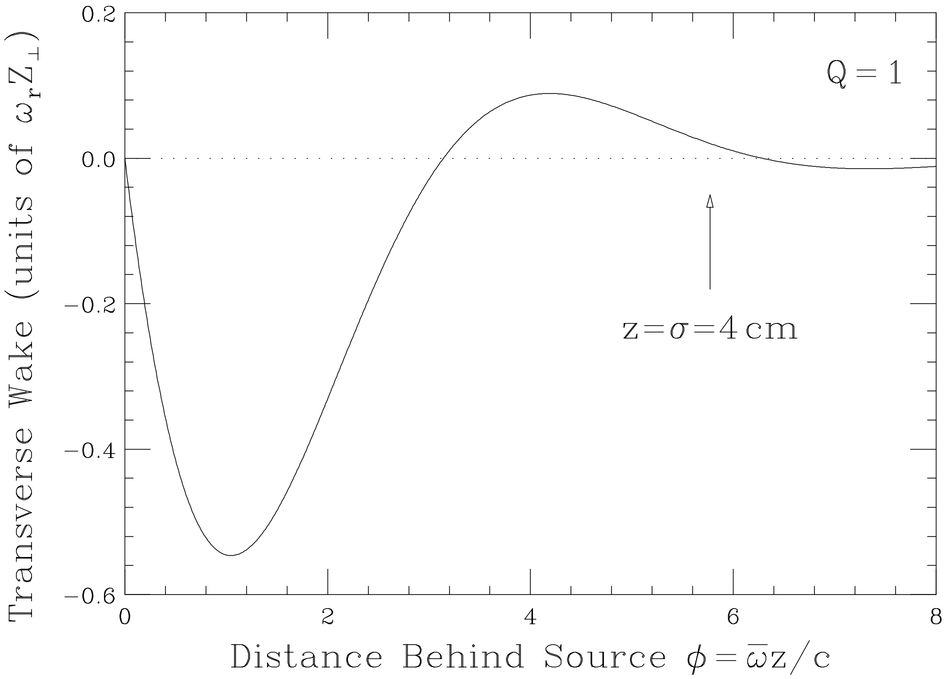

Figure 13.2: Transverse wake function for a broadband impedance with $Q = 1$ in units of $\omega_r Z_\perp$ as a function of $\phi = \bar{\omega}z/c$ behind the source. With resonant angular frequency $\omega_r = 50$ GHz, the position for $z = \sigma_\ell$ for the 4-cm bunch is marked, which is certainly outside the linear region and the 2-particle model will not apply.

at

$$\phi = \frac{\pi}{2} - \phi_0 \quad \text{or} \quad \frac{\alpha z}{c} = \left(\frac{\pi}{2} - \phi_0\right) \tan\phi_0 \ . \tag{13.12}$$

After that it oscillates with amplitude decaying at the rate of $\alpha = \omega_r/(2Q)$, crossing zero at steps of $\Delta\phi = \bar{\omega}z/c = \pi$. This is illustrated in Fig. 13.2.

Obviously, the growth expression of Eq. (13.6) does not apply to all bunch lengths. For example, if $\hat{z}$ just happens to fall on the first zero of $W_1(\hat{z})$, Eq. (13.6) says there is no growth at all. However, particles in between will be deflected and they will certainly affect the tail particle. Thus, the criterion for Eq. (13.6) to hold is the variation of the wake function along the bunch must be smooth. In other words, we must be in the linear region of the wake function, or

$$\phi = \frac{\bar{\omega}z}{c} \ll 1 \qquad \longrightarrow \qquad \sigma_\ell \ll \frac{1}{2}\frac{\lambda}{2\pi} \ , \tag{13.13}$$

i.e., the rms bunch length must be less than half the reduced wave length of the resonant impedance. As an example, if the broad-band impedance with $Q \sim 1$ has resonant frequency 7.96 GHz ($\omega_r = 50$ GHz), the two-particle model works only when the rms bunch length $\sigma_\ell \ll 3$ mm. Therefore, the model cannot be applied to the usual proton bunches. For the 50 GeV on 50 GeV muon collider, the muon bunches have a rms length of 4 cm, and will not be able to fit into this model also.

13.2 LONG BUNCH

For a bunch with linear density $\rho(z)$, the transverse motion $y(z,s)$ at a distance z behind the bunch center and *time* s is given by

$$\frac{d^2y(z,s)}{ds^2} + k_\beta^2 y(z,s) = -\frac{e^2N}{LE}\int_{-\infty}^{z} dz'\rho(z')W_1(z-z')y(z',s) \ . \tag{13.14}$$

This equation can be solved first by letting $y(z,s)$ be a free oscillation on the right-hand side and solving for the displacement $y(z,s)$ on the left-hand side. Then, iterations are made until the solution becomes stable. Therefore, when Υ is large, the growth will be proportional to powers of Υ and even exponential in Υ. Thus, $\langle\beta\rangle Z_\perp$, ω_r, as well as Q can be very sensitive to the growth.

Simulations have been performed for the 4-cm and 13-cm muon bunches in a quasi-isochronous collider ring, with a betatron tune $\nu_\beta \sim 6.24$, interacting with a broadband impedance with $Q = 1$ and $Z_\perp = 0.1$ MΩ/m at the angular resonant frequency $\omega_r =$ 50 GHz. Initially, a bunch is populated with a vertical Gaussian spread of $\sigma_y = 3$ mm and $y' = 0$ for all particles. There is no offset for the center of the bunch. Ten thousand macro-particles are used to represent the bunch intensity of 4×10^{12}. The half-triangular bin size is 15 ps (or 0.45 cm). In Fig. 13.3 we show the total growth of the *normalized* beam size $\sigma_y \equiv \langle y^2 + (\langle\beta\rangle y')^2\rangle^{1/2}$ relative to the initial beam size up to 1000 turns for various values of $\langle\beta\rangle$, respectively, for the 13-cm and 4-cm bunches. The turn-by-turn decay of the muons has been taken into account. We see that the beam size grows very much faster for larger betatron function. Also the growths for the 4-cm bunch are much larger than those for the 13-cm bunch because the linear charge density of the former is larger.

13.2.1 BALAKIN-NOVOKHATSKY-SMIRNOV DAMPING

Kim, Wurtele, and Sessler [2] suggested to suppress the growth of the transverse

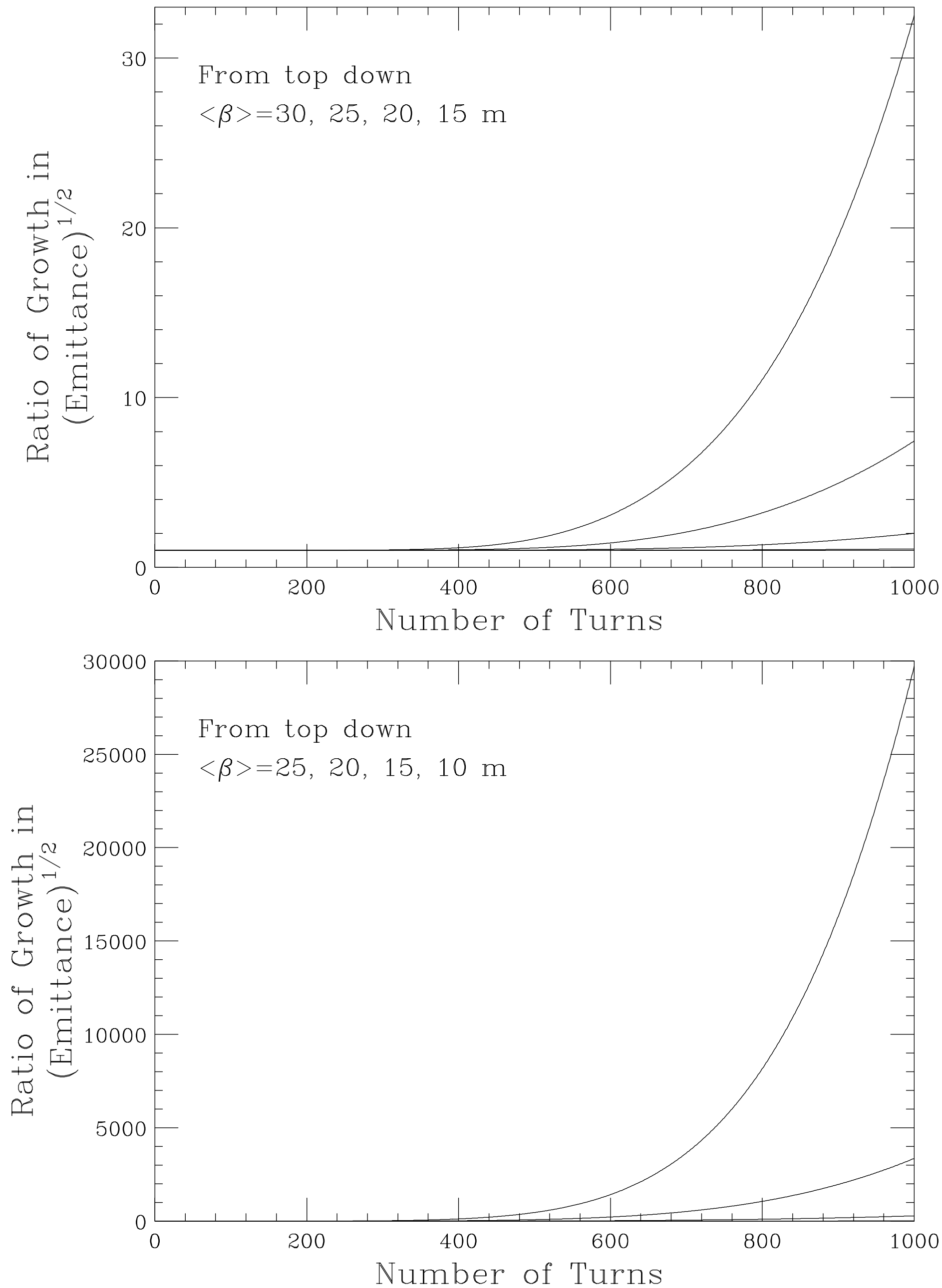

Figure 13.3: Beam-breakup growth for 1000 turns of a muon bunch interacting with a broadband impedance of $Q = 1$, $Z_\perp = 0.1$ MΩ/m at the angular resonant frequency of $\omega_r = 50$ GHz. Top: rms 13 cm bunch has total growths of 32.50, 7.4, 2.0, 1.09, 1.006, respectively for $\langle\beta\rangle = 30, 25, 20, 15, 10$ m. Bottom: rms 4 cm bunch has total growths of 29713, 3361, 287, 16.2, respectively for $\langle\beta\rangle = 25, 20, 15, 10$ m.

beam breakup by a small tune spread in the beam, coming either through chromaticity, amplitude dependency, or beam-beam interaction. This is because a beam particle will be resonantly driven by only a small number of particles in front that have the same betatron tune. This is a form of Balakin-Novokhatsky-Smirnov (BNS) damping suggested in 1983 [3].

