Fermi National Accelerator Laboratory

FERMILAB-FN-673

Explicit Expressions of Impedances and Wake Functions

K.Y. Ng

Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

November 1998

Operated by Universities Research Association Inc. under Contract No. DE-AC02-76CH03000 with the United States Department of Energy

FN-673

Explicit Expressions of Impedances and Wake Functions

K.Y. Ng

Fermi National Accelerator Laboratory, PO Box 500, Batavia, IL 60540

To be published in

Handbook of Accelerator Physics and Engineering

Ed. A. Chao and M. Tigner

World Science

1998

0.0.1 Explicit Expressions of Impedances and Wake Functions

K.Y. Ng, FNAL

References

[1] A.W. Chao, Physics of Collective Beam Instabilities in High Energy Accelerators, Wiley (1993) ch.2

[2] K.Y. Ng, PA 23 (1988) 93

[3] S.A. Heifets, G. Sabbi, SLAC/AP-104 (1996)

[4] E. Keil, B. Zotter, PA 3 (1972) 11; K.Y. Ng, Fermilab Report FN-389 (1981)

[5] S.S. Kurennoy, PR E55 (1997) 3529; S.S. Kurennoy, R.L. Gluckstern, *ibid* (1997) 3533

[6] S.S. Kurennoy, G.V. Stupakov, PA 45 (1994) 95

[7] A.M. Sessler, unpublished citation in E. Keil, NIM 100 (1972) 419; L.A. Veinshtein, Sov. Phys. JETP 17 (1963) 709; D. Brandt, B. Zotter, CERN-ISR/TH/82-13 (1982)

[8] C.C. Johnson, Field and Wave Electrodynamics, McGraw-Hill, Ch.6

[9] G. Nassibian, F. Sacherer, NIM 59 (1971) 21; G. Nassibian, CERN/PS 84-25 (BR) (1984); CERN 85-68 (BR) (1986)

[10] M. Sands, SLAC note PEP-253 (1977); H.A. Bethe, PR 66 (1944) 163

[11] S.S. Kurennoy, PA 39 (1992) 1; PA 50 (1995) 167; R.L. Gluckstern, PRA 46, 1106, 1110 (1992); S.S. Kurennoy, R.L. Gluckstern, G.V. Stupakov, PR E52 (1995) 4354

[12] A. Fedotov, PhD Thesis, U. Maryland (1997)

[13] G.V. Stupakov, SLAC-PUB-7908 (1998), submitted to PR Special Topics

[14] S.A. Heifets, PRD 40 (1989) 3097; S.A. Heifets, S.A. Kheifets, Rev. Mod. Phys. 63 (1990) 631

General Remarks and Notations:

W_m' denotes mth azimuthal longitudinal wake function as a function of distance z for $z<0$. When $z>0$, $W_m'(z)=0$ and $W_m'(0)=\lim_{z\to0-} W_m'(z)$. Similar for transverse wake W_m.

The mth azimuthal longitudinal impedance $Z_m^{\parallel}(\omega) = \int e^{-i\omega z/v} W_m^{\parallel}(z)dz/v$ is related to the transverse impedance of the same azimuthal $Z_m^{\perp}(\omega) = \int e^{-i\omega z/v} W_m^{\perp}(z)idz/(\beta v)$ by $Z_m^{\parallel} = (\omega/c)Z_m^{\perp}$ (valid when $m\neq 0$). In many cases, $\beta=v/c$ has been set to 1.

Unless otherwise stated, round beam pipe of radius b is assumed. $C=2\pi R$ is the ring circumference and n is the revolution harmonic. $Z_0 \approx 377\ \Omega$ is the free-space impedance. ϵ_0 and μ_0 are the free-space dielectric constant and magnetic permeability.

