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Analytical Theory of Multipass Crystal Extraction

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Abstract

An analytical theory for the efficiency of particle extraction from an accelerator by means of a bent crystal is proposed. The theory agrees with all the measurements performed in the broad energy range of 14 to 900 GeV, where the efficiency range also spans over two decades, from ~0.3% to ~30%.

Crystal extraction experiments have greatly progressed at high energy accelerators in recent years[1, 2]. The experimental data are in good agreement with predictions from the detailed Monte-Carlo simulations [3, 4]. Although the transmission of particles by a bent crystal can be described analytically with good accuracy, the process of extraction involves essentially multiple encounters of circulating particles with the crystal and many turns in the accelerator, and therefore its efficiency cannot be scaled easily, for instance with energy. An analytical theory of multipass crystal extraction would be highly helpful in understanding the existing experimental results, in extrapolation to future applications, and in optimization. Below we derive an analytical formula for the crystal extraction efficiency.

Suppose that a beam with divergence σ, Gaussian distribution, is aligned to the crystal planes. Then as many as

$$(2\theta_c/\sqrt{2\pi}\sigma)(\pi x_c/2d_p) \tag{1}$$

particles get channeled in the initial straight part of the crystal. Here θ_c stands for the critical angle of channeling, d_p the interplanar spacing, $x_c \approx d_p/2 - a_{TF}$ the critical distance, a_{TF} being the Thomas-Fermi screening distance.

We shall first consider the case typical for the geometry used e.g. at Protvino[5] and CERN SPS[1], where particles first come to the crystal with nearly zero divergence, due to very small impact parameters. As experiments indicate [1, 2], in this first passage the channeling is suppressed, apparently due to the poor quality of the crystal structure near surface. In our model we assume that the first passage of particle through the crystal is always "inefficient", i.e. there is no channeling, but there is scattering and possibly nuclear interactions.

After some turns in the accelerator ring, the scattered particles come to the crystal with rms divergence as defined by scattering in the first pass:

$$\sigma_1 = (E_s/pv)(L/L_R)^{1/2}, \tag{2}$$

where E_s=13.6 MeV, L is the crystal length, L_R the radiation length, pv the particle momentum times velocity.

In a real experiment, σ_1 may be affected by betatron oscillations, and also by the fact that in the first passage the particle may enter the bent crystal quite near its surface and hence leave the crystal before crossing its full length L. These complications are certainly to be taken into account in the detailed Monte Carlo simulations [3, 4], as well as the possibility of scattering in the crystal holder, the aperture restrictions, the thickness of a "septum width" (i.e. the amplitude of irregularities in the crystal structure near the surface), and a possible angular distortion of the crystal faces (such as "anticlastic bending"). However, our objective is to derive a simple analytical theory which includes only the basic physical parameters of crystal extraction process, and to see how far it goes. We assume then that any particle always crosses the full crystal length; that pass 1 is like through an amorphous matter but any further pass is like through a crystalline matter; that there are no aperture restrictions; and that the particles interact only with the crystal not a holder.

After k passes the divergence is $\sigma_k = k^{1/2}\sigma_1$. The number of particles lost in nuclear interactions is $1-\exp(-kL/L_N)$ after k passes; L_N is the interaction length. In what follows we shall first assume that the crystal extraction efficiency is substantially smaller than 100 % (which has actually been the case so far), i.e. the circulating particles are removed from the ring predominantly through the nuclear interactions, not through channeling.

That pulled together, we obtain the multipass channeling efficiency by summation over k passes, from 1 to infinity:

$$F_C = \left(\frac{\pi}{2}\right)^{1/2} \frac{\theta_c x_c}{\sigma_1 d_p} \times \Sigma(L/L_N) \tag{3}$$

where

$$\Sigma(L/L_N) = \Sigma_{k=1}^{\infty} k^{-1/2} \exp(-kL/L_N) \tag{4}$$

may be called a "multiplicity factor" as it just tells how much the single-pass efficiency is amplified in multipasses.

A fraction of channeled particles is to be lost along the bent crystal due to scattering processes and centripetal effects. The transmission factor for the channeled particles in a bent crystal we denote as T. Then the multipass extraction efficiency is

$$F_E = F_C \times T = \left(\frac{\pi}{2}\right)^{1/2} \frac{\theta_c x_c}{\sigma_1 d_p} \times \Sigma(L/L_N) \times T \tag{5}$$

We shall use an analytical approximation (as used also in [6]) for silicon

$$T = (1 - p/3R)^2 \exp\left(-\frac{L}{L_d(1-p/3R)^2}\right), \tag{6}$$

where p is in GeV/c, and R is in cm; L_d is dechanneling length for a straight crystal.

