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On the Construction of Scattering Amplitudes for Spinning Massless Particles

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Abstract

In this paper the general form of scattering amplitudes for massless particles with equal spins s ($ss \to ss$) or unequal spins ($s_a s_b \to s_a s_b$) are derived. The imposed conditions are that the amplitudes should have the lowest possible dimension, have propagators of dimension m^{-2}, and obey gauge invariance. It is shown that the number of momenta required for amplitudes involving particles with $s > 2$ is higher than the number implied by 3-vertices for higher spin particles derived in the literature. Therefore, the dimension of the coupling constants following from the latter 3-vertices has a smaller power of an inverse mass than our results imply. Consequently, the 3-vertices in the literature cannot be the first interaction terms of a gauge-invariant theory. When no spins $s > 2$ are present in the process the known QCD, QED or (super) gravity amplitudes are obtained from the above general amplitudes.

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1 Introduction

In the last ten years attempts have been made to prove or disprove the existence of field theories involving massless particles with spin higher than 2. Free field theories for spin s particles exist [1, 2] and the fields obey a gauge invariance which is a generalization of spin 1 and spin 2 gauge invariances. The introduction of spin 1 and spin 2 self interactions is related to a deformation of the original algebra of gauge transformations. Constructing a Lagrangian for massless spin 1 or spin 2 particles can be done by extending step by step the Lagrangian with trilinear, quadrilinear, ... interactions and at the same time adding terms to the original gauge transformation. Symbolically, the Lagrangian $\mathcal{L}$ and gauge transformations on the fields ϕ are series in the coupling constant g:

$$\mathcal{L} = \mathcal{L}_0 + g\mathcal{L}_1 + g^2\mathcal{L}_2 + \cdots \tag{1}$$

$$\delta\phi = \delta_0\phi + g\delta_1\phi + g^2\delta_2\phi + \cdots \ , \tag{2}$$

where $\delta_0\phi = \delta\xi$, with ξ a gauge parameter. At every order in g there should be gauge invariance:

$$\delta\mathcal{L} = 0 \ , \tag{3}$$

or

$$\delta_0\mathcal{L}_0 = 0 \tag{4}$$

$$\delta_0\mathcal{L}_1 + \delta_1\mathcal{L}_0 = 0 \tag{5}$$

$$\delta_2\mathcal{L}_0 + \delta_1\mathcal{L}_1 + \delta_0\mathcal{L}_2 = 0, \ \text{etc.} \tag{6}$$

In the literature constructions of various $\mathcal{L}_1$'s involving fields with $s > 2$ have been presented. Some studies have used the light-front gauge [3, 4] and others [5, 6, 7] a general covariant form. The former makes it easier to obtain results for general s, the latter gives the possibility to study the gauge transformations in detail.

Since the simplest theories are evidently preferred, the construction of trilinear interactions always aims to find a minimal number of derivatives on the fields. From various studies a pattern of the dimensionality of the coupling constants for s_1–s_2–s_2 trilinear interactions can be established. In table 1 those cases are listed which were constructed in a covariant form [5, 6, 7]. They turned out to be unique. Moreover, the spin 3 self interaction requires structure constants like those in spin 1 self interactions. Results in the light-front gauge [3, 4] confirm and extend the pattern of the table: $d = s_1 - 1$ ($s_1 \geq s_2$) and $d = 2s_2 - s_1 - 1$ ($s_1 \leq s_2$). Structure constants are required for all odd spin self interactions. It can be verified that the covariant results reduce to the others when inserting the light-front gauge.

In principle the next term $\mathcal{L}_2$ in eq. (1) should be constructed. One can try to do this directly or one can first study the algebra of the gauge transformations of eq. (2). The latter approach [6] applied to spin 3 reveals that the algebra requires additional gauge transformations of a different type, hinting at the need of again including higher spin fields.

$s_1 =$	$s_2 = 0$	$s_2 = \frac{1}{2}$	$s_2 = 1$	$s_2 = \frac{3}{2}$	$s_2 = 2$	$s_2 = \frac{5}{2}$	$s_2 = 3$	$s_2 = \frac{7}{2}$
0	-1	0	1	2	3			
1	0	0	0	1	2			
2	1	1	1	1	1	2		
3	2	2	2	2	2	2	2	
4								

Table 1: The inverse dimension d of the coupling constant: $[g] = m^{-d}$ for an s_1–s_2–s_2 interaction $\mathcal{L}_1$.

An explicit construction of an interaction $\mathcal{L}_2$ for four spin 3 fields was unsuccessful [8]. From these results a self interacting spin 3 theory with a coupling constant dimension m^{-2} does not seem to exist. This makes it unlikely that self interacting theories of even higher spin would exist.

So far we only mention the attempts to find interacting massless higher spin theories in flat space. In the literature it has been argued that higher spin gravitational interactions are non-analytic in the cosmological constant and therefore an expansion over the flat background is not possible [9]. Thus the lack of success to couple spin 2 fields to higher spin fields could be understood. It then also seems unlikely that higher spin ($s > 2$) self interacting theories would exist. When expanding near the anti-de Sitter background, higher spin gravitational interactions are well defined [9].

The present paper tries to shed light on these questions in a different way. The reasoning assumes for the time being that a gauge theory describing spin s (and maybe other spins) exists, so that a scattering amplitude can be calculated. The scattering amplitude will originate from 3-vertices between two spin s particles and one other particle and a 4-vertex between four spin s particles. The propagators give all kinds of contractions between the momenta and polarization tensors in the vertices and give in principle poles in the Mandelstam variables $1/s$, $1/t$ or $1/u$. Scattering amplitudes for equal and unequal spin scattering will be constructed in such a way that a minimum number of momenta are required and that for different gauges in the polarization the same amplitudes arise.

It then turns out that for all spins $s \leq 2$ amplitudes are obtained which correspond to those of known theories, with the exception of one case ($12 \rightarrow 12$). As soon as particles participate with a spin $s > 2$ the dimension of the amplitude increases, and in such a way that the 3-vertices are required to have a dimension d higher than in table 1. In other words, the problems encountered for the specific case of spin 3 self interactions with a 3-vertex with three derivatives are confirmed: for spins $s > 2$ the constructed interactions of table 1 do not lead to a gauge theory.

We show that scattering amplitudes involving particles with $s > 2$ exist when a suitable number of derivatives is allowed for. This number is lower than trivial interactions involving only gauge invariant field strengths would require. However, whether there exists a field theory that gives these scattering amplitudes is an unanswered question. The construction method automatically shows that certain helicity amplitudes in known theories vanish at tree level, but can exist at one-loop level. The loop integral provides the additional momenta required for a non-vanishing amplitude. The method can also be used to derive general decay amplitudes of a massive particle into massless ones, e.g. $\pi_0 \to 3\gamma$.

