QED2 as a testbed for interpolations between quenched and full QCD

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Lattice QED2 with the Wilson formulation of fermions is used as a convenient model system to study artifacts of the quenched approximation on a finite lattice. The quenched functional integral is shown to be ill-defined in this system as a consequence of the appearance of exactly real modes for physical values of the fermion mass. The location and frequency of such modes is studied as a function of lattice spacing, lattice volume, topological charge and improved action parameters. The efficacy of the recently proposed modified quenched approximation is examined, as well as a new approach to the interpolation from the quenched to full dynamical theory employing a truncated form of the fermion determinant.

1. Introduction

In this talk, some general features of the Wilson-Dirac spectrum in quenched lattice gauge theory are discussed using 2-dimensional QED as a convenient model system [1]. The specific focus will be the dependence of the real part of the spectrum on the parameters of the theory. The nonexistence of the quenched functional integral is found to arise from a complicated analytic structure induced by these real modes. The relation of quenched, pole-shifted [2] and full dynamical amplitudes is also discussed. Finally, the usefulness, accuracy and feasibility of an interpolating determinant approach to the full theory can be studied in detail in this model.

2. General Features of the Wilson-Dirac spectrum

In QED2 quark propagators are inverses of a matrix $D-rW+m \equiv \mathcal{M}+m$, with D, W and m the naive Dirac matrix, W the Wilson term, and m a quark mass parameter:

$$\mathcal{M} \equiv D - rW \tag{1}$$

$$D_{a\vec{m},b\vec{n}} = \frac{1}{2}(\gamma_\mu)_{ab}U_{\vec{m}\mu}\delta_{\vec{n},\vec{m}+\hat{\mu}} - \frac{1}{2}(\gamma_\mu)_{ab}U^\dagger_{\vec{n}\mu}\delta_{\vec{n},\vec{m}-\hat{\mu}} \tag{2}$$

$$W_{a\vec{m},b\vec{n}} = \frac{1}{2}\delta_{ab}(U_{\vec{m}\mu}\delta_{\vec{n},\vec{m}+\hat{\mu}} + U^\dagger_{\vec{n}\mu}\delta_{\vec{n},\vec{m}-\hat{\mu}}) \tag{3}$$

where a,b are Dirac indices, $\vec{m},\vec{n}$ lattice sites, U the unimodular link variables, and the Wilson parameter r is usually taken to be unity. Quite a lot is known about the spectrum of $\mathcal{M}$, which is complex as W is hermitian while D is skew-hermitian:

(1) The norm of the quadratic form $\mathcal{M}$ is less than or equal to 2 for arbitrary gauge fields [3], so the spectrum is contained inside a circle of radius 2 in the complex plane. In fact, a typical spectrum (see Fig. 1) has an elliptical shape with four critical branches, two in the center and one on either side. Conventionally the left critical branch represents the chiral (zero fermion mass) limit.
(2) The secular polynomial for $\mathcal{M}$ has real coefficients and only even terms, so eigenvalues necessarily appear as real doublets $\lambda, -\lambda$ or as complex quartets $\lambda, \lambda^*, -\lambda, -\lambda^*$. In particular, the appearance of exactly real eigenvalues (despite the fact that $\mathcal{M}$ is not a normal matrix) is generic, and such eigenvalues persist in finite neighborhoods of any gauge configuration point with a real mode. (Note the 2 exactly zero modes associated with each critical branch in Fig. 1).
(3) The appearance of exactly real modes for

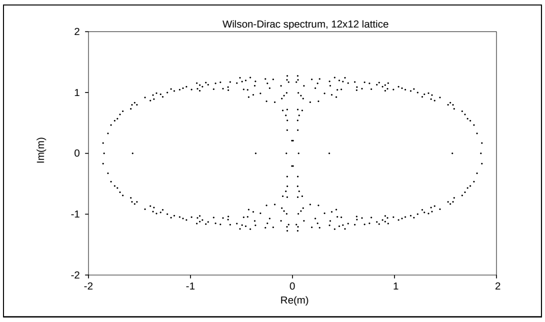

Figure 1. A typical Wilson-Dirac spectrum in QED2

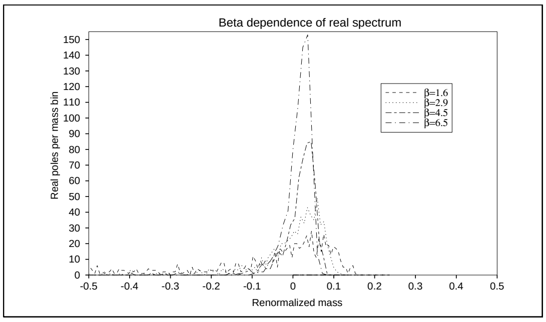

Figure 2. Beta dependence of histogram of real modes

$-2 \leq \lambda \leq -\frac{1}{2\kappa_c}$ (i.e. for physical naive fermion masses using the left critical branch) will lead to nonintegrable singularities in the quenched functional integral involving lattice Wilson-Dirac propagators. The integral can be defined by analytical continuation from the nonsingular region $|\lambda| > 2$, but the region inside the spectral ellipse is thoroughly infested with complicated branch cuts connecting a large number of branch points. A pinch argument shows that such branch points arise at any eigenvalue of $\mathcal{M}$ for gauge configurations where the link variables U are either +1 or -1. The noisy behavior of quenched simulations can be traced directly to this pathology.