To implement this, we add a detuning term

$$\Delta\nu_{\beta_i} = a[y_i^2 + (\langle\beta\rangle y_i')^2] \tag{13.15}$$

to the i-th particle, as if it is contributed by an octupole or sextupole. In Fig. 13.4, we plot the growths of the normalized beam size relative to the initial beam size with various rms tune spreads $\sigma_{\nu_\beta} = a\langle\sigma_y^2 + (\langle\beta\rangle\sigma_{y'})^2\rangle$. Here, an average betatron function of $\langle\beta\rangle = 20$ m has been used. This is because BPMs, which contribute significantly to the transverse impedance, are usually installed at locations where the betatron function is large. We see that to damp the growth of the 13-cm bunch to less than 1%, we need a rms tune spread of $\sigma_{\nu_\beta} = 0.0008$ or a total tune spread of $\Delta\nu_\beta = 3\sigma_{\nu_\beta} = 0.0024$. On the other hand, to damp the growth of the 4-cm bunch to less than 1%, we need a rms tune spread of $\sigma_{\nu_\beta} = 0.006$ or a total tune spread of $\Delta\nu_\beta = 3\sigma_{\nu_\beta} = 0.024$. However, if the transverse impedance is larger, the average betatron function is larger, the resonant frequency is larger, or the quality factor is smaller, this required tune spread may become too large to be acceptable. This is because a large amplitude-dependent tune spread can lead to reduction of the dynamical aperture of the collider ring.

For the lattice of the muon collider ring designed by Trbojevic and Ng [1], in order to allow for a large enough momentum aperture, the amplitude-dependent tune shifts are

$$\begin{aligned} \nu_{\beta x} &= 8.126 - 100\epsilon_x - 4140\epsilon_y \\ \nu_{\beta y} &= 6.240 - 4140\epsilon_x - 50.6\epsilon_y \end{aligned} \tag{13.16}$$

for the on-momentum particles, where the unnormalized emittances ϵ_x and ϵ_y are measured in πm. For the 4-cm bunch, the normalized rms emittance is $\epsilon_{\rm Nrms} = 85\times10^{-6}\ \pi$m. Since the muon energy is 50 GeV, the unnormalized rms emittance is $\epsilon_{\rm rms} = 1.80\times 10^{-7}\ \pi$m, and becomes $1.62\times10^{-6}\ \pi$m when 3σ are taken. Thus, the allowable tune spread for the on-momentum particles is $\Delta\nu_\beta = 4140\,\epsilon_y = 0.0067$. Tune spreads larger than this will lead to much larger tune spreads for the off-momentum particles, thus reducing the momentum aperture of the collider ring. For 4-cm bunch, to damp beam breakup to about 1% when $Z_\perp = 0.1$ MΩ/m and $\langle\beta\rangle = 20$ m, one needs $\Delta\nu_\beta = 0.024$. However, we do not know exactly what $\langle\beta\rangle$ and $Z_\perp$ are. Simulations show that if $\langle\beta\rangle Z_\perp$

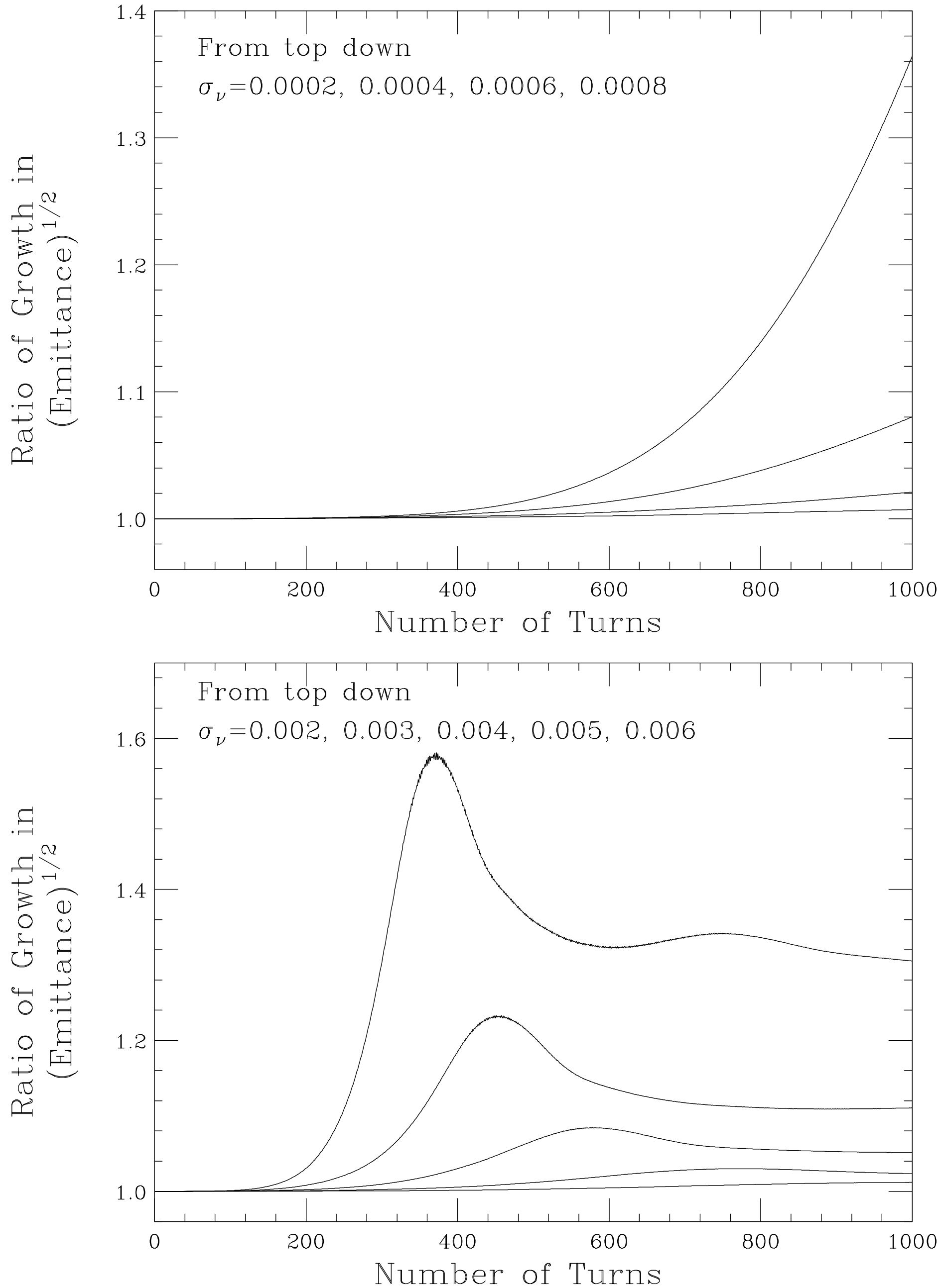

Figure 13.4: Total growth in 1000 turns in the presence of an amplitude dependent tune shift, such as provided by an octupole. An average betatron function of $\langle\beta\rangle =$ 20 m has been assumed. Top: growths of the rms 13 cm bunch are 1.36, 1.08, 1.02, 1.007, respectively for rms tune spread of $\sigma_{\nu_\beta} = 0.0002, 0.0004, 0.0006, 0.0008$. Bottom: growths of the rms 4 cm bunch are 1.58, 1.23, 1.08, 1.03, 1.012, respectively for rms tune spread of $\sigma_{\nu_\beta} = 0.002, 0.003, 0.004, 0.005, 0.006$.

becomes doubled, 2.5 times, 5 times, and 10 times, the tune spreads required jump to, respectively, ~ 0.054, 0.073, 0.18, and 0.54. Thus, it appears that pure tune spread may be able to damp beam breakup for the 13-cm bunch but may not work for the 4-cm bunch. Although tune spreads due to chromaticity and beam-beam interaction will also damp beam breakup, it is unclear how much the momentum aperture will be reduced due to these tune spreads.

13.2.2 AUTOPHASING

The transverse beam breakup can be cured by varying the betatron tune of the beam particles along the bunch, so that resonant growth can be avoided. In the two-particle model, we can set

$$\Delta\nu_\beta = -\frac{e^2 N W_1(\hat{z})}{2LE\bar{k}_\beta} \ , \tag{13.17}$$

in Eq. (13.4), so that the tail will be oscillating in phase and with the same amplitude and tune as the head. The is another form of BNS damping known as *autophasing* [4].