Description	Impedances	Wakes
Space-charge: [1] beam radius a in a length L of perfectly conducting beam pipe of radius b.	$\frac{Z_0^{\parallel}}{n} = i\frac{Z_0 L}{2C\beta\gamma^2}\left[1+2\ln\frac{b}{a}\right]$ $Z_{m\neq0}^{\perp} = i\frac{Z_0 L}{2\pi\beta^2\gamma^2 m}\left[\frac{1}{a^{2m}}-\frac{1}{b^{2m}}\right]$	$W_0' = \frac{Z_0 cL}{4\pi\gamma^2}\left[1+2\ln\frac{b}{a}\right]\delta'(z)$ $W_{m\neq0} = \frac{Z_0 cL}{2\pi\gamma^2 m}\left[\frac{1}{a^{2m}}-\frac{1}{b^{2m}}\right]\delta(z)$
Resistive Wall: [1] pipe length L, wall thickness t, conductivity σ_c, skin depth $\delta_{\rm skin}$.	$\frac{Z_m^{\parallel}}{L} = \frac{\omega}{c}\frac{Z_m^{\perp}}{L} = \frac{Z_0 c/(\pi b^{2m})}{[1+{\rm sgn}(\omega)i](1+\delta_{m0})bc\sqrt{\frac{\sigma_c Z_0 c}{2\|\omega\|}}-\frac{ib^2\omega}{m+1}+\frac{imc^2}{\omega}}$ $t\gg\delta_{\rm skin}=\sqrt{2c/(\|\omega\|Z_0\sigma_c)},\quad \|\omega\|\gg c\chi/b,\quad \chi=1/(Z_0\sigma_c b)$	
For $t\gg\delta_{\rm skin}$ and $b/\chi\gg\|z\|\approx c/\|\omega\|\gg b\chi^{1/3}$.	$Z_m^{\parallel} = \frac{\omega}{c}Z_m^{\perp}$ $Z_m^{\parallel} = \frac{1-{\rm sgn}(\omega)i}{1+\delta_{0m}}\frac{L}{\pi\sigma_c\delta_{\rm skin}b^{2m+1}}$	$W_m = -\frac{c}{\pi b^{m+1}(1+\delta_{m0})}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{L}{\|z\|^{1/2}}$ $W_m' = -\frac{c}{2\pi b^{m+1}(1+\delta_{m0})}\sqrt{\frac{Z_0}{\pi\sigma_c}}\frac{L}{\|z\|^{3/2}}$
For $t\ll\delta_{\rm skin}$ or very low freq., and $b/\chi\gg\|z\|\approx c/\|\omega\|\gg\sqrt{bt}$.	$\frac{Z_0^{\parallel}}{L} = -\frac{iZ_0 t\omega}{2\pi bc},\quad \frac{Z_1^{\perp}}{L} = -\frac{iZ_0 t}{\pi b^3}$	$\frac{W_0'}{L} = -\frac{Z_0 tc}{2\pi b}\delta'(z),\ \frac{W_1}{L} = -\frac{Z_0 tc}{\pi b^3}\delta(z)$
A *pair* of strip-line BPM's: [2] length L, angle each subtending to pipe axis ϕ_0, forming transmission lines of characteristic impedance Z_c with pipe.	$Z_0^{\parallel} = 2Z_c\left[\frac{\phi_0}{2\pi}\right]^2\left[2\sin^2\frac{\omega L}{c}-i\sin\frac{2\omega L}{c}\right]$ $Z_1^{\perp} = \left[\frac{Z_0^{\parallel}}{\omega}\right]_{\rm pair}\frac{c}{b^2}\left[\frac{4}{\phi_0}\right]^2\sin^2\frac{\phi_0}{2}$	$W_0' = 2Z_c c\left[\frac{\phi_0}{2\pi}\right]^2[\delta(z)-\delta(z+2L)]$ $W_1 = \frac{8Z_c c}{\pi^2 b^2}\sin^2\frac{\phi_0}{2}[H(z)-H(z+2L)]$
	The strip-lines are assumed to terminate with impedance Z_c at the upstream end.	
Heifets inductive impedance: [3] low freq. pure inductance $\mathcal{L}$. $Z_0^{\parallel}$ rolls off as $\omega^{-1/2}$.	$Z_0^{\parallel} = -\frac{i\omega\mathcal{L}}{(1-i\omega a/c)^{3/2}}$ $\longrightarrow -i\omega\mathcal{L}$ as $\omega\to0$	$W_0' = \frac{c^2\mathcal{L}}{a\sqrt{\pi az}}\left[1+\frac{2z}{a}\right]e^{z/a}$ $\longrightarrow c^2\mathcal{L}\delta'(z)$ as $a\to0$
Pill-box cavity at low frequencies with length g and depth h, where $g\ll h$ [6].	$Z_0^{\parallel} = -i\frac{\omega Z_0}{2\pi cb}\left[gh-\frac{g^2}{2\pi}\right]$ $Z_1^{\perp} = -i\frac{Z_0}{\pi b^3}\left[gh-\frac{g^2}{2\pi}\right]$	$W_0' = -\frac{Z_0 c}{2\pi b}\left[gh-\frac{g^2}{2\pi}\right]\delta'(z)$ $W_1 = -\frac{Z_0 c}{\pi b^3}\left[gh-\frac{g^2}{2\pi}\right]\delta(z)$