The first factor in T describes a centripetal dechanneling. E.g., at pv/R=0.75 GeV/cm (which is close to the highest values used in extraction) our approximation gives $(1 - p/3R)^2$=0.563 whereas Forster et al.[7] measured 0.568±0.027.

The dechanneling length for a straight crystal of Si(110) was measured by several authors, being e.g. 103 mm at 200 GeV (Forster et al. [8]). We shall use the theoretical formula for L_d by Biryukov et al. [9], which is in good agreement with measurements.

Now we have explicit formulas with one exception for the sum (4) $\Sigma k^{-1/2} \exp(-kL/L_N)$. Although it is elementary for a program calculation, we shall also suggest an analytical expression. Since we are interested in the region of multipass channeling, which assumes $L \ll L_N$, we can approximate this sum by integral $\int_{k=1}^{\infty} k^{-1/2} \exp(-kL/L_N) \mathrm{d}k$ and then obtain:

$$\Sigma(L/L_N) \simeq (\pi L_N/L)^{1/2} - 1.5 \tag{7}$$

E.g., with L=4 cm and L_N=30 cm the miltiplicity factor is $\Sigma(L/L_N)$=3.41, whereas our approximation equals 3.35; the agreement is even better at smaller L, e.g. with L=1 cm we have $\Sigma(L/L_N)$=8.25 and Eq.(7) gives 8.21. In calculations we use L_N=30 cm for silicon, which includes both inelastic and elastic interactions of protons; this is necessary , because the r.m.s. angle of elastic scattering is much greater than θ_c [9].

Let us check the theory, first against the CERN SPS data [10] where the crystal extraction efficiency was measured at 14, 120, and 270 GeV, making use of the same 4-cm long Si(110) crystal, deflecting at 8.5 mrad. The crystal had 3-cm long bent part with two 5-mm straight ends, having in the center pv/R=(0.34 GeV/cm)$\times$(pv/120 GeV).

We take θ_c=13.8 μrad$\times$(120 GeV/pv)$^{1/2}$ (as used by the authors of Ref.[10]). The dechanneling length for a straight crystal is taken as 0.569$\times$270=154 mm at 270 GeV, 0.603$\times$120=72.4 mm at 120 GeV, and 0.718$\times$14=10.1 mm at 14 GeV. This length is reduced in a bent crystal by a factor of $(1-p/3R)^2$, Eq.(6). Only the dechanneling over 35 mm is taken into acount, as the last 5-mm end is unbent.

Table 1 shows good agreement of theory with measurements. Another

Table 1: Extraction efficiencies (%) from the SPS experiment, Eq.(5), and detailed simulations [11].

pv(GeV)	SPS	Eq.(5)	Monte Carlo
14	0.55±0.30	0.30	0.35±0.07
120	15.1±1.2	13.5	13.9±0.6
270	18.6±2.7	17.6	17.8±0.6

check may be the SPS extraction experiment with Pb nuclei, performed with the same crystal at same energy of 270 GeV [12], although here it may be difficult to evaluate the reduction in L_N precisely. In proton-nucleus interactions the cross-section scales roughly as $A^{2/3}$ with atomic weight A. We should compare p+Si with Pb+Si. As $A_{Pb} \gg A_{Si} \gg 1$, we can simply write the reduction in interaction length as $(A_{Si}/A_{Pb})^{2/3}$, i.e. 1:3.8 times. With L_N=30/3.8=7.9 cm, the extraction efficiency becomes 5.8% from Eq.(5). In the experiment it was between 4 and 11 % (on "average" 8±3 %). From the theory, the efficiency ratio, p to Pb, should be 3.0; the experimental ratio was something like 2.4±1.2.

The Tevatron extraction experiment at 900 GeV provides another check at a substantially higher value of efficiency. Here a slight modification of the formulas is needed to account for the non-zero starting divergence, namely σ_0=11.5 μrad (rms). This results in the change in Eq.(4):

$$\Sigma(L/L_N) = \Sigma_{k=1}^{\infty}(k + \sigma_0^2/\sigma_1^2)^{-1/2}\exp(-kL/L_N) \tag{8}$$

Since in this experiment Si(111) planes were used, consisting of narrow (1/4 weight) and wide (3/4 weight) channels, this is to be taken into account in Eq.(5) with respective change in d_p and x_c.