For a particle-distributed bunch, in order that all particles will perform betatron oscillation with the same frequency and same phase after the consideration of the perturbation of the transverse wake, special focusing force is required to compensate for the variation of unperturbed betatron tune along the bunch. With the linear distribution $\rho(z)$, the equations of motion of Eq. (13.2) in the two-particle model generalize to

$$\frac{d^2 y(z,s)}{ds^2} + \big[k_\beta + \Delta k_\beta(z)\big]^2 y(z,s) = -\frac{e^2 N}{LE}\int_{-\infty}^{z} dz' \rho(z') W_1(z-z') y(z',s) \ , \tag{13.18}$$

where $z > 0$ denotes the tail and $z < 0$ the head, or the bunch is traveling towards the left. We need to choose the compensation $\Delta k_\beta(z)$ along the bunch in such a way that the betatron oscillation amplitude

$$y(z,s) \sim \sin\big(k_\beta s + \varphi_0\big) \tag{13.19}$$

is independent of z, the position along the bunch, with φ_0 being some phase, because only in this way any particle will not be driven by a resonant force from any particle in front. The solution is then simply

$$2k_\beta \Delta k_\beta + \Delta k_\beta^2(z) = -\frac{e^2 N}{LE}\int_{-\infty}^{z} dz' \rho(z') W_1(z-z') \ , \tag{13.20}$$

or, for small compensation $\Delta k_\beta(z)$,

$$\frac{\Delta k_\beta(z)}{k_\beta} = -\frac{e^2NR}{2LEk_\beta^2}\int_{-\infty}^{z} dz'\rho(z')W_1(z-z') \ . \tag{13.21}$$

If the linear bunch distribution $\rho(z)$ is a Gaussian interacting with a broadband impedance, the integration can be performed exactly to give

$$\frac{\Delta k_\beta(z)}{k_\beta} = \frac{e^2N}{2LEk_\beta^2E}\frac{\omega_r^2 Z_\perp}{2\bar\omega Q}e^{-z^2/(2\sigma_\ell^2)}\,\mathcal{I}m\, w\left[\frac{ve^{j\phi_0}}{\sqrt{2}} - \frac{jz}{\sqrt{2}\sigma_\ell}\right] \ , \tag{13.22}$$

where w is the complex error function while $\sin\phi_0 = 1/(2Q)$ and $v = \omega_r\sigma_\ell/c$ as defined in Eqs. (13.8) and (13.9). For long bunches and high resonant frequency, or $v \gg Q$, the complex error function behaves as

$$w(z) = \frac{j}{\sqrt{\pi}z} + \mathcal{O}\left(\frac{1}{|z|^3}\right) \ . \tag{13.23}$$

This is certainly satisfied by both the 4-cm and 13-cm muon bunches, where $v = 6.67$ and 21.7, respectively, but not by the short electron bunches. Let us first discuss the long muon bunches in a storage ring. For convenience, we convert the betatron number to betatron tune by $k_\beta = \nu_\beta$ and the length L to the ring circumference $C = 2\pi R$. Thus Δk_β, the shift in betatron wave number in a cavity length L, becomes $\Delta\nu_\beta$, the betatron tune shift in a turn. Then, the relative tune-shift compensation in Eq. (13.22) can be simplified to

$$\frac{\Delta\nu_\beta(z)}{\nu_\beta} \approx \frac{e^2N\omega_r Z_\perp R}{2(2\pi)^{3/2}\nu_\beta^2 QvE}\left[1 + \frac{z}{vQ\sigma_\ell}\right]e^{-z^2/(2\sigma_\ell^2)} \ . \tag{13.24}$$

The relative tune-shift compensations required for the two bunches are shown in the top plot of Fig. 13.5. This is the situation for autophasing of short electron bunches, which is very different from the autophasing for the longer muon bunches. Note that in Eq. (13.24), vQ controls the asymmetry of the tune-shift compensation curve. When $vQ \to \infty$, there is no asymmetry and the compensation curve reduces to just a Gaussian, and, at the same time, $\Delta\nu_\beta/\nu_\beta$ decreases to zero. On the other hand, when $v \ll Q$ for short bunches or low broadband resonant frequency, the relative tune-shift becomes rather linear as depicted by the 1.8 mm ($v = 0.3$) curve in the lower plot of Fig. 13.5. The curves for the 5.0 mm, 1.0 cm, and 4 cm bunch ($v = 0.83$, 1.67, and 6.67) are also shown for comparison. Note that as the bunch length gets shorter, the frequency components of the tune compensation become much lower. For a very short bunch, the compensation becomes nearly linear in the region of the bunch.

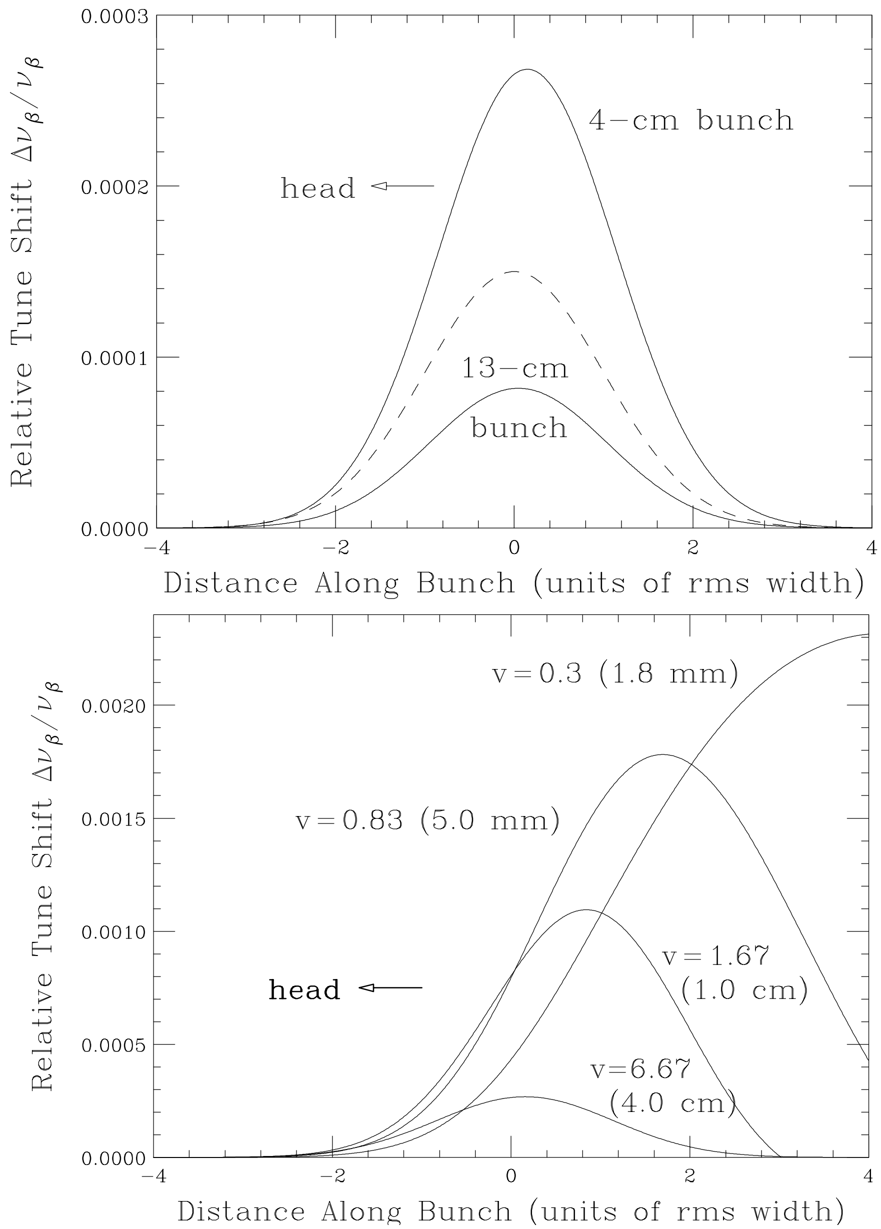

Figure 13.5: Relative tune shift autophasing compensation at distance z/σ_ℓ behind the bunch center (or bunch going to the left) to cure beam breakup. Impedance is broad-band resonating at $\omega_r = 50$ GHz. Top: for the rms 4-cm and 13-cm bunches, where $v = \omega_r\sigma_\ell/c = 6.67$ and 21.7 respectively, with bunch profile plotted in dashes as a reference. Bottom: for short bunches, rms 1.8, 5.0, 10.0 mm, with $v = 0.3$, 0.83, 1.67, respectively. The curve for the 4-cm bunch is plotted as comparison. Note that when v is small, the compensation is of much lower frequencies.

To cure beam breakup with autophasing damping in an electron linac, the electron bunch is usually placed off the crest of the rf wave so that the head and tail of the bunch will acquire slightly different energies, and therefore slightly different betatron tunes through chromaticity. For muon bunches in the collider ring, however, this method cannot be used. If one insists on having autophasing, a rf quadrupole must be installed and pulsed according to the compensation curve for each bunch as the bunch is passing through it. The variation of a quadrupole field at such high frequencies is not possible at all. Another method is to install cavities that have dipole oscillations at these frequencies, which is not simple either. For this reason, autophasing for long bunches is not practical at all.