Description	Impedances	Wakes
Pill-box cavity at low freq.: length g, radial depth h, where $b \le g \ll h$ [6].	$Z_0^{\parallel} = -i\dfrac{\omega Z_0 h^2}{\pi^2 cb}\left[\ln\dfrac{2\pi g}{h}+\dfrac{1}{2}\right]$ $Z_1^{\perp} = -i\dfrac{2Z_0 h^2}{\pi^2 b^3}\left[\ln\dfrac{2\pi g}{h}+\dfrac{1}{2}\right]$	$W_0' = -\dfrac{Z_0 c h^2}{\pi^2 b}\left[\ln\dfrac{2\pi g}{h}+\dfrac{1}{2}\right]\delta'(z)$ $W_1 = -\dfrac{2Z_0 c h^2}{\pi^2 b^3}\left[\ln\dfrac{2\pi g}{h}+\dfrac{1}{2}\right]\delta(z)$
Pill-box cavity: length g, radial depth d. At freq. $\omega \gg c/b$, diffraction model applies [1].	$Z_m^{\parallel} = \dfrac{[1+\mathrm{sgn}(\omega)i]Z_0}{(1+\delta_{m0})\pi^{3/2}b^{2m+1}}\sqrt{\dfrac{cg}{\lvert\omega\rvert}}$ $Z_m^{\parallel} = \dfrac{\omega}{c}Z_m^{\perp}$	$W_m = -\dfrac{2Z_0 c\sqrt{2g}}{(1+\delta_{m0})\pi^2 b^{2m+1}}\lvert z\rvert^{1/2}$ $W_m' = \dfrac{Z_0 c\sqrt{2g}}{(1+\delta_{m0})\pi^2 b^{2m+1}}\lvert z\rvert^{-1/2}$
Optical model: [7] A series of cavities of periodic length L. Each cavity has width g, high Q resonances of freq. $\omega_n/(2\pi)$ and loss factor $k_n^{(m)}$ for azimuthal mode m.	$\mathrm{Re}Z_m^{\parallel} = \displaystyle\sum_{n=1}^{N}\pi k_n^{(m)}\delta(\omega-\omega_n) + \dfrac{2\pi C_{\rm SV}G(\bar\nu)F(\nu)}{(1+\delta_{m0})b^{2m}}H(\omega-\omega_N)$ $W_m' = \displaystyle\sum_{n=1}^{N}2k_n^{(m)}\cos\dfrac{\omega_n z}{c} + \dfrac{2C_{\rm SV}G(\bar\nu)}{(1+\delta_{m0})b^{2m}}\int_{\omega_N}^{\infty}d\omega F(\nu)\cos\dfrac{\omega z}{c}$ where $C_{\rm SV} = 2Z_0 j_{m1}^2/(\pi^2\zeta^2\beta) \approx 650\ \Omega$ for $m=0$ and $1650\ \Omega$ for $m=1$, j_{m1} is first zero of Bessel function J_m, $\zeta = 0.8237$. $G(\bar\nu)=\bar\nu^2K_1^2(\bar\nu),\ \ F(\nu)=\dfrac{\sqrt{\nu}+1}{(\nu+2\sqrt{\nu}+2)^2},\ \bar\nu=\dfrac{\omega b}{\beta\gamma c},\ \ \nu=\dfrac{\omega}{\omega_{\rm SV}}=\dfrac{4b^2\omega}{\zeta^2 c\sqrt{gL}}$	