The outline of the paper is as follows. In section 2 the polarization tensors for massless spin s particles are described and various gauges for polarization vectors are listed. Section 3 is devoted to the actual construction of scattering amplitudes. In section 4 comparisons with some known scattering and decay amplitudes are made. Section 5 summarizes our conclusions.

2 Polarization states and gauge choice

For the description of massless particles with spin s we make repeated use of the polarization vector e_μ. For a boson with spin s the decomposition of the polarization tensor into $e_{\mu_1\cdots\mu_s} = e_{\mu_1}\cdots e_{\mu_s}$ will be used; for fermions this quantity will be multiplied by a spinor in order to describe an $s+\frac{1}{2}$ state. First, we shall deal with bosons.

It has been shown that the scattering amplitudes for massless particles are conveniently described in the Weyl-van der Waerden spinor formalism [10, 11]. Vectors are translated into bispinors through

$$V_{\dot{A}B} = \sigma^\mu_{\dot{A}B} V_\mu \ , \tag{7}$$

where $\sigma^\mu_{\dot{A}B}$ are Pauli matrices. For a null vector $V_{\dot{A}B}$ becomes a product of two Weyl spinors. Thus the momentum K_μ of a massless particle becomes

$$K_{\dot{A}B} = \sigma^\mu_{\dot{A}B} K_\mu = k_{\dot{A}} k_B \ . \tag{8}$$

The polarization vector of a spin 1 outgoing massless particle can be described by a bispinor [11] (for positive/negative helicity):

$$e^+_{\dot{A}B} = \frac{k_{\dot{A}} b_B}{\langle kb\rangle} \tag{9}$$

$$e^-_{\dot{A}B} = \frac{b_{\dot{A}} k_B}{\langle kb\rangle^*} \ , \tag{10}$$

where k is the spinor related to the momentum K of the particle (cf. eq. (8)) and b is an arbitrary spinor. The arbitrariness reflects the freedom of gauge choice and therefore we call b a gauge spinor. A proper normalization would require an overall factor $\sqrt{2}$, which we shall omit, since overall normalizations will not be important for our arguments. The antisymmetric spinor "in-product" is denoted by

$$\langle pq\rangle = p_A q_B \epsilon^{BA} = p_A q^A \ , \tag{11}$$

with the antisymmetric 2×2 matrix ϵ. Complex conjugation gives spinors with dotted indices, e.g.

$$\langle pq\rangle^* = p_{\dot{A}} q^{\dot{A}} \ . \tag{12}$$

The gauge freedom in eqs. (9) and (10), and therefore in $e_{\mu_1} \cdots e_{\mu_s}$, will play an essential role in our discussion. For processes involving massless particles with spin it is useful to introduce the concept of a minimal gauge [11]. It is that choice of gauge spinors which minimizes the number of non-vanishing inner products $(e_i \cdot e_j)$, where e_i and e_j are polarization vectors describing the polarization states of particle i and j in the process.

As an example, which will be often used in the following, let us consider a scattering process involving four massless particles with equal spins ($ss \to ss$) or with pair-wise equal spins ($s_a s_a \to s_b s_b$). In order to simplify the expressions all particles will be considered as outgoing. In principle for the above processes one has three different scattering amplitudes: A^{++++}, A^{+++-} and A^{++--}, where $\pm$ denotes the helicity $\pm s$. The opposite helicity amplitudes are obtained by complex conjugation.

For A^{++++} a minimal gauge is the one where all b_i are the same. Then all $e_i \cdot e_j$ will vanish. For A^{+++-} a minimal gauge is obtained when $b_1 = b_2 = b_3 = k_4$ and $b_4 = k_1$: again all $e_i \cdot e_j$ vanish. For A^{++--} minimal gauges give just one non-vanishing $e_i \cdot e_j$. In this case there are four minimal gauges of which we list the main characteristics in table 2 for subsequent use. For each of the minimal gauges there are two non-vanishing $K_m \cdot e_j$ for fixed j. When imposing momentum conservation it means that there are two non-vanishing $K_m \cdot e_j$ for each j with opposite values.

Although processes with four massless particles will be the main focus of the paper, let us comment on minimal gauges for a different number of particles. For two or three particles the minimal gauges give all $e_i \cdot e_j$ vanishing. For processes with n ($n \geq 5$) particles the $A^{++\cdots+}$ and $A^{++\cdots+-}$ amplitudes can be dealt with as in the $n = 4$ case. The $A^{++\cdots+--}$ case has only one non-vanishing $e_i \cdot e_j$, but $n - 2$ non-vanishing $K_m \cdot e_j$ for fixed j.

For massless particles with spin $s + \frac{1}{2}$ the polarization multispinors are a product of the bispinors of eqs. (9) and (10) and of a single spinor. For outgoing particles with helicity $\pm(s + \frac{1}{2})$ and for outgoing antiparticles with helicity $\pm(s + \frac{1}{2})$ we have respectively