13.3 LINAC

13.3.1 ADIABATIC DAMPING

Let us come back to the short electron bunches in a linac. An expression was given in Eq. (13.6) for the deflection of the tail particle in the two-particle model. In a linac, the bunches are accelerated and the energy change of the beam particles cannot be neglected. The equations of motion of the head and tail macro-particles now become

$$\frac{1}{\gamma}\frac{d}{ds}\left(\gamma\frac{dy_1}{ds}\right)+k_\beta^2 y_1=0\ , \tag{13.25}$$

$$\frac{1}{\gamma}\frac{d}{ds}\left(\gamma\frac{dy_2}{ds}\right)+k_\beta^2 y_2=-\frac{e^2NW_1(\hat{z})}{2L\gamma E_0}\,y_1\ , \tag{13.26}$$

where E_0 is the electron rest energy. The betatron wave number, which we have set to be the same for the two macro-particles, can have different dependency on energy. One way is to have k_β energy independent or the particle makes the same number of betatron oscillations per unit length along the linac. This is actually the operation of a synchrotron, where the quadrupole fields are ramped in the same way as the dipole field. If we further assume a constant acceleration

$$\gamma(s)=\gamma_i(1+\alpha s)\ , \tag{13.27}$$

where γ_i is the initial gamma and α is a constant, the equation of motion of the head becomes

$$\frac{d}{du}\left(u\frac{dy_1}{du}\right)+\frac{k_\beta^2}{\alpha^2}\,uy_1=0\ , \tag{13.28}$$

where $u = 1 + \alpha s$. Usually the acceleration gradient α is much slower than the betatron wave number k_β. For example, in the $L_0 = 3$ km SLAC linac where electrons are accelerated from $E_i = 1$ GeV to $E_f = 50$ GeV, $\alpha = 0.0163\ \mathrm{m}^{-1}$, while the betatron wave number is $k_\beta = 0.06\ \mathrm{m}^{-1}$. In that case, the solution is (Exercise 13.1)

$$y_1(s) = \frac{\hat{y}}{\sqrt{1+\alpha s}} \cos k_\beta s \ , \tag{13.29}$$

which is obtained by letting $y_1 = A \cos k_\beta s$ with A a slowly varying function of u. The equation of motion of the tail becomes

$$\frac{d}{du}\left(u \frac{dy_2}{du}\right) + \frac{k_\beta^2}{\alpha^2} u y_2 = -\frac{e^2 N W_1(\hat{z})}{2LE_i\alpha_2} \frac{\hat{y}}{\sqrt{u}} \cos k_\beta s \ . \tag{13.30}$$

To obtain the particular solution, we try $y_1 = D \sin k_\beta s$ with D a slowly varying function of u. The final solution is

$$y_2(s) = \frac{\hat{y}}{\sqrt{1+\alpha s}} \left[\cos k_\beta s - \frac{e^2 N W_1(\hat{z})}{2LE_i\alpha} \ln(1+\alpha s)\ \sin k_\beta s\right] \ . \tag{13.31}$$

Noticing that $E_i\alpha \approx E_f/L_0$, the growth for the whole length L_0 of the linac is

$$\Upsilon = -\frac{e^2 N W_1(\hat{z}) L_0}{4k_\beta E_f L} \ln \frac{E_f}{E_i} \ . \tag{13.32}$$

This is to be compared with Eq. (13.6), where we gain here a factor of

$$\mathcal{F} = \frac{E_i}{E_f} \ln \frac{E_f}{E_i} \tag{13.33}$$

For the SLAC linac, this factor is 7.8, meaning that the tail will be deflected by 7.8 less with the acceleration. This effect is called acceleration damping.

13.3.2 DETUNED CAVITY STRUCTURE

The dipole wake function of a cavity structure is given by

$$W_1(z) = -2 \sum_n K_n \sin \frac{2\pi\nu_n z}{c} e^{-\pi\nu_v z/(cQ_n)} \qquad z > 0 \ , \tag{13.34}$$

where K_n, ν_n, and Q_n are the kick factor, resonant frequency, and quality factor of the nth eigenmode in the structure. The kick factor is defined as

$$K_n = \frac{\pi R_n \nu_n}{Q_n} \ , \tag{13.35}$$

with R_n being the dipole shunt impedance of the nth mode. To reduce beam break up, it is important to reduce this dipole wake function.

One way to reduce the dipole wake is to manufacture the cavity structure with cell varying gradually so that each cell has a slightly different resonant frequency. In this case, the effect of the wake due to each individual cell will not add together and the wake of the whole structure will be reduced. Such a structure is called a *detuned cavity structure* [5].

Let us first study the short-range part of the dipole wake. The assumption that all the cells do not couple can be made, and the wake function of Eq. (13.34) can be considered as the summation of the wake due to each individual cell. Thus K_n, ν_n, and Q_n become the kick factor, resonant frequency, and quality factor of the nth cell. Since the variation from cell to cell is small, the summation can be replaced by an integral

$$W_1(z) \approx -2\int d\nu\, K\frac{dn}{d\nu}\sin 2\pi\nu zc\ . \tag{13.36}$$

Some comments are in order. First, the decays due to the quality factors have been neglected, because these are high-Q cavity and we are interested in the short-range wake only. Second, $K(dn/d\nu)$ is considered as a function of ν and the normalization of $dn/d\nu$ is unity because in Eq. (13.36), we refer $W_1(z)$ to the *dipole wake per cell.* Since $K(dn/d\nu)$ must be a narrow function centered about the average resonant frequency of the cells $\bar\nu$, the wake can be rewritten as

$$W_1(z) \approx -2\,\mathcal{I}m\left[e^{2\pi\bar\nu z/c}\int d\nu\, K(\bar\nu+x)\frac{dn}{d\nu}(\bar\nu+x)e^{2\pi ixz/c}\right]\ . \tag{13.37}$$

We see that the wake consists of a rapidly varying part, oscillating at frequency $\bar\nu$, and a slowly varying part, the envelope, that is given by the Fourier transform of the function $K(dn/d\nu)$ after it has been centered about zero. For uniform frequency distribution with full frequency spread $\Delta\nu$, the wake is given by

$$W_1(z) \approx -2\bar K\,\sin\frac{2\pi\bar\nu z}{c}\,\frac{\sin(\pi\Delta\nu z/c)}{\pi\Delta\nu z/c}\ , \tag{13.38}$$

with $\bar K$ the average value of K. If the frequency distribution is Gaussian with rms width σ_ν, then

$$W_1(z) \approx -2\bar K\,\sin\frac{2\pi\bar\nu z}{c}\,e^{-2(\pi\sigma_\nu z/c)^2}\ . \tag{13.39}$$

In this case, the envelope also drops as a Gaussian. It seems reasonable to expect that the proper Gaussian frequency distribution is near ideal in the sense of giving a rapid drop in the wake function for a given total frequency spread, and this is the motivation for choosing the Gaussian detuning.

Take the example of the Next Linear Collider (NLC). Consider a detuned structure with $N = 200$ cells. The central frequency is $\bar{\nu} = 15.25$ GHz. The detuned frequency distribution is Gaussian with $\pm 2.5\sigma_\nu$, where the rms spread σ_ν 2.5% of $\bar{\nu}$. It is found that the average kick factor is $\bar{K} = 40$ MV/nC/m^2. Such a wake is shown in the top plot Fig. 13.6. Notice that the wake function in fact does start from zero and has a first peak around 80 MV/nC/m^2 at $z \approx c/(4\bar{\nu}) = 4.91$ mm It is important to point out that the dipole wake function defined in this way differs from our usual definition; it is equal to our usual W_1/L with $L = 1$ m. The designed rms bunch length is $\sigma_\ell = 0.150$ mm which is much less than the fist peak. Therefore, the detuned structure will not help the single-bunch breakup at all. The bunch spacing is 42 cm. Over there the wake has dropped by more than two orders of magnitude. Thus, this lowering of the wake will definitely help the multi-bunch train beam breakup.

There are some comments on the wake depicted in Fig. 13.6. First, the wake does not continue to drop exponentially after about 0.4 m. Instead, it rises again having another peak around 4.2 m, although this peak is very much less than the first one. The main reason is due to the finite number of cells in the structure so that the Gaussian distribution has to be truncated. It is easy to understand the situation of the uniform frequency distribution of Eq. (13.38). The envelope is dominated by the $\sin x/x$ term which gives a main peak at $x = 0$ and starts to oscillate after the first zero at $z = c/\Delta\nu$. Second, the coupling of the cells will nevertheless become important at some larger distance. Thus the long-range part of the wake cannot be trusted at all. Bane and Gluckstern [5] used a circuit model with coupled resonators to give a more realistic computation of the long range wake. Later, Kroll, Jones, *et al.* [6] introduce a four-hole manifold in the cells to carry away the dipole wave generated by the beam. The final wake is shown in the bottom plot of Fig. 13.6. We see that the short-range part of the wake is almost the same as is given by the top plot of Fig. 13.6. On the other hand, the long-range wake has been kept much below 1 MV/nC/m^2. This wake has been computed first in the frequency domain as a spectral function and is then converted to the time or space domain via a Fourier transform. For this reason, we do not expect it to deliver the correct values at very short distances. The interested readers are referred to Refs. [5] and [6].

For the NLC, assuming a uniform energy independent betatron focusing with 100 betatron oscillations in the linac, the betatron wave number is $k_\beta = 0.06283$ m^{-1}. The