Formulas for computation of W_m'. $\mathrm{erfc}(x)$ is the complementary error function.	$\displaystyle\int_{\hat\omega}^{\infty}d\omega F(\nu)\cos\dfrac{\omega z}{c} = \omega_{\rm SV}\tilde F_0(z/c) - \int_0^{\hat\omega}d\omega F(\nu)\cos\dfrac{\omega z}{c}$ $\tilde F_0(x) = \displaystyle\int_0^{\infty}d\omega F(\nu)\cos\omega x = \dfrac{\pi}{4}(1+4x)e^{2x}\mathrm{erfc}(\sqrt{2x}) - \sqrt{\dfrac{\pi x}{2}}$	
Resonator model for the mth azimuthal, with shunt imp. $R_s^{(m)}$, resonant freq. $\omega_r/(2\pi)$, quality factor Q [1].	$Z_m^{\parallel} = \dfrac{R_s^{(m)}}{1+iQ\left(\omega_r/\omega-\omega/\omega_r\right)}$ $Z_m^{\perp} = \dfrac{c}{\omega}\,\dfrac{R_s^{(m)}}{1+iQ\left(\omega_r/\omega-\omega/\omega_r\right)}$	$W_m = \dfrac{R_s^{(m)}c\,\omega_r}{Q\bar\omega_r}\,e^{\alpha z/c}\sin\dfrac{\bar\omega_r z}{c}$ where $\alpha = \omega_r/(2Q)$ $\bar\omega_r = \sqrt{\lvert\omega_r^2-\alpha^2\rvert}$
Res. freq. $\omega_{mnp}/(2\pi)$ and shunt impedance $(R_s)_{mnp}$ of a pill-box cavity for nth radial and pth longitudinal modes. Radial depth h and length g. x_{mn} is nth zero of Bessel function J_m [8].	$\dfrac{\omega_{mnp}^2}{c^2} = \dfrac{x_{mn}^2}{d^2}+\dfrac{p^2\pi^2}{g^2}$ $\left[\dfrac{R_s}{Q}\right]_{0np} = \dfrac{Z_0}{x_{0n}^2 J_0'^2(x_{0n})}\dfrac{8c}{\pi g\omega_{0np}}\begin{cases}\sin^2\dfrac{g\omega_{0np}}{2\beta c}\times(1+\delta_{0p}) & p \text{ even}\\ \cos^2\dfrac{g\omega_{0np}}{2\beta c} & p \text{ odd}\end{cases}$ $\left[\dfrac{R_s}{Q}\right]_{1np} = \dfrac{Z_0}{J_1'^2(x_{1n})}\dfrac{2c^2}{\pi g d^2\omega_{1np}^2}\begin{cases}\sin^2\dfrac{g\omega_{1np}}{2\beta c}\times(1+\delta_{0p}) & p \text{ even}\\ \cos^2\dfrac{g\omega_{1np}}{2\beta c} & p \text{ odd}\end{cases}$	