$$(e^+)^s k_{\dot{C}}, \text{ particle, } (s + \frac{1}{2}) \tag{13}$$

$$(e^-)^s k^C, \text{ particle, } -(s + \frac{1}{2}) \tag{14}$$

Gauge choice	1	2	3	4
$b_{1,2}$ $b_{3,4}$	k_3 k_1	k_4 k_1	k_3 k_2	k_4 k_2
Non-vanishing $e_i^+ \cdot e_j^-$	$2e_2^+ \cdot e_4^- = \frac{\langle 21\rangle^*\langle 34\rangle}{\langle 23\rangle\langle 41\rangle^*}$	$2e_2^+ \cdot e_3^- = \frac{\langle 21\rangle^*\langle 43\rangle}{\langle 24\rangle\langle 31\rangle^*}$	$2e_1^+ \cdot e_4^- = \frac{\langle 12\rangle^*\langle 34\rangle}{\langle 13\rangle\langle 42\rangle^*}$	$2e_1^+ \cdot e_3^- = \frac{\langle 12\rangle^*\langle 43\rangle}{\langle 14\rangle\langle 32\rangle^*}$
$2K_2 \cdot e_1^+$	$\frac{\langle 21\rangle^*\langle 23\rangle}{\langle 13\rangle}$	$\frac{\langle 21\rangle^*\langle 24\rangle}{\langle 14\rangle}$	$\frac{\langle 21\rangle^*\langle 23\rangle}{\langle 13\rangle}$	$\frac{\langle 21\rangle^*\langle 24\rangle}{\langle 14\rangle}$
$2K_1 \cdot e_2^+$	$\frac{\langle 12\rangle^*\langle 13\rangle}{\langle 23\rangle}$	$\frac{\langle 12\rangle^*\langle 14\rangle}{\langle 24\rangle}$	$\frac{\langle 12\rangle^*\langle 13\rangle}{\langle 23\rangle}$	$\frac{\langle 12\rangle^*\langle 14\rangle}{\langle 24\rangle}$
$2K_4 \cdot e_3^-$	$\frac{\langle 43\rangle\langle 41\rangle^*}{\langle 31\rangle^*}$	$\frac{\langle 43\rangle\langle 41\rangle^*}{\langle 31\rangle^*}$	$\frac{\langle 43\rangle\langle 42\rangle^*}{\langle 32\rangle^*}$	$\frac{\langle 43\rangle\langle 42\rangle^*}{\langle 32\rangle^*}$
$2K_3 \cdot e_4^-$	$\frac{\langle 34\rangle\langle 31\rangle^*}{\langle 41\rangle^*}$	$\frac{\langle 34\rangle\langle 31\rangle^*}{\langle 41\rangle^*}$	$\frac{\langle 34\rangle\langle 32\rangle^*}{\langle 42\rangle^*}$	$\frac{\langle 34\rangle\langle 32\rangle^*}{\langle 42\rangle^*}$
Non-vanishing $\langle m\|e_i\|n\rangle$	$\langle 2\|e_1^+\|4\rangle = \frac{\langle 21\rangle^*\langle 43\rangle}{\langle 13\rangle}$	$\langle 2\|e_1^+\|3\rangle = \frac{\langle 21\rangle^*\langle 34\rangle}{\langle 14\rangle}$	$\langle 2\|e_1^+\|4\rangle = \frac{\langle 21\rangle^*\langle 43\rangle}{\langle 13\rangle}$	$\langle 2\|e_1^+\|3\rangle = \frac{\langle 21\rangle^*\langle 34\rangle}{\langle 14\rangle}$
	$\langle 1\|e_2^+\|4\rangle = \frac{\langle 12\rangle^*\langle 43\rangle}{\langle 23\rangle}$	$\langle 1\|e_2^+\|3\rangle = \frac{\langle 12\rangle^*\langle 34\rangle}{\langle 24\rangle}$	$\langle 1\|e_2^+\|4\rangle = \frac{\langle 12\rangle^*\langle 43\rangle}{\langle 23\rangle}$	$\langle 1\|e_2^+\|3\rangle = \frac{\langle 12\rangle^*\langle 34\rangle}{\langle 24\rangle}$
	$\langle 2\|e_3^-\|4\rangle = \frac{\langle 21\rangle^*\langle 43\rangle}{\langle 31\rangle^*}$	$\langle 2\|e_3^-\|4\rangle = \frac{\langle 21\rangle^*\langle 43\rangle}{\langle 31\rangle^*}$	$\langle 1\|e_3^-\|4\rangle = \frac{\langle 12\rangle^*\langle 43\rangle}{\langle 32\rangle^*}$	$\langle 1\|e_3^-\|4\rangle = \frac{\langle 12\rangle^*\langle 43\rangle}{\langle 32\rangle^*}$
	$\langle 2\|e_4^-\|3\rangle = \frac{\langle 21\rangle^*\langle 34\rangle}{\langle 41\rangle^*}$	$\langle 2\|e_4^-\|3\rangle = \frac{\langle 21\rangle^*\langle 34\rangle}{\langle 41\rangle^*}$	$\langle 1\|e_4^-\|3\rangle = \frac{\langle 12\rangle^*\langle 34\rangle}{\langle 42\rangle^*}$	$\langle 1\|e_4^-\|3\rangle = \frac{\langle 12\rangle^*\langle 34\rangle}{\langle 42\rangle^*}$

Table 2: Minimal gauges for the amplitude A^{++--}

$$(e^+)^s k^{\dot{C}}, \text{ antiparticle, } (s+\frac{1}{2}) \tag{15}$$

$$(e^-)^s k_C, \text{ antiparticle, } -(s+\frac{1}{2}) \ . \tag{16}$$

The spinors of the fermions can be combined into spinorial products amongst themselves or with vectors, e.g.

$$\langle 1|V|2\rangle \equiv k_{1\dot{A}} V^{\dot{A}B} k_{2B} = \langle 1v\rangle^* \langle 2v\rangle \ , \tag{17}$$

when V is a null vector. In table 2 the non-vanishing $\langle m|e_i|n\rangle$ are also listed for a case with two positive and two negative helicity particles outgoing, giving the spinorial factor $k_{1\dot{A}} k_{2\dot{B}} k_{3C} k_{4D}$.

3 The structure of massless spin scattering amplitudes

In this section we shall consider scattering of massless particles with spin, where all spins are the same or are pair-wise the same. We shall construct those scattering amplitudes where a minimal number of momenta is involved and which are still obeying gauge invariance. This number of momenta will be determined by the spins of the particles. Once the spins are chosen, the number of polarization vectors is fixed. Since in a minimal gauge many $e_i \cdot e_j$ vanish, those $e_{i,j}$ have to be contracted with external momenta, of which a minimum number will be required. Moreover, the amplitudes must be constructed in such a way that the different minimal gauges give the same amplitude. Initially only bosons will be considered and later on also fermions. For the bosons, let us start with $ss \to ss$ scattering, where for convenience all particles are again assumed to be outgoing.