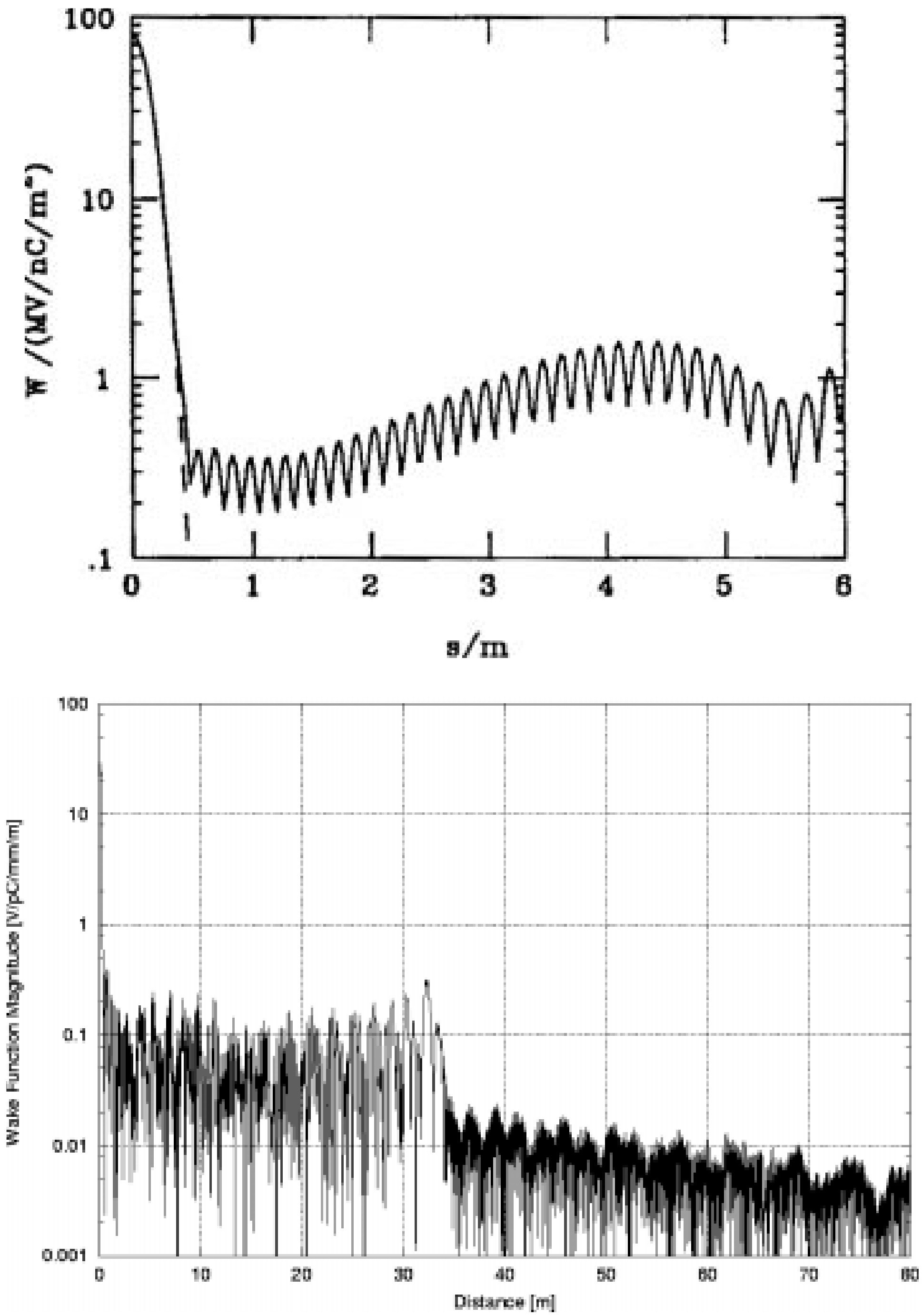

Figure 13.6: Envelope of the dipole wake function of a Gaussian detuned structure. Top: Coupling between cells has been ignored. Bottom: Coupling between cells has been included using a circuit model. Also the structure is coupled to a manifold.

NLC bunch has a vertical rms beam size of $\sigma_{y0} = 4.8$ μm, or the normalized rms vertical emittance $\epsilon_y = 0.028$ μm. The deflection of the tail particle in the two-particle model is multiplied only $\Upsilon \sim 2.1$ fold per unit offset of the head particle (see Exercise 13.3). Assuming 1 μm initial offset of the head particle, and conservation of normalized emittance in the absence of beam breakup, the normalized vertical emittance becomes $\epsilon_y = 0.30$ μm. For autophasing, assuming a chromaticity $\xi = 1$ defined by

$$\frac{\Delta k_\beta}{k_\beta} = \xi\delta \ , \tag{13.40}$$

an energy spread of 0.34% will be enough to damp the growth of the tail. These values are in close agreement of the simulations performed by Stupakov [10], as illustrated in Fig. 13.7.

13.3.3 MULTI-BUNCH BEAM BREAKUP

The NLC delivers a train of 95 bunches with bunch spacing 42 cm. Even if there is not beam breakup for a single bunch, the bunches in the train can also suffer beam breakup driven by the bunches preceding them. The first thing to do to ameliorate the situation is to design the linac cavities in such a way that the long-range dipole wake function will be as small as possible. The Gaussian detuned structure has been a way to lower the dipole wake by as much as two orders of magnitudes. According to the lower plot of Fig. 13.6, at 42 cm, the dipole wake is only ~ 0.21 MV/nC/m^2.

The two-particle model can be extended to accommodate the study of multi-bunch beam breakup. Each bunch is visualized as a macro-particle containing N electrons. Then the equation governing the displacement of the first bunch is

$$\frac{d^2y_1}{ds^2} + k_\beta^2 y_1 = 0 \ , \tag{13.41}$$

and that of the second bunch is

$$\frac{d^2y_2}{ds^2} + k_\beta^2 y_2 = -\frac{e^2NW_1(\hat{z})}{LE}\, y_1 \ , \tag{13.42}$$

The first equation is the free betatron oscillation and is the same as Eq. (13.1). For the second equation differs slightly from Eq. (13.2) in not having the factor 2 in the denominator. This is because in the two-particle model of a bunch, each macro-particle contains $\frac{1}{2}N$ electrons and here each macro-particle represents one bunch which is composed of N

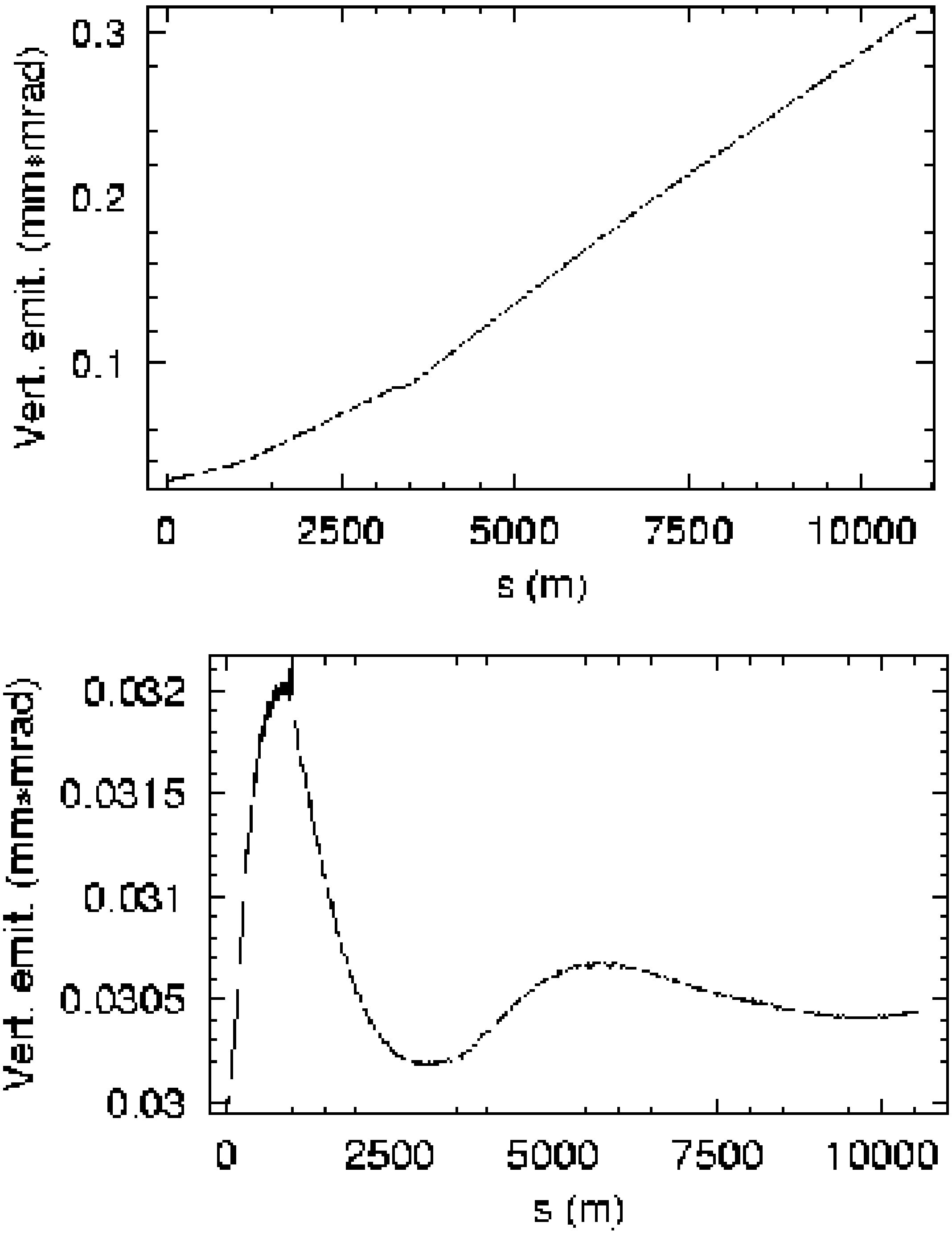

Figure 13.7: The normalized vertical emittance of a NLC bunch from the beginning to the end of the main linac, assuming an initial vertical offset of 1 μm. Top: The emittance increases to $\sim$ 0.3 μm because of beam breakup. Bottom: An energy spread of $\sim$ 0.8% is added across the bunch by offsetting the rf phase. The emittance increase has been damped.

electrons. Also the dipole wake $W_1(\hat{z})$ in Eq. (13.42) is evaluated at the bunch spacing $\hat{z}$. Recall that the two-particle model will not work when the bunch length is too long and falls out of the linear region of the dipole wake, because some particles in between the head and the tail will suffer more beam-breakup deflections than the tail. However, this model still works for a long train of bunches, because unlike a long bunch, there are no particles between the point bunches.