<table>
<tr><th>Description</th><th>Impedances</th><th>Wakes</th></tr>
<tr><td rowspan="2">Low-freq. response of a pill-box cavity: [4] length g, radial depth d. When $g \gg 2(d-b)$, replace g by $2(d-b)$. Here, $S=d/b$.</td><td>$$\frac{Z_0^{\parallel}}{n} = -i\frac{Z_0 g}{2\pi R}\ln S$$ $$Z_1^{\perp} = -i\frac{Z_0 g}{\pi b^2}\frac{S^2-1}{S^2+1}$$</td><td>$$W_0' = -\frac{Z_0 c g}{2\pi}\ln S\,\delta'(z)$$ $$W_1 = -\frac{Z_0 c g}{\pi b^2}\frac{S^2-1}{S^2+1}\,\delta(z)$$</td></tr>
<tr><td colspan="2">Effect will be one half for a step in the beam pipe from radius b to radius d, or vice versa, with g replaced by $2(d-b)$.</td></tr>
<tr><td>Iris of half elliptical cross section at low freq.: width $2a$, maximum protruding length h [5].</td><td>$$Z_0^{\parallel} = -i\frac{\omega Z_0 h^2}{4cb}$$ $$Z_1^{\perp} = -i\frac{Z_0 h^2}{2b^3}$$</td><td>$$W_0' = -\frac{Z_0 c h^2}{4b}\,\delta'(z)$$ $$W_1 = -\frac{Z_0 c h^2}{2b^3}\,\delta(z)$$</td></tr>
<tr><td>Pipe transition at low freq.: tapering angle θ, transition height h. γ is Euler's constant and ψ is the psi-function [6].</td><td colspan="2">$$Z_0^{\parallel}=\frac{\omega b^2 Z_1^{\perp}}{2c}=-i\frac{\omega Z_0 h^2}{2\pi^2 cb}\left\{\ln\left[\frac{b\theta}{h}-2\theta\cot\theta\right]+\frac{3}{2}-\gamma-\psi\left(\frac{\theta}{\pi}\right)-\frac{\pi}{2}\cot\theta-\frac{\pi}{2\theta}\right\}$$ $$W_0' = -\left|\frac{Z_0^{\parallel}}{\omega}\right| c^2\delta'(z)\,,\quad W_1 = -\left|Z_1^{\perp}\right| c\delta(z)\,,\quad h\cot\theta \ll b$$</td></tr>
<tr><td>Pipe transition at low frequencies with transition height $h \ll b$ [6].</td><td colspan="2">$$Z_0^{\parallel} = \frac{\omega b^2}{2c}Z_1^{\perp} = -i\frac{\omega Z_0 h^2}{2\pi^2 cb}\left(\ln\frac{2\pi b}{h}+\frac{1}{2}\right)$$ $$W_0' = -\left|\frac{Z_0^{\parallel}}{\omega}\right| c^2\delta'(z)\,,\quad W_1 = -\left|Z_1^{\perp}\right| c\delta(z)$$</td></tr>
<tr><td rowspan="2">Kicker with window-frame magnet [9]: width a, height b, length L, beam offset x_0 horizontally, and all image current carried by conducting current plates.</td><td>$$Z_0^{\parallel} = \frac{\omega^2\mu_0^2 L^2 x_0^2}{4a^2 Z_k}$$ $$Z_1^{\perp} = \frac{c\omega\mu_0^2 L^2}{4a^2 Z_k}$$</td><td>$$W_0' = -\frac{c^3\mu_0^2 L^2 x_0^2}{4a^2 Z_k}\,\delta_0''(z)$$ $$W_1 = -\frac{c^3\mu_0^2 L^2}{4a^2 Z_k}\,\delta'(z)$$</td></tr>
<tr><td colspan="2">$Z_k = -i\omega\mathcal{L} + Z_g$ with $\mathcal{L} \approx \mu_0 bL/a$ the inductance of the windings and Z_g the impedance of the generator and the cable. If the kicker is of C-type magnet, x_0 in $Z_0^{\parallel}$ should be replaced by (x_0+b).</td></tr>
<tr><td>Traveling-wave kicker with characteristic impedance Z_c for the cable, and a window magnet of width a, height b, and length L [9].</td><td colspan="2">$$Z_0^{\parallel}=\frac{Z_c}{4}\left[2\sin^2\frac{\theta}{2}-i(\theta-\sin\theta)\right]\,,\quad Z_1^{\perp}=\frac{Z_c L}{4ab}\left[\frac{1-\cos\theta}{\theta}-i\left(1-\frac{\sin\theta}{\theta}\right)\right]$$ $$W_0' = \frac{Z_c c}{4}\left[\delta(z)-\delta\left(z-\frac{Lc}{v}\right)-\frac{Lc}{v}\delta'(z)\right]$$ $$W_1 = \frac{Z_c v}{4ab}\left[H(z)-H\left(z-\frac{Lc}{v}\right)-\frac{Lc}{v}\delta(z)\right]$$ $\theta = \omega L/v$ denotes the electrical length of the kicker windings and $v = Z_c ac/(Z_0 b)$ is the matched transmission-line phase velocity of the capacitance-loaded windings.</td></tr>
<tr><td>Bethe's electric and magnetic moments of a hole of radius a in beam pipe wall [10].</td><td colspan="2">Electric and magnetic dipole moments when wavelength $\gg a$: $\vec{d}=-\frac{2\epsilon_0}{3}a^3\vec{E}\,,\ \vec{m}=-\frac{4}{3\mu_0}a^3\vec{B}$ $\vec{E}$ and $\vec{B}$ are electric and magnetic flux density at hole when hole is absent. This is a diffraction solution for a thin-wall pipe.</td></tr>
</table>