From the discussion in section 2 it is clear that the amplitudes A^{++++} and A^{+++-} will require $4s$ momenta factors, denoted symbolically as $(K)^{4s}$ in order not to be identically zero. A lower number is required for non-vanishing A^{++--} amplitudes, namely $(K)^{2s}$. Depending on which of the minimum gauges is chosen we obtain the general expressions

$$\begin{aligned}
A^{++--} &= c_1 \left(e_2^+ \cdot e_4^-\right)^s \left(K_2 \cdot e_1^+\right)^s \left(K_4 \cdot e_3^-\right)^s \\
&= c_2 \left(e_2^+ \cdot e_3^-\right)^s \left(K_2 \cdot e_1^+\right)^s \left(K_3 \cdot e_4^-\right)^s \\
&= c_3 \left(e_1^+ \cdot e_4^-\right)^s \left(K_1 \cdot e_2^+\right)^s \left(K_4 \cdot e_3^-\right)^s \\
&= c_4 \left(e_1^+ \cdot e_3^-\right)^s \left(K_1 \cdot e_2^+\right)^s \left(K_3 \cdot e_4^-\right)^s \ .
\end{aligned} \tag{18}$$

The quantities $c_i = c_i(s,t,u)$ are determined by the possible pole structures. At this stage of the construction the above number of momenta will be obtained in a theory where the 3-vertex for spin s particles has s derivatives. Note that a 4-vertex vanishes in these minimal gauges since it should have two derivatives less than the product of the two 3-vertices. In the next stage we impose gauge invariance, implying that it should be possible to choose the quantities c_i in such a way that all four expressions in eq. (18) give the same result. Using the explicit expression from table 1, the four equations simplify to

$$A^{++--} = \langle 12\rangle^{*2s}\langle 34\rangle^{2s}\left(\frac{c_1}{t^s}, \frac{c_2}{u^s}, \frac{c_3}{u^s}, \frac{c_4}{t^s}\right) \ , \tag{19}$$

where

$$\begin{aligned} t &= 2K_1 \cdot K_3 = 2K_2 \cdot K_4 \\ u &= 2K_1 \cdot K_4 = 2K_2 \cdot K_3 \\ s &= 2K_1 \cdot K_2 = 2K_3 \cdot K_4 \ . \end{aligned} \tag{20}$$

The quantities c_i arise from the propagator of the exchanged massless particles and have the general form

$$c_i = \frac{\alpha_i}{s} + \frac{\beta_i}{t} + \frac{\gamma_i}{u} = \frac{P_i}{stu} \ , \tag{21}$$

where P_i is a polynomial of second degree in s, t, u. Since $s+t+u=0$, the polynomial can be written as an arbitrary polynomial of second degree in just two Mandelstam variables, taking the final form

$$P_i = A_i t^2 + B_i tu + C_i u^2 \ . \tag{22}$$

The requirement

$$\frac{P_1}{t^s} = \frac{P_2}{u^s} \ (s \geq 1) \tag{23}$$

can only be satisfied for

$$s = 1 : \ C_1 = A_2 = 0\,, \ A_1 = B_2\,, \ B_1 = C_2 \tag{24}$$

$$s = 2 : \ A_1 = C_2\,, \ B_1 = C_1 = A_2 = B_2 = 0 \ . \tag{25}$$

Higher s values are inconsistent with eq. (23). The amplitudes now take the form

$$A^{++--}_{s=1} = \langle 12\rangle^{*2}\langle 34\rangle^{2}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{tu}\right) \tag{26}$$

$$A^{++--}_{s=2} = \frac{\langle 12\rangle^{*4}\langle 34\rangle^{4}}{stu} \ . \tag{27}$$

For spins $s > 2$ a gauge invariant theory with 3-vertices with s derivatives does therefore not exist. One may wonder what happens when the dimension of the propagator is m^{-4} in the cases of $s = 3, 4$. One would obtain eq. (21), with the denominator squared and a polynomial of the 4th degree with 5 coefficients related to 3 arbitrary parameters. For the case $s = 4$ the coefficient of t^4 in P_1 and of u^4 in P_2 should survive, the others should vanish. This cannot be realized. For $s = 3$ eq. (23) requires at least $\beta_1 = 0$, $\alpha_1 + \gamma_1 = 0$ and $\gamma_2 = 0$, $\alpha_2 + \beta_2 = 0$, but then it is still impossible to choose α_1 and α_2 such that eq. (23) is satisfied.

When the number of derivatives is increased we can construct a polynomial P_i of higher degree

$$P_i = a^{(i)}_{2n} t^{2n} + a^{(i)}_{2n-1} t^{2n-1} u + \cdots + a^{(i)}_0 u^{2n} \ . \tag{28}$$

Note that with the increased number of derivatives the contact term between four spin s particles is not necessarily zero and therefore contributes to the polynomial (28). For spin s the lowest degree $2n$, which could give a gauge invariant theory, will be for $n = s/2$. For even spins we take this condition, but for odd spins we take $n = (s+1)/2$, which generalizes eqs. (26) and (27). This is necessary since a 3-vertex for odd spin s requires an odd number of derivatives. Gauge invariance now demands

$$s = 2n-1\,,\; a_{2n}^{(1)} = a_1^{(2)}\,,\; a_{2n-1}^{(1)} = a_0^{(2)}\,,\; \text{all others vanish} \tag{29}$$

$$s = 2n\,,\; a_{2n}^{(1)} = a_0^{(2)}\,,\; \text{all others vanish}\,. \tag{30}$$

The general forms will be

$$A_{s=2n-1}^{++--} = \langle 12\rangle^{*2s}\langle 34\rangle^{2s}\left(\frac{1}{su}\,,\ \frac{1}{st}\,,\ \frac{1}{tu}\right) \tag{31}$$

$$A_{s=2n}^{++--} = \frac{\langle 12\rangle^{*2s}\langle 34\rangle^{2s}}{stu}\,. \tag{32}$$

From the above arguments we see that a priori a higher spin ($s > 2$) massless gauge theory is not excluded. The 3-vertex between odd spin s particles should have $2s-1$ derivatives ($s \geq 3$), for even spin $2s-2$ derivatives ($s \geq 4$). Strictly speaking, the exchanged particle is not necessarily a spin s particle, but the 3-vertex connecting the external spin s particles with the exchanged one should have the above number of derivatives.

At this point we would like to make three comments. One is that we are only looking for amplitudes with the lowest number of derivatives. So we do not consider “charged” even spin particles, which would have $2s-1$ derivatives in 3-vertices. The next comment is that even with this higher number of derivatives it is still not enough to make the $A^{++\cdots+}$ or $A^{++\cdots+-}$ amplitudes non-vanishing. The third comment is that for spin 1 and 2 the n-particle amplitude will have in the numerator $n-2$ and $2n-4$ momenta, which is less than the n or $2n$ needed for non-vanishing $A^{++\cdots+}$ and $A^{++\cdots+-}$ amplitudes.