Now the solution for the first bunch is

$$y_1(s) = \mathcal{R}e\, \hat{y} e^{ik_\beta s} \ . \tag{13.43}$$

The solution for the second bunch is

$$y_2(s) = \mathcal{R}e\, \hat{y}\, \Gamma s e^{ik_\beta s} \ , \tag{13.44}$$

where

$$\Gamma = \frac{ie^2 N W_1(\hat{z})}{2k_\beta L E} \ , \tag{13.45}$$

and we have neglected the general solution

$$y_2(s)\big|_{\text{general}} = \hat{y} e^{\pm ik_\beta s} \ , \tag{13.46}$$

which is much smaller than the particular solution in Eq. (13.44) which grows linearly as s. The equation for the deflection of the third bunch is

$$\frac{d^2 y_3}{ds^2} + k_\beta^2 y_3 = -\frac{e^2 N W_1(2\hat{z})}{LE}\, y_1 - \frac{e^2 N W_1(\hat{z})}{LE}\, y_2 \ . \tag{13.47}$$

Here, we are going to retain only the largest driving force on the right-side. This means that the driving force from y_1 can be neglected and so is the force from the general solution of y_2. Substituting Eq. (13.44) in Eq. (13.47), we solve for the most divergent solution

$$y_3(s) = \mathcal{R}e\, \hat{y}\, \frac{1}{2}\, \Gamma^2 s^2 e^{ik_\beta s} \ . \tag{13.48}$$

Continuing this way, the deflection for the mth bunch will be (Exercise 13.4)

$$y_m(s) = \mathcal{R}e\, \hat{y} \frac{1}{(m-1)!} \Gamma^{m-1} s^{m-1} e^{ik_\beta s} \ . \tag{13.49}$$

Stupakov [11] tries to estimate how much energy spread will be required to BNS damp the multi-bunch beam breakup. In order to damp the deflection of the second bunch the amount of tune spread is

$$\frac{\Delta k_\beta}{k_\beta} = -\frac{e^2 N W_1(\hat{z})}{2k_\beta^2 E_f} \ln \frac{E_f}{E_i} \ , \tag{13.50}$$

taking the linac acceleration into account. It is reasonable to assume that n_b times spread will be required for n_b bunches. Next the natural chromaticity for a FODO lattice of phase advance μ is

$$\xi = -\frac{2}{\pi}\tan\frac{\mu}{2} \ . \tag{13.51}$$

For 95 bunches, one gets the required energy spread of 2.7% (Exercise 13.5). The simulations by Stupakov are shown in Fig. 13.8. The initial bunch offset is 1 μm and it takes an rms energy spread of 0.8% among the bunches to damp the growth.

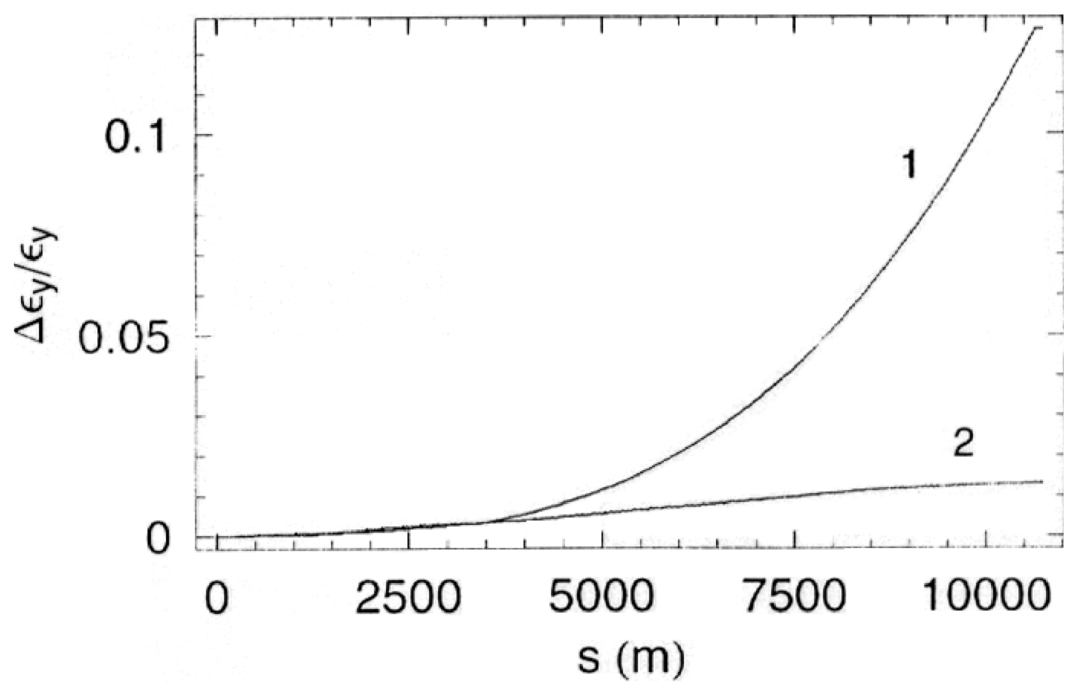

Figure 13.8: The relative change in vertical emittance of the 95th bunch, taking the vertical size as the vertical offset of the bunch center added to the actual rms vertical size in quadrature. The initial vertical offset is 1 μm. Curve 1 shows the growth without any energy variation in the bunches. Curve 2 shows the beam-breakup growth has been damped with a 0.8% rms energy spread from the first to the 95th bunch.

13.3.4 ANALYTIC TREATMENT

Analytic computation of beam breakup for a bunch train has been attempted by many authors [8, 7]. In all these papers, the dipole wake has been taken as a single dipole resonance and BNS damping has not been included. Recently, Bohn and Ng [9] have been able to include an energy chirp and derive analytic expressions for the BNS damping of a train of point bunches. Essentially, the energy chirp gives rise to a spread in betatron wave number among the bunches. In order for the derivation to go through analytically, it has been assumed that the betatron wave number decreases as $\gamma^{-1/2}$. This focusing arrangement implies that all the quadrupoles are identical and they can be on one common bus, because the focusing field gradient will be exactly the same along the linac. This implies the focusing becomes weaker as the energy increases. In fact, the NLC quadrupoles are deployed roughly in this way, although the quadrupoles there are all on separate buses for the ease of beam alignment. An outline of the analytic derivation is given below.

Introduce the dimensionless *spatial* parameter $\sigma = s/L_0$ normalized to the total linac length L_0. Introduce also the dimensionless *time* parameter $\zeta = \omega_r(t - s/c)$, with ω_r being the dipole resonant angular frequency, to describe the arrival of the first particle of the beam at position s along the linac. Thus, ζ measures the longitudinal position of the particle inside the beam. The transverse displacement of a particle in the beam, represented by $y(\sigma,\zeta)$, depends on both σ and ζ and its motion is governed by

$$\left[\frac{1}{\gamma}\frac{\partial}{\partial\sigma}\left(\gamma\frac{\partial}{\partial\sigma}\right)+\kappa^2(\sigma,\zeta)\right]y(\sigma,\zeta) = -\epsilon(\sigma)\int_0^{\zeta} d\zeta' w(\zeta-\zeta')F(\zeta')y(\sigma,\zeta')\ , \tag{13.52}$$

which is just another way of writing Eq. (13.14) with acceleration included as in Eq. (13.26). Here, the normalized betatron wave number is $\kappa = k_\beta L_0$. The beam profile $F(\zeta)$ will be defined in Eq. (13.55) below. The normalized dipole wake is

$$w(\zeta) = -H(\zeta)\, e^{-\zeta/(2Q)} \sin\zeta\ , \tag{13.53}$$

where Q is its quality factor and $H(\zeta)$ is the Heaviside step function. All the rest is lumped into the dimensionless beam-breakup coupling strength

$$\epsilon(\sigma) = \frac{e^2 N w_0 L_0^2}{\gamma E_0 \omega_r \tau}\ , \tag{13.54}$$

where w_0 is the sum-wake amplitude or twice the kick factor of the dipole resonance measured in V/C/m^2 and $N/(\omega_r\tau)$ is the number of electrons per *longitudinal time* ζ. For a train of bunches with temporal spacing τ, N becomes the number per bunch. When these bunches are further considered as points, the beam profile in above is represented by

$$F(\zeta) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{\zeta}{\omega_r \tau} - n\right) \ . \tag{13.55}$$

A betatron linear chirp is now introduced,

$$\kappa(\sigma, \zeta) = \kappa_0(\sigma) + \kappa_1(\sigma, 0)\zeta \ , \tag{13.56}$$

where $\kappa_0(\sigma)$ is the normalized betatron wave number without the chirp and $\kappa_1(\sigma, 0)$ represents the strength of the chirp. With the assumption that the acceleration gradient is much less than the betatron wave number, we can introduce a new transverse offset variable

$$\xi(\sigma, \zeta) = \sqrt{\gamma(\sigma)}\, y(\sigma, \zeta)\, e^{-i\zeta\Delta(\sigma)} \ , \tag{13.57}$$

where $\Delta(\sigma) = \int_0^\sigma d\sigma' K_1(\sigma', 0)$. Now Eq. (13.52) can be rewritten as

$$\left[\frac{\partial^2}{\partial\sigma^2} + \kappa_0^2(\sigma)\right] \xi(\sigma, \zeta) \simeq -\epsilon(\sigma) \int_0^\zeta d\zeta' w_\Delta(\sigma, \zeta - \zeta') F(\zeta') \xi(\sigma, \zeta') \ , \tag{13.58}$$

where the assumption of strong focusing, $\partial\xi(\sigma, \zeta)/\partial\sigma \simeq i\kappa_0 \xi(\sigma, \zeta)$, has been used. The chirped-modified wake in Eq. (13.58) is defined as

$$w_\Delta(\sigma, \zeta) = w(\zeta) e^{-i\zeta\Delta(\sigma)} \ , \tag{13.59}$$

where obviously the exponential comes from the definition of $\xi(\sigma, \zeta)$. This exponential, when combined with the exponential of the original wake of Eq. (13.53), gives an *effective quality factor* Q_{eff}, where

$$\frac{1}{2Q_{\text{eff}}} = \frac{1}{2Q} + i\Delta \ . \tag{13.60}$$

Immediately, a result can be drawn that the chirp will be important if Q is high, but will be masked if Q is sufficiently low.