Description	Impedances	Wakes
<u>Small obstacle</u> [5, 11] on beam pipe, size $\ll$ pipe radius, freq. below cutoff. α_e and α_m are electric polarizability and magnetic susceptibility of the obstacle.	$Z_0^{\parallel} = -i\frac{\omega Z_0}{c}\frac{\alpha_e+\alpha_m}{4\pi^2 b^2}$ $Z_1^{\perp} = -i\frac{Z_0(\alpha_e+\alpha_m)}{\pi^2 b^4}\cos\Delta\varphi$	$W_0' = -Z_0 c\frac{\alpha_e+\alpha_m}{4\pi^2 b^2}\,\delta'(z)$ $W_1 = -Z_0 c\frac{\alpha_e+\alpha_m}{\pi^2 b^4}\cos\Delta\varphi\,\delta(z)$
	$\Delta\varphi$ is the azimuthal angle between the obstacle and the direction concerning $Z_1^{\perp}$ and W_1.	
Polarizabilities for various geometry: beam pipe radius is b and wall thickness is t.		
Elliptical hole: major and minor radii are a and d. $K(m)$ and $E(m)$ are complete elliptical functions of the first and second kind, with $m = 1–m_1$ and $m_1 = (d/a)^2$. For long ellipse $\perp$ beam, major axis $a \ll b$, beam pipe radius, because the curvature of the beam pipe has been neglected here [12].	$\alpha_e+\alpha_m=\begin{cases}\frac{\pi a^3 m_1^2[K(m)-E(m))]}{3E(m)[E(m)-m_1K(m)]}\\ \frac{\pi a^3[E(m)-m_1K(m)]}{3[K(m)-E(m)]}\end{cases}\underset{\text{long ellipse}}{\overset{\text{m}\to 1}{\Longrightarrow}}\begin{cases}\frac{\pi d^4[\ln(4a/d)-1]}{3a} & \parallel\text{ beam } d\ll b\\ \frac{\pi a^3}{3\,[\ln(4a/d)-1]} & \perp\text{ beam } a\ll b\end{cases}$ $\alpha_e+\alpha_m \underset{m\to 0}{\overset{\text{circular}}{\Longrightarrow}} \frac{2a^3}{3}$ circular hole $a=d\ll b$ Above are for $t \ll a$, $\times 0.56$ (circular) or $\times 0.59$ (long ellipse) when $t \geq a$. For higher frequency correction, add to $\alpha_e+\alpha_m$ the extra term, $+\frac{2\pi a^3}{3}\left[\frac{11\omega^2 a^2}{30c^2}\right]$ circular, $\begin{cases}-\frac{\pi a d^2}{3}\left[\frac{\omega^2 a^2}{5c^2}\right] & \parallel\text{ beam long ellipse}\\ +\frac{2\pi a^3}{3}\left[\frac{2\omega^2 a^2}{5c^2[\ln(4a/d)-1]}\right] & \perp\text{ beam long ellipse}\end{cases}$	
Rectangular slot: length L, width w.	$\alpha_e+\alpha_m = w^3(0.1814-0.0344w/L)$ $t\ll a$, $\times 0.59$ when $t\geq a$	
Rounded-end slot: length L, width w.	$\alpha_e+\alpha_m = w^3(0.1334-0.0500w/L)$ $t\ll a$, $\times 0.59$ when $t\geq a$	
Annular-ring-shaped cut: inner and outer radii a and $d = a+w$ with $w \ll d$.	$\alpha_e+\alpha_m = \frac{\pi^2 d^2 a}{2\ln(32d/w)-4} - \frac{\pi^2 w^2(a+d)}{16}$ $t\ll d$ $\alpha_e+\alpha_m = \pi d^2 w - \frac{1}{2}w^2(a+d)$ $t\geq d$	
Half ellipsoidal protrusion with semi axes h radially, a longitudinally, and d azimuthally. ${}_2F_1$ is the hypergeometric function.	$\alpha_e+\alpha_m = 2\pi a h d\left[\frac{1}{I_b}+\frac{1}{I_c-3}\right]$ $I_b = {}_2F_1\left(1,1;\frac{5}{2};1-\frac{h^2}{a^2}\right)$, $I_c = {}_2F_1\left(1,\frac{1}{2};\frac{5}{2};1-\frac{a^2}{h^2}\right)$, if $a=d$ $\alpha_e+\alpha_m = \pi a^3$ if $a=d=h$, $\frac{2\pi h^3}{3[\ln(2h/a)-1]}$ if $a=d\ll h$ $\alpha_e+\alpha_m = \frac{8h^3}{3}\left[1+\left(\frac{4}{\pi}-\frac{\pi}{4}\right)\frac{a}{h}\right]$ if $a\ll h=d$ $\alpha_e+\alpha_m = \frac{8\pi h^4}{3a}\left[\ln\frac{2a}{h}-1\right]$ if $a\gg h=d$	