Next we consider the scattering process with two particles with spin s_a and two with s_b ($s_b > s_a > 0$) and look for the lowest number of momenta required for a gauge invariant amplitude. Again this will come from an A^{++--} amplitude for which we now distinguish two cases

$$A_1^{++--} = A^{++--}(aabb) \tag{33}$$

$$A_2^{++--} = A^{++--}(abab)\,. \tag{34}$$

The general form for A_1 will be for gauge choice 1

$$A_1^{++--} = c_1\left(e_2^+\cdot e_4^-\right)^{s_a}\left(K_2\cdot e_1^+\right)^{s_a}\left(K_4\cdot e_3^-\right)^{s_b}\left(K_3\cdot e_4^-\right)^{s_b-s_a} \tag{35}$$

and similar forms for the other minimal gauges. The lowest possible number of momenta is $(K)^{2s_b}$. In a similar fashion we find, as before,

$$A_1^{++--} = \langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b}\left(\frac{c_1}{t^{s_a}}, \frac{c_2}{u^{s_a}}, \frac{c_3}{u^{s_a}}, \frac{c_4}{t^{s_a}}\right)\,, \tag{36}$$

giving scattering amplitudes for $s_a = 1, 2$ that take the form

$$A_1{}^{++--}_{s_a=1} = \langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b}\left(\frac{1}{su}\,,\ \frac{1}{st}\,,\ \frac{1}{tu}\right) \tag{37}$$

$$A_1{}^{++--}_{s_a=2} = \frac{\langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b}}{stu}\ . \tag{38}$$

The latter formulae generalize to odd and even spins s_a when we increase the number of momenta in the amplitudes (35) from $2s_b$ to $2s_b + 2(s_a - 1)$ or $2s_b + 2(s_a - 2)$. In terms of 3-vertices the product of two vertices has $2s_b + 2(s_a - 1)$ or $2s_b + 2(s_a - 2)$ derivatives.

For the other amplitude A_2^{++--} we start with the minimal gauge 4:

$$A_2^{++--} = c_4\left(e_1^+\cdot e_3^-\right)^{s_a}\left(K_1\cdot e_2^+\right)^{s_b}\left(K_3\cdot e_4^-\right)^{s_b}\ . \tag{39}$$

This gauge requires in the amplitude $2s_b$ momenta; the minimal gauge 1 would require $2s_a$, but since both gauges should be acceptable we need at least $2s_b$ momenta. For gauges 2 and 3 we have the general form

$$A_2^{++--} = c_2\left(e_2^+\cdot e_3^-\right)^{s_a}\left(K_2\cdot e_1^+\right)^{s_a}\left(K_1\cdot e_2^+\right)^{s_b-s_a}\left(K_3\cdot e_4^-\right)^{s_b} \tag{40}$$

$$= c_3\left(e_1^+\cdot e_4^-\right)^{s_a}\left(K_1\cdot e_2^+\right)^{s_b}\left(K_4\cdot e_3^-\right)^{s_a}\left(K_3\cdot e_4^-\right)^{s_b-s_a}\ . \tag{41}$$

For the given number of momenta we have in the minimal gauge 1 more possibilities to write a general expression for A_2^{++--}. The condition that c_4, c_2 and c_3 should be chosen in such a way that (39)–(41) give the same results fixes the amplitude. The remaining amplitude with c_1 can also be constructed. The resulting amplitudes can be found for $s_b = 1, 2$ and take the form

$$A_2{}^{++--}_{s_b=1} = \langle 12\rangle^{*2s_b}\langle 14\rangle^{2(s_b-s_a)}\langle 34\rangle^{2s_a}\left(\frac{1}{su}\,,\ \frac{1}{st}\,,\ \frac{1}{tu}\right) \tag{42}$$

$$A_2{}^{++--}_{s_b=2} = \frac{\langle 12\rangle^{*2s_b}\langle 14\rangle^{2(s_b-s_a)}\langle 34\rangle^{2s_a}}{stu}\ . \tag{43}$$

For higher s_b values we find the same expressions for odd and even s_b, the number of momenta in the amplitudes (39) being $4s_b - 2$ for odd s_b and $4s_b - 4$ for even $s_b \geq 2$. These numbers again are the number of derivatives in the product of two 3-vertices.

For completeness one also should consider the case $s_a = 0$ for the amplitude $A^{00+-} = A^{+-}$. There is only one minimal gauge with $b_3 = k_4$, $b_4 = k_3$. Thus the amplitude will require at least $(K)^{2s_b}$ momenta:

$$A^{+-} = c\left(K_1\cdot e_3^+\right)^{s_b}\left(K_1\cdot e_4^-\right)^{s_b} = c\frac{\left[\langle 13\rangle^*\langle 14\rangle\right]^{2s_b}}{s^{s_b}}\ , \tag{44}$$

where c contains the propagator poles. In another gauge it should be possible to obtain an expression compatible with eq. (44). Take for instance $b_{3,4} = k_1$, with

$$e_3^+\cdot e_4^- = \frac{\langle 31\rangle^*\langle 14\rangle}{\langle 31\rangle\langle 41\rangle^*} = -\frac{\left[\langle 31\rangle^*\langle 14\rangle\right]^2}{tu}\ . \tag{45}$$

The amplitude in this gauge is of the form

$$A^{+-} = d\frac{[\langle 13\rangle^* \langle 14\rangle]^{2s_b}}{t^{s_b} u^{s_b}} (K)^{2s_b} \ , \tag{46}$$

where the polynomial $(K)^{2s_b}$ is still arbitrary and d contains the poles. When we take for $(K)^{2s_b}$ the specific form u^{s_b}, compatibility between (44) and (46) can be achieved for $s_b = 1$ or $s_b = 2$, leading to

$$A^{+-}_{s_b=1} \;=\; [\langle 13\rangle^* \langle 14\rangle]^{2s_b} \left(\frac{1}{su} , \ \frac{1}{st} , \ \frac{1}{tu}\right) \tag{47}$$

$$A^{+-}_{s_b=2} \;=\; \frac{[\langle 13\rangle^* \langle 14\rangle]^{2s_b}}{stu} \ . \tag{48}$$

These forms generalize to higher spin when more momenta are allowed for. The amplitude (43) requires not $2s_b$ momenta, but $4s_b - 2$ momenta for odd s_b or $4s_b - 4$ momenta for even s_b. Thus (47) and (48) reduce to A_2^{0+0-} of eqs. (42) and (43).

Next we consider fermion–fermion scattering for equal and unequal spins and finally fermion–boson scattering. For fermion–fermion scattering with equal $s + 1/2$ spins we have at our disposal for the construction of the amplitudes not only $4s$ polarization vectors but also 4 spinors. For the construction of amplitudes, the occurrence of these spinors could in principle reduce the number of required momenta, since spinorial forms can arise:

$$k_{\dot{A}} e^{\dot{A}B} p_B \ , \ k_{\dot{A}} e^{\dot{A}B} e_{\dot{C}B} p^{\dot{C}} \ , \tag{49}$$

with an odd or even number of e's. We first ask whether this possibility really reduces the number of momenta required. Here again, we consider A^{++++} and A^{+++-}. In the first case we need even strings of polarization bispinors, which will be zero when all b_i are the same. For the second case we need one odd string and/or an even one. They cannot be made non-vanishing. For these amplitudes one still needs $4s$ momenta for a non-vanishing result.