The above statements are analytically demonstrable, but even more can be learned from detailed explicit simulations. For example:

(4) The integer part of the topological charge $Q_1 \equiv \frac{1}{2\pi}\sum_P \sin(\theta_P)$, (where θ_P is the plaquette angle for plaquette P), tracks quite closely the number of exactly zero modes per critical branch. Transitions between different topological charge sectors in the course of the simulation are accompanied by movement of complex eigenvalue quartets towards and then along the real axis.
(5) Histograms of the exactly real modes accumulated over many (typically 1000) decorrelated configurations for different beta values, but keeping the physical lattice volume fixed show that the spread of real modes into the physical mass region becomes acute at strong coupling, and that the probability at fixed physical fermion mass of encountering exceptional configurations in which a nearby real propagator pole introduces large fluctuations in measured hadronic amplitudes decreases rapidly as beta is increased (see Fig. 2).
(6) With increasing lattice volume at fixed β, the probability of encountering an exceptional configuration as one approaches the left crtical line decreases with increasing volume if one keeps a fixed offset from the critical line to maintain a fixed physical quark mass. However, exceptional configurations necessarily appear at any volume once one goes sufficiently close to the critical point.
(7) The frequency and distribution of exactly real modes is not substantially affected by a clover improved action. Of course, on any individual configuration, the location of real modes (if present) will change with the value of the clover coefficient chosen. But the statistical noise introduced by exceptionals in any large ensemble remains.

3. Comparison of Quenched, MQA and Full Dynamical Simulations

Recently, we have proposed a modified quenched approximation (MQA) in which the quenched functional integral is made well-defined by a pole-shifting procedure which incorporates the correct spectral behavior in the continuum limit (see [4] for a more detailed description). QED2 offers a convenient model for comparison of naive quenched, MQA and full dynamical results. A typical result is shown in Fig. 3, where the pseudoscalar correlator ("pion propagator") is shown at a bare quark mass of 0.08 (at β=4.5, 10x10 lattice) for these 3 cases. The statistical noise in the quenched correlators is essentially eliminated in the MQA results, which also are found to interpolate between the naive quenched and full dynamical results. This is gratifying- the MQA, in addition to rendering quenched amplitudes meaningful on coarse lattices, appears to move us closer to the unquenched theory.

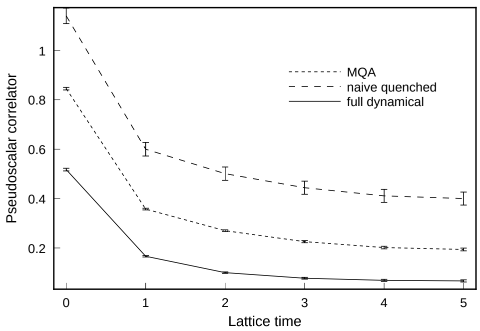

Figure 3. Pseudoscalar correlators in QED2-quenched, MQA and full dynamical

4. Interpolating Determinant Approach to Dynamical Fermions

We have recently begun a study of an alternative approach to the problem of interpolating between quenched and unquenched gauge theory, inspired by the insights gained in the MQA work on the role of small eigenvalues. The idea is to separate off and include explicitly in the simulation the infrared contributions to the determinant. In superrenormalizable QED2, the lowest $2N_\lambda$ eigenvalues contribute essentially all of the fluctuations to $\ln\det(\gamma_5(\mathcal{M}-m))$, as indicated in Fig.4, while the remaining 200-$2N_\lambda$ (on a 10x10 lattice) hardly contribute to the determinantal variation. As a consequence correlators computed using just the lowest 10% of the spectrum are essentially exact (see Fig.5).

In QCD4 the UV part of the quark spectrum certainly contributes importantly to a renormalization of coupling, visible as a substantial shift of scale in lattice amplitudes. However, work in progress shows that all the important infrared physics (e.g. the correct chiral structure, eliminating quenched chiral logs), say up to a scale of

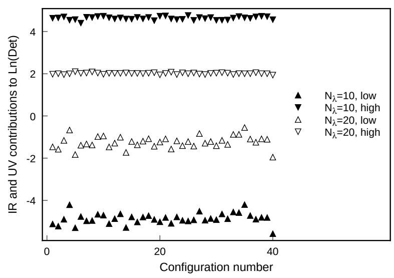

Figure 4. Fluctuations in Log(Det) from low and high eigenvalues (dynamical simulation at β=4.5)

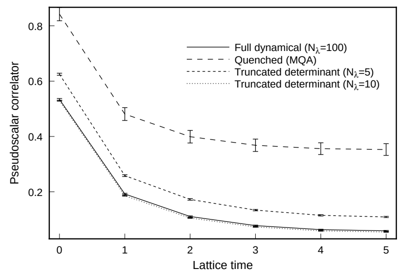

Figure 5. Comparison of quenched, truncated determinant and unquenched correlators

300 MeV , can be built in by inclusion of a few hundred eigenvalues of $\gamma_5(\mathcal{M}-m)$ which are readily accessible by a Lanczos scheme [5]. It seems possible that the remaining determinant effects not simply reducible to a change of scale may be included at the end by a reweighting scheme, or perhaps by using an appropriate loop representation [6] for the intermediate part of the quark spectrum. A study of these issues in QCD4 is in progress.

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