The transformation into Eq. (13.58) is important, because the operator on the left side no longer depends on ζ, and the chirp has been incorporated into the dipole wake. In this form, the WKB method can be employed to give a formal result for $\xi(\sigma, \zeta)$. Denoting the displacement for the $(m+1)^{\text{th}}$ bunch as $y_m(\sigma) = y(\sigma, m\omega_r\tau)$, the solution is [8, 12]

$$y_m(\sigma) = \frac{1}{2\pi} \sum_{n=0}^{m} e^{-n\omega_r\tau/(2Q)} \int_{-\pi}^{\pi} d\theta\, e^{-in\theta} \left\{ y_{m-n}(0)\mathcal{C}(\sigma, \theta; m) + y'_{m-n}(0) \frac{\mathcal{S}(\sigma, \theta; m)}{\Lambda(0, \theta)} \right\} \ , \tag{13.61}$$

in which

$$\Lambda(\sigma,\theta) = \kappa_0(\sigma)\left\{1 - \frac{\epsilon(\sigma)}{4\kappa_0^2(\sigma)}\,\frac{\omega_r\tau\sin\omega_r\tau}{\cos[\theta+\omega_r\tau\Delta(\sigma)] - \cos\omega_r\tau}\right\} \tag{13.62}$$

is an auxiliary function reflecting the coupling between the bunch spacing and the deflecting-mode frequency, and

$$\left\{\begin{array}{c}\mathcal{C}(\sigma,\theta;m)\\ \mathcal{S}(\sigma,\theta;m)\end{array}\right\} = \sqrt{\frac{\Lambda(0,\theta)}{\Lambda(\sigma,\theta)}}\left\{\begin{array}{c}\mathcal{R}e\\ \mathcal{I}m\end{array}\right\}\exp\left[im\omega_r\tau\Delta(\sigma) + \int_0^\sigma d\sigma'\Lambda(\sigma',\theta)\right] \tag{13.63}$$

are cosine-like and sine-like functionals, respectively.

It is evident from Eq. (13.61) that upon taking $\theta \to -\theta$ and remembering that y_m is real, the algebraic sign of $\Delta(\sigma)$ affects only the phase of $y_m(\sigma)$ but not the envelope. This demonstrates that, as expected intuitively, the effect of a linear increase in focusing from head to tail is the same as a linear decrease.

For further discussion, let us set the initial conditions $y_m(0) = y_0$ and $y'_m(0) = 0$ for every bunch, and assume a constant acceleration gradient in the linac. The sum in Eq. (13.61) can be decomposed into two parts: $\sum_0^m = \sum_0^\infty - \sum_m^\infty$. The first part pertains to the *steady-state* displacement y_{ss} that would arise were the deflecting wake first seeded with an infinitely long bunch train immediately preceding the actual bunch train. Given strong focusing, the steady-state displacement is

$$y_{ss}(\sigma, m\omega_r\tau) \simeq y_0\left[\frac{E_i}{E_\sigma}\right]^{1/4}\cos\left[m\omega_r\tau\Delta(\sigma) + \int_0^\sigma d\sigma'\kappa_0(\sigma')\right], \tag{13.64}$$

where we have written, for convenience, the energy of the beam particle at location σ as $E_\sigma = \gamma(\sigma)E_0$ and the initial energy as $E_i = \gamma(0)E_0$. Later we will also write the energy at linac exit as $E_f = \gamma(1)E_0$.

The second part pertains to the *transient* displacement $\delta y_m = y_m - y_{ss}$. Saddle-point integration gives a closed-form solution for δy_m, whose bounding envelope takes the form:

$$\frac{|\delta y_m|}{y_0} \simeq \left[\frac{E_i}{E_\sigma}\right]^{\frac{1}{4}}\frac{\sqrt{P}\ \exp\left[q(\eta)P - m\omega_r\tau/(2Q)\right]}{4m\sqrt{2\pi}\,|\sin(\omega_r\tau/2)|}\times\left\{\begin{array}{ll}|1-\eta^2|^{-\frac{1}{4}} & \eta \text{ not near } 1\\ \left(\frac{4}{3}\right)^{\frac{1}{6}}P^{\frac{1}{6}}\,\frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt{2\pi}} & \eta = 1.\end{array}\right. \tag{13.65}$$

The auxiliary relations comprising Eq. (13.65) are:

$$
\begin{aligned}
P(\sigma, m) &= \left[\frac{4mw_0e^2NL_0^2}{\bar{\kappa}_0 E_i}\right]^{1/2} \frac{\left[\left(\sqrt{E_f/E_i}-1\right)\left(\sqrt{E_\sigma/E_i}-1\right)\right]^{1/2}}{E_f/E_i - 1} , \\
\eta(\sigma, m) &= \frac{\bar{\kappa}_0|f_\gamma|}{2P}\frac{m}{M}\frac{\sqrt{E_\sigma/E_i}-1}{\sqrt{E_f/E_i}-1} , \\
q(\eta) &= \begin{cases} \dfrac{\sqrt{1-\eta^2}}{2} + \dfrac{1}{4\eta}\tan^{-1}\left(\dfrac{2\eta\sqrt{1-\eta^2}}{1-2\eta^2}\right) & \eta < 1 \\ \dfrac{\pi}{4\eta} & \eta \geq 1 , \end{cases}
\end{aligned}
$$

in which $\bar{\kappa}_0$ is the focusing strength averaged over the linac and is related to the focusing strength at entrance $\kappa_0(0)$ by

$$
\bar{\kappa}_0 = \frac{2\kappa_0(0)}{\sqrt{E_f/E_i}+1} , \tag{13.66}
$$

M is the total number of bunches in the train, $|f_\gamma|$ is the magnitude of the total fractional energy spread across the bunch train, or twice the total fractional focusing variation.

The expression for $|\delta y_m|$ in Eq. (13.65) reflects a number of physical processes. The coefficient involving beam energy manifests adiabatic damping. The factor $|\sin(\omega_r\tau/2)|$ is a relic of a resonance function deriving from the coupling between the bunch spacing and the deflecting-mode frequency. Resonances lie near even-order wake zero-crossings [8]; because the solution is valid only away from zero-crossing, resonance is removed. The focusing variation represented by $|f_\gamma|$ regulates exponential growth, and finite Q yields exponential damping. Yet "$\eta = 1$" does have special physical significance; it demarks the onset of saturation of exponential growth and, with infinite Q, algebraic decay of the envelope. For $\eta \geq 1$ the "growth factor" $q(\eta)P$ is independent of bunch number m and of linac coordinate σ; temporal "damping" then ensues through a negative power of m, and spatial "damping" ensues adiabatically as already mentioned. Therefore $\eta = 1$ corresponds to a global maximum in the envelope $|\delta y_m|$. The effect of the focusing variation is the saturation of the exponential growth, not damping; its action distinctly differs from that of a real effective Q.

The special significance of $\eta = 1$ translates into a criterion for the focusing variation to be effective. Specifically, one should choose a value of f_γ that ensures $\eta(1, M) > 1$, *i.e.*, that $\eta = 1$ is reached somewhere along the bunch train before it leaves the linac.

According to the auxiliary relations to Eq. (13.65), the criterion is

$$|f_\gamma| > \frac{2P(1,M)}{\bar{\kappa_0}} \ . \tag{13.67}$$

The analytic envelope at the linac exit is plotted in Fig. 13.9 with total energy spread of 1.5% at the top and 3% at the bottom. Simulations are also displayed showing excellent agreement with theory. The plots are made with the scenario that the linac is $L_0 = 10$ km long, accelerating 90 bunches with bunch spacing $\tau = 2.8$ ns from 10 GeV to 1 TeV. Each bunch contains 1 nC of charges or $N = 6.24 \times 10^9$ electrons, making 1000 betatron oscillations along the linac. The dipole wake is of resonant frequency $\omega_r/(2\pi) = 14.95$ GHz with infinite quality factor Q and sum-wake amplitude $w_0 = 1$ MV/nC/m^2.

The steady-state and transient displacements, being uncorrelated, comprise a measure of the total projected normalized emittance as

$$\varepsilon \equiv \left(|y_{ss}|^2 + |\delta y_m|^2_{max}\right) \frac{\gamma\kappa_0}{L_0} \ , \tag{13.68}$$

wherein $|y_{ss}| = y_0[E_i/E_\sigma]^{1/4}$ per Eq. (13.64), and $|\delta y_m|_{\rm max}$ is the maximum value of the transient envelope reached along the bunch train. If $\eta < 1$ always, then the maximum is reached at the last bunch $m = M$. Otherwise, the maximum corresponds to the value of $|\delta y_m|$ at which $\eta = 1$. Imposing a focusing variation will reduce the transient envelope, but it also will establish a harmonic variation of y_{ss} with m and thereby introduce a nonzero steady-state emittance ε_{ss}. For this reason the quantity of interest is the ratio

$$(\varepsilon - \varepsilon_{ss})/\varepsilon_{ss} = \left(\frac{|\delta y_m|_{max}}{|y_{ss}|}\right)^2 \ , \tag{13.69}$$

from which one sees the benefit of keeping the ratio of envelopes small. This quantity, calculated from the analytic expressions given in Eqs. (13.64) and (13.65), is plotted against $|f_\gamma|$ in Fig. 13.10 for various values of the sum-wake amplitude w_0. Fig. 13.10 points to the region of parameter space that, respecting multibunch beam breakup, admits viable linear-collider designs. In particular it shows that to achieve low multibunch emittance without aid from a focusing variation requires small sum-wake amplitudes, $w_0 \lesssim 0.5$ V/pC/mm/m. Otherwise, as depicted, a few-percent energy spread relieves the constraint on sum-wake amplitude. There are, of course, practical limitations on the energy spread, to include longitudinal beam requirements at the interaction point, lattice chromaticity, degradation in acceleration, etc. Nonetheless, introducing a modest energy spread constitutes a backup in case sufficiently low wake amplitudes prove generally infeasible.