For A^{++--} the spinors are $k_{1\dot{A}} {k_2}^{\dot{B}} {k_3}^{C} k_{4D}$ or $k_{1\dot{A}} k_{2\dot{B}} k_{3C} k_{4D}$ for particles 2, 4 or 3, 4 being antiparticles. The question is whether for a minimal gauge the above spinors can reduce the number of momenta for a non-vanishing matrix element. Take gauge 1, so e_1^+ and e_3^- should be contracted with the spinors. This can happen only with $k_{2\dot{B}} k_{4D}$, but then $k_{1\dot{A}} k_{3C}$ survive and must be contracted with a momentum, e.g. K_2. So the number of required momenta remains the same. This makes the minimum number of momenta for $s+1/2\, s+1/2 \to s+1/2\, s+1/2$ and $s\, s \to s\, s$ scattering the same. In the latter case the amplitudes have to be multiplied with a factor $\langle 12\rangle^* \langle 34\rangle$ to get the amplitudes of the former case. Explicitly, the amplitudes could exist for 3/2 and 5/2, which are obtained from eqs. (26) and (27). For spin $s = 3/2$

$$A^{++--}_{s=3/2} = \langle 12\rangle^{*2s} \langle 34\rangle^{2s} \left(\frac{1}{su} , \ \frac{1}{st} , \ \frac{1}{tu}\right) \ . \tag{50}$$

When the particles are identical, we have symmetry under $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ or $t \leftrightarrow u$, and consequently

$$A^{++--}_{s=3/2} = \frac{\langle 12\rangle^{*3}\langle 34\rangle^3}{ut} \ , \tag{51}$$

and for $s = 5/2$

$$A^{++--}_{s=5/2} = \frac{\langle 12\rangle^{*2s}\langle 34\rangle^{2s}}{stu} \ . \tag{52}$$

These formulae generalize to higher non-integral spins, when $s - 1/2$ is odd or even. For completeness we note that for spin $1/2$, again taking all particles outgoing, the lowest number of momenta is required for

$$A^{++--}_{s=1/2} = \langle 12\rangle^*\langle 34\rangle \left(\frac{1}{s}\ ,\ \frac{1}{t}\ ,\ \frac{1}{u}\right) \ . \tag{53}$$

For unequal spin scattering for fermions $s_a + 1/2\, s_a + 1/2 \to s_b + 1/2\, s_b + 1/2$ the case of eq. (33) will still involve the same number of momenta. The reason is that not all the spinors can be contracted with e_1, e_3, e_4 in a non-vanishing way. Here again, the modification for the change of $s_{a,b}$ into $s_{a,b} + 1/2$ is a multiplication of the amplitudes (37) and (38) by the factor $\langle 12\rangle^*\langle 34\rangle$. For the amplitude A_2 the starting point (39) does not leave any room for a reduction of the number of momenta. For fermions one has to multiply the expressions (42) and (43) again with $\langle 12\rangle^*\langle 34\rangle$. This also applies when $s_a = 0$.

In summary, for fermion–fermion scattering with spins s, s_a or s_b the formulae of equal and unequal spin scattering (31), (32), (37), (38), (42) and (43) are valid. The choice between odd/even cases is made depending on $s - 1/2$, $s_a - 1/2$, $s_b - 1/2$ being odd or even.

It is for fermion–boson scattering that the arguments differ. We distinguish the cases

$$s_a + \frac{1}{2}\ ,\ \ s_a + \frac{1}{2}\ ,\ \ s_b\ ,\ \ s_b\ , \tag{54}$$

$$s_a\ ,\ \ s_a\ ,\ \ s_b + \frac{1}{2}\ ,\ \ s_b + \frac{1}{2}\ . \tag{55}$$

For the A_1^{++--} we get the spinorial factor $k_{1\dot{A}}k_{2\dot{B}}$ whereas for A_2^{++--} we get $k_{1\dot{A}}k_{3B}$ (or $k_{3A}k_{4B}$ and $k_{2\dot{A}}k_{4B}$ for (55)). These factors should be used to reduce the number of momenta in the expressions (35) and (39), respectively. For A_1^{++--}, only in case (55) is this possible, leading to

$$A_1{}^{++--} = \langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b-1}\left(\frac{1}{su}\ ,\ \frac{1}{st}\ ,\ \frac{1}{tu}\right) \tag{56}$$

$$A_1{}^{++--} = \frac{\langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b-1}}{stu}\ , \tag{57}$$

for odd and even s_a. For case (54) the two spinors give a factor $\langle 12\rangle^*$ to expressions (37) and (38).

For the amplitude A_2^{++--} the starting point is (39) where either e_2^+ or e_4^- can be contracted with $k_{1\dot{A}}k_{3B}$ for case (54) or with $k_{2\dot{A}}k_{4B}$ for case (55). Only in the former case can this be done with the result

$$A_2^{++--} = \langle 12\rangle^{*2s_b}\langle 14\rangle^{2s_b-2s_a-1}\langle 34\rangle^{2s_a+1}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{tu}\right) \tag{58}$$

$$A_2^{++--} = \frac{\langle 12\rangle^{*2s_b}\langle 14\rangle^{2s_b-2s_a-1}\langle 34\rangle^{2s_a+1}}{stu}, \tag{59}$$

for odd and even s_b. For case (55) the remaining two spinors should be contracted with a momentum giving a factor $\langle 12\rangle^*\langle 14\rangle$ to eqs. (42) and (43).

Summarizing, for $s_a\, s_a \to s_b\, s_b$ scattering we can take formulae (42), (43) and (37), (38) with one exception. When s_b is non-integer one should take in eqs. (37) and (38) a factor $\langle 34\rangle^{2s_b-1}$ instead of $\langle 34\rangle^{2s_b}$.

4 Comparison with known results

Summarizing, we have derived scattering amplitudes for boson–boson, fermion–fermion and boson–fermion scattering.