The crystal used at the Tevatron was 4 cm long with 8-mm straight ends, having in the center pv/R=0.29 GeV/cm. The theoretical dechanneling length for a straight crystal of Si(111) is 0.646×900 GeV=581 mm; notice that for (111) it is factor of d_p^{111}/d_p^{110}=1.23 higher than for (110). We take into account the dechanneling over 32 mm, as the last 8-mm part is unbent.

Eq.(5) then gives an extraction efficiency of 40.8 %. However, a minor correction to the theoretical value is discussed below.

Let us note that as the extraction efficiency is getting high, our earlier assumption that the nuclear interactions dominate over the crystal channeling may need correction. To take into account the fact that the circulating particles are efficiently removed from the ring by a crystal extraction as well, one would require a *recurrent* procedure of summation: instead of ΣF_k one has to sum ΣF_k^*, where F_k^*=$F_k(1 - F_{k-1}^*)$. This requires just a bit of elementary programming, but analytical investigation of the formula would then be not easy. This "recurrent" correction doesn't affect our earlier SPS calculation at 14 GeV and makes ~1% drop to the efficiencies at 120 and 270 GeV listed in Table 1; for Pb nuclei the correction is −0.1 % (accordingly, the p to Pb ratio of efficiencies becomes 2.8 instead of 3.0). For Tevatron this correction constitutes −6.7 %, converting 40.8 % into 34.1 %, whereas the measured value is on the order of 30 % [13], and the Monte Carlo simulation predicted about 35 % [3].

To see the dependence of extraction efficiency on the microscopic properties of the crystal material and on the particle energy, let us use the well-known theoretical expressions for θ_c=$(4\pi N d_p Z e^2 a_{TF}/pv)^{1/2}$, radiation length

L_R=137/[$4Z(Z+1)r_e^2 N \ln(183Z^{-1/3})$], and E_s=$2\sqrt{2\times 137}m_e c^2$, where N is the number of atoms per unit volume of crystal. The multipass extraction efficiency is then

$$F_E = \frac{\pi}{4}\left(\frac{x_c^2 a_{TF}}{L(Z+1)d_p r_e \ln(183Z^{-1/3})}\right)^{1/2}\left(\frac{pv}{m_e c^2}\right)^{1/2} T\Sigma(L/L_N) \qquad (9)$$

here m_e is the electron mass, r_e the classical electron radius. Despite of the simplifications done, this equation still predicts the SPS efficiency of 15.7 % at 120 GeV which is within the experimental error limits.

To uncover the Z dependence in the channeling efficiency, let us substitute $a_{TF} \approx r_e \times 137^2/Z^{1/3}$:

$$F_E \simeq \frac{137\pi}{4}\frac{x_c Z^{-1/6}(Z+1)^{-1/2}}{(d_p L \ln[183Z^{-1/3}])^{1/2}}\left(\frac{pv}{m_e c^2}\right)^{1/2} T\Sigma(L/L_N) \qquad (10)$$

With handy formula it is easy to investigate the dependence on p, L, Z etc. Interestingly, with the crystal used at the SPS the efficiency is maximal at just 270 GeV, as shows Fig. 1.

For any given energy one can optimize the crystal length L. In optimization we assumed the same proportion between the bent part and the full crystal length, 3 to 4. At 270 GeV the length used at the SPS was close to optimal, 3.0±0.5 cm. At 120 GeV the optimal length is 1.5±0.5 cm resulting in the efficiency of 28%; at 70 GeV it is 0.8±0.2 cm with best efficiency of 38%. A recurrent process of summation was used for Eq.(5). The qualitative lesson is that a lower energy permits a shorter crystal, and then the multiplicity factor becomes substantial. Figure 2 shows the $F_E(L)$ dependence for extraction at the 120-GeV SPS, 900-GeV Tevatron, and 7-TeV Large Hadron Collider (where 0.7 mrad deflection angle is assumed); in all the cases the crystal bent part was 0.75 of the full length. One can see that the analytical dependences $F_E(L)$ are very close to those obtained earlier in Monte Carlo simulations [4]. The same maxima at the same optimal lengths are predicted. One obvious conclusion is that the crystal extraction experiments at the SPS and Tevatron have been working rather far from the optimum, so there is a good possibility for improvement.