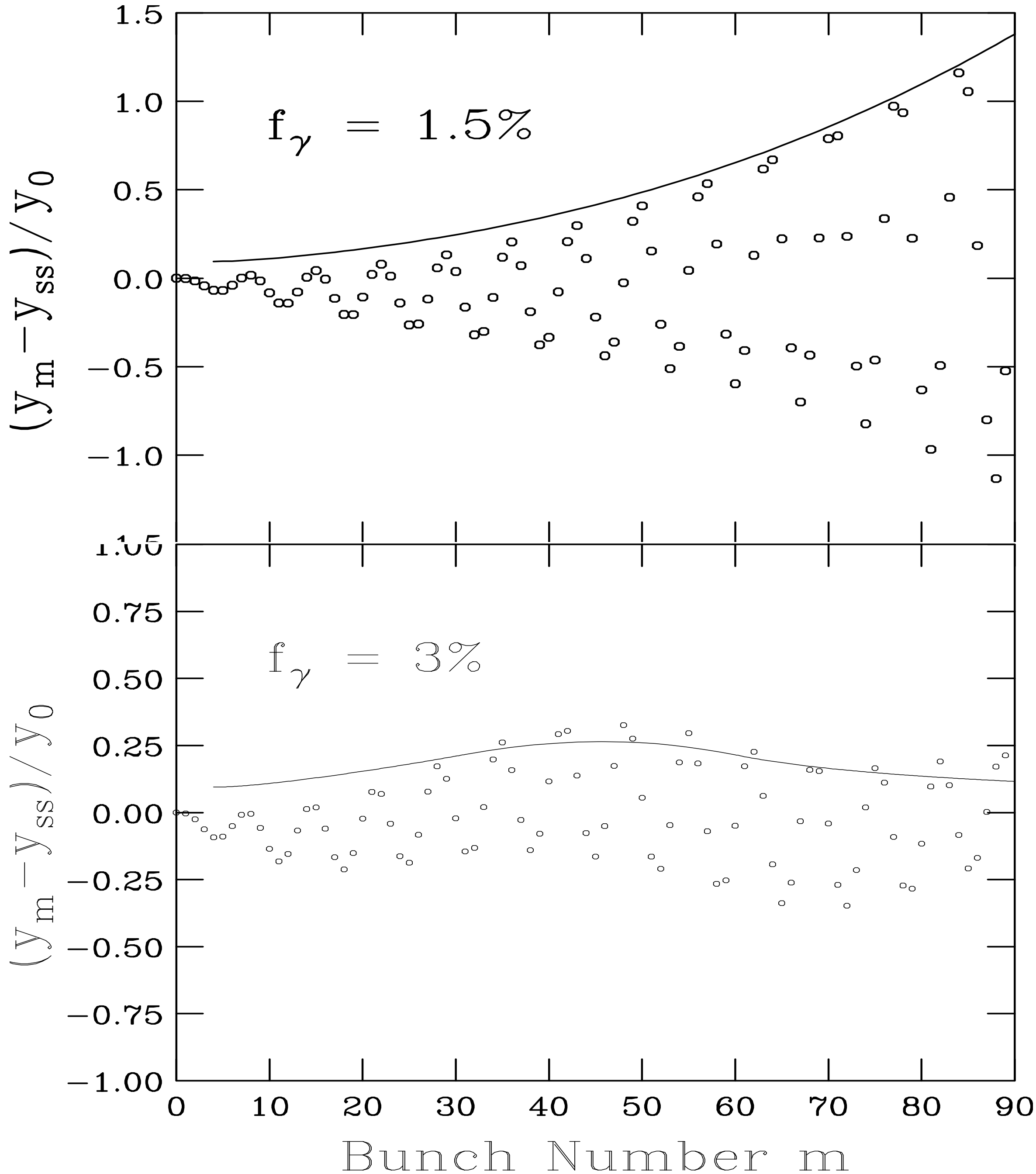

Figure 13.9: Analytic envelope at the linac exit (solid curve) plotted against the transverse displacement of bunches calculated numerically, with total energy spreads of 1.5% (top) and 3% (bottom).

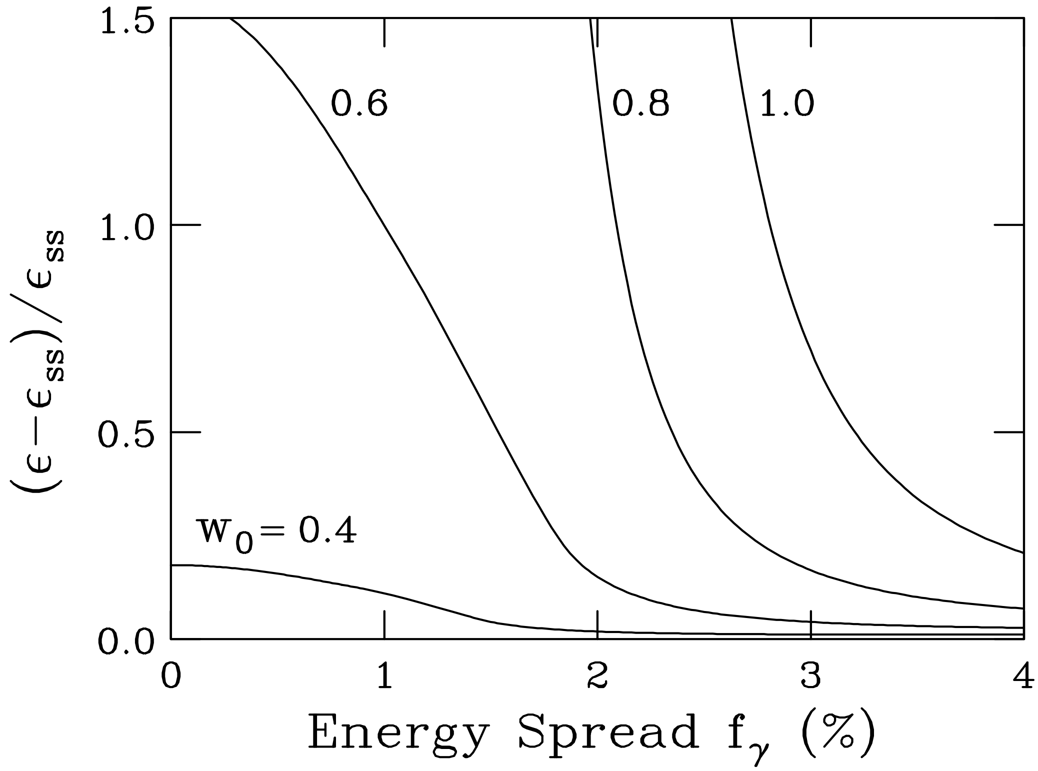

Figure 13.10: Total normalized transverse multibunch emittance at the linac exit, referenced to its steady-state value, versus total energy spread across the bunch train, plotted for various sum-wake amplitudes $w_0 = 2K$.

It is worth mentioning that the plots in Figs. 13.9 and 13.10 have been performed with the data of the upgraded NLC. If we use the present lower energy design of accelerating the bunches up to only 500 GeV, we lose a lot of acceleration adiabatic damping and the growths of the bunch deflections at the linac exit will be increased tremendously. To BNS damp such growths, an energy chirp close to 50% will be necessary. Certainly this is not workable because of the large momentum spread of the bunches which later translates into unacceptable transverse bunch sizes at the interaction point. The acceleration gradient will also be largely reduced. Needless to say, the linac itself will hardly have such large energy aperture for the bunches to pass through. What we actually want to point out is BNS damping is only feasible when the actual beam breakup is not too large, because only a small amount of energy chirp is acceptable in reality.

13.4 EXERCISES

13.1. (1) Assuming that the acceleration gradient is much less than the betatron wave number, derive the beam-breakup solutions, Eqs. (13.29) and Eq. (13.31), for the displacements of the head and tail in the two-particle model.

(2) The dipole transverse wake function of the SLAC linac per cavity cell at 1 mm is 62.9 V/pC/m. The bunch is of rms length 1 mm containing 5×10^{10} electrons. The cavity accelerating frequency is 2.856 GHz, with each cavity having the length of $\frac{1}{3}$ wavelength. The betatron wave number is $k_\beta = 0.06$ m^{-1}. In a two-particle model, compute the ratio of the deflection of the tail particle versus that of the head particle along the whole linac. Compute the same ratio if the linac stays at 1 GeV without acceleration.

13.2. A linac has a lattice consisting of N FODO cells. In between two consecutive quadrupoles, there is an acceleration structure of length ℓ, which is half of the FODO cell length. The acceleration is linear with $E_f/E_i = 1 + 2N\alpha\ell$ where E_i and E_f are, respectively, the initial and final energy across the N FODO cells.

(1) Show that the transverse transfer matrix across the nth acceleration structure is

$$\begin{pmatrix} 1 & \dfrac{1+n\alpha\ell}{\alpha} \ln \dfrac{1+(n+1)\alpha\ell}{1+n\alpha\ell} \\ 0 & \dfrac{1+n\alpha\ell}{1+(n+1)\alpha\ell} \end{pmatrix} . \tag{13.70}$$

(2) Is the transfer matrix symplectic? Give a physical answer.

Hint: Solve Eq. (13.28) with $k_\beta = 0$.

13.3. The NLC bunch has a rms length of $\sigma_\ell = 150$ μm containing 1.1×10^9 electrons. The linac has a length of 10 km, accelerating electrons from 10 GeV to 500 GeV. Assume a uniform betatron focusing with 100 betatron oscillations in the linac The accelerating structure has a transverse mode at the mean frequency of $\bar{\nu} =$ 15.25 GHz with a rms spread $\sigma_\nu = 25\%$ of $\bar{\nu}$.

(1) Use Eq. (13.39) to compute the transverse wake function at a distance σ_ℓ, assuming that the average kick factor is $\bar{K} = 40$ MV/nC/m^2.

(2) Compute the multiplication factor of the tail particle in the two-particle model at the end of the linac.

(3) Assuming the natural chromaticity of $\xi = (\Delta k_\beta/k_\beta)/\delta = -1$ for the FODO-cell

lattice, compute the energy spread between the head and tail of the bunch in order to damp the deflection of the tail.

13.4. (1) Complete the derivation of the beam-breakup deflection of the mth bunch as given by Eq. (13.49).
(2) For the NLC with 95 bunches with spacing 42 cm, estimate the deflection of the last bunch if the first bunch has an initial offset of 1 μm. You may take the mean energy of the linac in the computation and the dipole wake at one bunch spacing as 0.21 MV/nC/m^2.

13.5. Fill in the steps and give the estimate of the energy spread from the first to the 95th bunch in order to damp the beam breakup instability of the bunch train as outlined in Sec. 13.3.3.

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