For equal spins we have, both for boson–boson and fermion–fermion scattering with all particles outgoing:

- s or $s-1/2$ is odd:

$$\begin{aligned} A^{++--} &= \langle 12\rangle^{*2s}\langle 34\rangle^{2s}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{ut}\right) \\ &\simeq s^{2s}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{ut}\right), \end{aligned} \tag{60}$$

- s or $s-1/2$ is even:

$$\begin{aligned} A^{++--} &= \frac{\langle 12\rangle^{*2s}\langle 34\rangle^{2s}}{stu} \\ &\simeq \frac{s^{2s}}{stu}, \end{aligned} \tag{61}$$

where the symbol $\simeq$ means an equality modulo a complex phase. The latter formulae are given to facilitate the comparison with the literature where the phase factors often are different. These cases should be compared to 4-gluon, 4-graviton and 4-gravitino amplitudes. In the comparison we shall omit overall constants.

For the 4-gluon amplitude we use the expressions written in terms of spinorial products from [11]:

$$\begin{aligned}
A^{++--} &= \langle 34\rangle^4 \left[\frac{(a_1a_2a_3a_4)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right. \\
&\quad \left.+\frac{(a_1a_3a_4a_2)}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle}+\frac{(a_1a_4a_2a_3)}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}\right] \\
&= -\langle 12\rangle^{*2}\langle 34\rangle^2 \left[\frac{(a_1a_2a_3a_4)}{\langle 12\rangle\langle 12\rangle^*\langle 14\rangle\langle 14\rangle^*}\right. \\
&\quad \left.+\frac{(a_1a_3a_4a_2)}{\langle 12\rangle\langle 12\rangle^*\langle 13\rangle\langle 13\rangle^*}+\frac{(a_1a_4a_2a_3)}{\langle 13\rangle\langle 13\rangle^*\langle 14\rangle\langle 14\rangle^*}\right] , \qquad (62)
\end{aligned}$$

where $(a_1a_2a_3a_4)$ denotes the trace of a product of $SU(N)$ matrices T^{a_i}. The amplitude is evidently a combination of the three expressions of eq. (60).

The 4-graviton amplitude from ref. [12] reads s^3/ut, which agrees with (61). The 4-gravitino amplitude reads s^3/ut [13], which for identical particles is the only $t \leftrightarrow u$ invariant form following from (60).

For unequal spins we have, for both fermion–fermion and boson–boson scattering with $0 < s_a < s_b$ for $A_1^{++--} = A^{++--}(aabb)$:

- s_a or $s_a - 1/2$ odd:

$$\begin{aligned}
A_1^{++--} &= \langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{ut}\right) \\
&\simeq s^{s_a+s_b}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{ut}\right) , \qquad (63)
\end{aligned}$$

- s_a or $s_a - 1/2$ even:

$$\begin{aligned}
A_1^{++--} &= \frac{\langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b}}{stu} \\
&\simeq \frac{s^{s_a+s_b}}{stu} . \qquad (64)
\end{aligned}$$

For the other case, $A_2^{++--} = A^{++--}(abab)$ with $0 \le s_a \le s_b$, we have for

- s_b or $s_b - 1/2$ odd:

$$\begin{aligned}
A_2^{++--} &= \langle 12\rangle^{*2s_b}\langle 14\rangle^{2s_b-2s_a}\langle 34\rangle^{2s_a}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{ut}\right) \\
&\simeq s^{s_a+s_b}u^{s_b-s_a}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{ut}\right) , \qquad (65)
\end{aligned}$$

- s_b or $s_b-1/2$ even:

$$\begin{aligned} A_2^{++--} &= \frac{\langle 12\rangle^{*2s_b}\langle 14\rangle^{2s_b-2s_a}\langle 34\rangle^{2s_a}}{stu} \\ &\simeq \frac{s^{s_a+s_b}u^{s_b-s_a}}{stu} \ . \end{aligned} \tag{66}$$

The case of spin 1–spin 2 scattering can arise from eqs. (63) and (66)

$$A_1^{+1\,+1\,-2\,-2} \simeq s^3\left(\frac{1}{su}\,,\ \frac{1}{st}\,,\ \frac{1}{ut}\right) \tag{67}$$

$$A_2^{+1\,+2\,-1\,-2} \simeq \frac{s^3u}{stu}=\frac{s^2}{t} \ . \tag{68}$$

In quantum gravity (68) is found for photon–graviton scattering ([12] (where the two formulae of eq. (21) should be interchanged). The amplitude (67) does not arise in quantum gravity, it vanishes. That such an amplitude exists on general grounds has also been noticed in [14]. In that paper even more non-vanishing amplitudes were found: however, these vanish here by the condition of gauge invariance.

Spin 0–spin 1, 2 scattering can give rise to amplitude A_2

$$A_2^{0\,+1\,0\,-1} \simeq su\left(\frac{1}{su}\,,\ \frac{1}{st}\,,\ \frac{1}{ut}\right) \tag{69}$$

$$A_2^{0\,+2\,0\,-2} \simeq \frac{(su)^2}{stu}=\frac{su}{t} \ . \tag{70}$$

The former agrees with scalar–photon scattering and the latter is in agreement with scalar–graviton scattering [12]. For spin 1/2–spin 3/2 scattering one expects from (65)

$$A_2^{+1/2\,+3/2\,-1/2\,-3/2} \simeq s^2u\left(\frac{1}{su}\,,\ \frac{1}{st}\,,\ \frac{1}{ut}\right) \ , \tag{71}$$

which we could not compare with other results in the literature.

For boson–fermion scattering we have

- s_b non-integer, s_a odd

$$\begin{aligned} A_1^{++--} &= \langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b-2}\left(\frac{1}{su}\,,\ \frac{1}{st}\,,\ \frac{1}{ut}\right) \\ &\simeq s^{s_a+s_b-1}\left(\frac{1}{su}\,,\ \frac{1}{st}\,,\ \frac{1}{ut}\right) \ , \end{aligned} \tag{72}$$

- s_b non-integer, s_a even

$$\begin{aligned} A_1^{++--} &= \frac{\langle 12\rangle^{*2s_a}\langle 34\rangle^{2s_b-2}}{stu} \\ &\simeq \frac{s^{s_a+s_b-1}}{stu} \ , \end{aligned} \tag{73}$$

where s_b is the spin of the fermions and s_a is odd or even. For all other cases, eqs. (63), (64) (s_a integer, s_b half-integer) and eqs. (65), (66) (s_a integer, s_b half-integer or reverse) apply. Thus from eq. (66) one gets

$$A_2^{1/2\ 2\ -1/2\ -2} \simeq \frac{s^{5/2}u^{3/2}}{stu} = \frac{s\sqrt{su}}{t} \tag{74}$$
$$A_2^{3/2\ 2\ -3/2\ -2} \simeq \frac{s^{7/2}u^{1/2}}{stu} = \frac{s^2\sqrt{su}}{tu} \tag{75}$$

in agreement with gravity (at least with the unpolarized cross section in refs. [15, 16]) and supergravity [13].