Wall roughness [13] 1-D axisymmetric bump, $h(z)$ or 2-D bump $h(z,\theta)$. Valid for low frequency $k=\omega/c \ll$ (bump length or width)$^{-1}$, $h \ll b$, pipe radius, and $\lvert\nabla h\rvert \ll 1$.	1-D: $Z_0^{\parallel} = -\frac{2ikZ_0}{b}\int_0^\infty \kappa \lvert\tilde{h}(\kappa)\rvert^2 d\kappa$ with spectrum $\tilde{h}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(z)e^{-ikz}dz$ 2-D: $Z_0^{\parallel} = -\frac{4ikZ_0}{b}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\kappa^2}{\sqrt{\kappa^2+m^2/b^2}}\lvert\tilde{h}_m(\kappa)\rvert^2 d\kappa$ with spectrum $\tilde{h}_m(k) = \frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dz\, h(z,\theta)e^{-ikz-im\theta}$
Heifets and Keifets formulas for tapered steps and tapered cavity at high frequencies [14].	
Taper in from radius h to $b\,(<h)$, out from radius b to h; tapering angle α. Tapering inefficient for a bunch of rms length σ, if $2(h-b)\tan\alpha \gg \sigma$. All formulas here and below are valid for *positive* $k=\omega/c$ only.	$\mathrm{Re}Z_0^{\parallel} = \pm\frac{Z_0}{2\pi}\ln\frac{h}{b} + \left(Z_0^{\parallel}\right)_{\text{step}},\ \mathrm{Re}Z_1^{\perp} = \pm\frac{Z_0 b}{4\pi}\left(\frac{1}{b^2}-\frac{1}{h^2}\right) + \left(Z_1^{\perp}\right)_{\text{step}}\ \begin{cases} +\text{ in} \\ -\text{ out}\end{cases}$ $\left(Z_0^{\parallel}\right)_{\text{step}} = \frac{Z_0}{2\pi}\ln\frac{h}{b},\ \tan\alpha > \frac{h-b}{kb^2},\quad \left(Z_0^{\parallel}\right)_{\text{step}} = \frac{Z_0}{4}kb\tan\alpha,\ \tan\alpha \ll \frac{1}{kb}$ $\left(Z_1^{\perp}\right)_{\text{step}} = \frac{Z_0}{4\pi b}\left[1-\frac{1}{(1+kb)^2}\,{}_2F_1\!\left(1,\tfrac{3}{2},3,\tfrac{4bh}{(b+h)^2}\right)\right],\ \tan\alpha > \frac{h-b}{kb^2},\ kb \gg 1$ $\left(Z_1^{\perp}\right)_{\text{step}} = \frac{Z_0 b}{4\pi}\left(\frac{1}{b^2}-\frac{1}{h^2}\right),\quad \tan\alpha > \frac{h-b}{kb^2},\ kb \gg 1,\ h \gg b$ $\left(Z_1^{\perp}\right)_{\text{step}} = \frac{Z_0}{16b}(kb)^3\tan\alpha,\quad \tan\alpha \ll \frac{1}{kb}$
Pill-box cavity: total length g, radial depth h without taper. Tapering angle α on both sides, $g \gg h$.	$Z_0^{\parallel} = \begin{cases} \frac{(1+i)Z_0}{2\pi b}\sqrt{\frac{g}{k\pi}} & g \ll kb^2 \\ \frac{Z_0}{\pi}\ln\frac{h}{b} & g \gg kb^2 \end{cases}$ $\mathrm{Re}Z_0^{\parallel} = 2\left(Z_0^{\parallel}\right)_{\text{step}},\qquad \mathrm{Re}Z_0^{\perp} = 2\left(Z_0^{\perp}\right)_{\text{step}}$ as given above