Formula (5) predicts a good efficiency of multipass extraction at a multi-TeV LHC, about 45 %, with the optimal length of Si(110) crystal being 6±1 cm.

Another interesting application is the extraction of a 900-GeV beam from the Tevatron with the crystal location being near A0 point [6]. Here a minimal deflection of 16.4 mrad is required. From Eq.(5) we find the optimal Si crystal length to be 13±3 cm, in accord with the proposal [6]. The efficiency is as low as 4% due to our conservative assumption about the particle losses over a very long crystal; a Monte Carlo simulation would clarify if this estimate is too conservative indeed.

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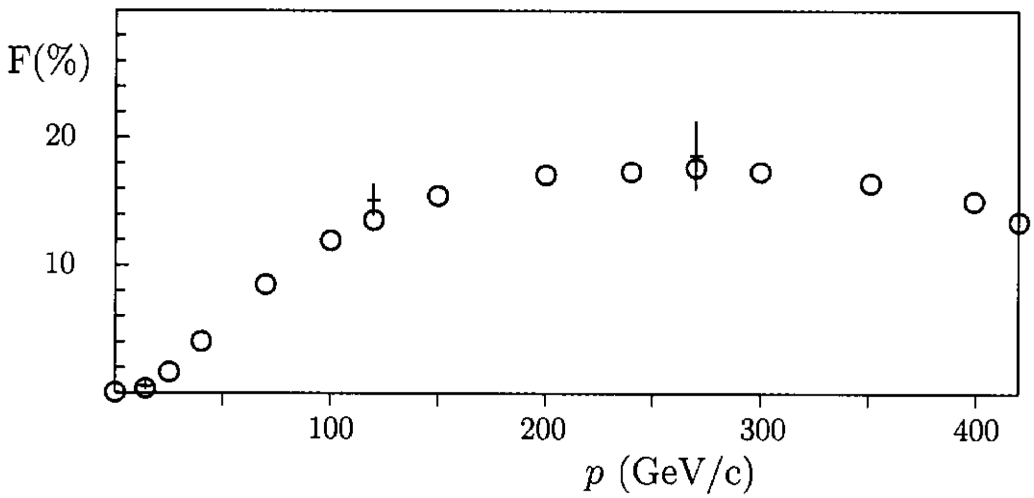

Figure 1: The SPS extraction efficiency as a function of momentum p. The curve (o) is for Eq.(5), the crosses at 14, 120 and 270 GeV/c are for the SPS experiment.

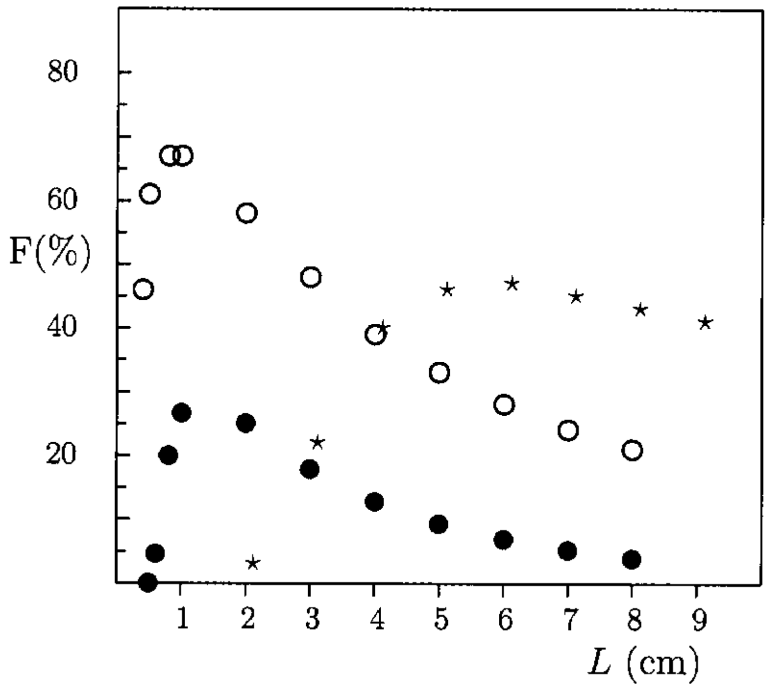

Figure 2: The extraction efficiency, Eq.(5), as a function of the crystal length L; for the SPS ($\bullet$), Tevatron (o), and Large Hadron Collider ($\star$).