From (65) we have

$$A_2^{1/2\ 1\ -1/2\ -1} \simeq s^{3/2}u^{1/2}\left(\frac{1}{su}, \frac{1}{st}, \frac{1}{ut}\right) , \tag{76}$$

which is of a form that arises in quark–gluon scattering or electron–photon scattering.

So far we have compared the general scattering amplitudes derived in section 3 to those obtained in existing gauge theories. We would like to stress that the type of reasoning used in section 3 can also be useful for finding amplitudes that arise from certain effective Lagrangians. We illustrate this by considering an explicit example. Suppose a particular theory makes $\pi^0 \to 3\,\gamma$ decay possible. One should find the simplest possible Lagrangian involving 3 fields A_μ and a pion field φ. From the Lagrangian one can derive the matrix element by the method of section 3. In an effective field theory the A_μ fields come in $F_{\mu\nu}$ combinations, i.e. $K_\mu\varepsilon_\nu - K_\nu\varepsilon_\mu$. In the Weyl-van der Waerden formalism these combinations become

$$K_\mu\varepsilon_\nu^+ - K_\nu\varepsilon_\mu^+ \to k_{\dot{A}}k_{\dot{C}}\epsilon_{BD} , \tag{77}$$
$$K_\mu\varepsilon_\nu^- - K_\nu\varepsilon_\mu^- \to k_B k_D \epsilon_{\dot{A}\dot{C}} . \tag{78}$$

From these expressions it is clear that an odd number of photons with equal helicity cannot arise. So we have to consider amplitudes A^{++-}. When using a minimal gauge 1 from table 2 the amplitude takes the form

$$\begin{aligned} A^{++-} &= c\left(K_2\cdot e_1^+\right)\left(K_1\cdot e_2^+\right)\left(K_2\cdot e_3^-\right) \\ &= c\frac{\langle 21\rangle^{*3}\langle 23\rangle\langle 13\rangle}{(1\cdot 3)} , \end{aligned} \tag{79}$$

where c contains polynomials in $K_i \cdot K_j = (i\cdot j)$ coming from derivatives on the field strengths. Since the artificial pole (it is not present in (77), (78)) should be absent, c should contain the inner product $(1\cdot 3)$, such that we have in general

$$A^{++-} = c'\langle 12\rangle^{*3}\langle 23\rangle\langle 13\rangle . \tag{80}$$

Bose symmetry requires $A^{++-}(123) = A^{++-}(213)$. Since $\langle 12\rangle^3$ is odd in $1 \leftrightarrow 2$ the expression c' should be as well. The simplest form for c' then leads to

$$A^{++-} = d\,(K_1 - K_2)\cdot K_3 \langle 12\rangle^{*3}\langle 23\rangle\langle 13\rangle \ , \tag{81}$$

with an overall constant d and similar expressions for A^{+-+} and A^{-++}. The opposite helicity amplitudes are obtained by complex conjugation and have the same constant d when parity is conserved. Summing over the helicities we find

$$\begin{aligned}\sum_{\text{hel.}} |A|^2 &= d^2\,(1\cdot 2)\,(2\cdot 3)\,(1\cdot 3)\,\Big[(1\cdot 2)^2\,[(1\cdot 3)-(2\cdot 3)]^2 \\ &+ (1\cdot 3)^2\,[(1\cdot 2)-(3\cdot 2)]^2 + (2\cdot 3)^2\,[(2\cdot 1)-(3\cdot 1)]^2\Big] \ ,\end{aligned} \tag{82}$$

which is the same form as found from a Lagrangian [17]

$$\mathcal{L} = \partial_\alpha \varphi \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}\,(\partial_\gamma \partial_\beta F_{\rho\sigma})\,\partial^\gamma F^{\alpha\beta} \tag{83}$$

5 Discussion and Conclusions

In order to place our results in a slightly different perspective, we reconsider the QCD 4-gluon amplitude. Separating the colour structure from the momentum/helicity structures we write the amplitude as

$$\mathcal{M}(1234) = 2ig^2 \sum_{P(123)} (a_1 a_2 a_3 a_4)\,\mathcal{C}(1234) \ , \tag{84}$$

where $(a_1 a_2 a_3 a_4)$ is a trace of a product of $SU(N)$ matrix T^{a_i}. It is the construction of the gauge invariant amplitude $\mathcal{C}(1234)$ in which we are interested. It arises from three ordered Feynman diagrams and reads

$$\begin{aligned}\mathcal{C}(1234) &= 2\sum_{P(123)} \frac{\Big(Tr\,(F_1F_2F_3F_4) - \frac{1}{4}Tr\,(F_1F_2)\,Tr\,(F_3F_4)\Big)}{su} \\ &= \frac{N_1}{su} \ .\end{aligned} \tag{85}$$

The diagrams with a $1/u$ and $1/s$ pole have numerators with two momenta, the constant term has none. When combining the three diagrams there are enough momenta to allow for four field strengths. The particular combination N_1 has the properties

$$\begin{aligned} N_1(++++) &= N_1(+++-) = 0 \ , \\ N_1(++--) &= [\langle 12\rangle^*\langle 34\rangle]^2 \ .\end{aligned} \tag{86}$$

For graviton–graviton scattering one has four diagrams, also a $1/t$ pole diagram. The diagrams with 3-vertices have four momenta in the numerator, the contact term has two momenta. Combining the four diagrams to an expression N_2/stu leads to 8 momenta in N_2. This is sufficient for four spin 2 field strengths, since a spin 2 field strength requires two derivatives. In momentum space a spin 2 field strength can be written as a product of spin 1 field strengths, since $e_{\mu\nu} = e_\mu e_\nu$. Thus a suitable form of N_2 would be $(N_1)^2$, being expressible in spin 2 field strengths. Moreover, the correct $++++$, $+++-$, $++--$ amplitudes would arise. In fact the construction of the graviton–graviton scattering amplitude along these lines is explicitly known [18, 19]:

$$\begin{aligned}\mathcal{A}(1234) &= s \times \mathcal{C}(1234)\,\mathcal{C}(1243) \\ &= \frac{N_1(1234)N_1(1243)}{stu} \\ &= \frac{N_2}{stu}\ .\end{aligned} \tag{87}$$

Note that eq. (85) is uniquely determined by demanding factorization of the expression in Feynman diagrams (i.e. a term with an s-pole, a u-pole and a contact term). So we could try to turn the reasoning around. Suppose we start with amplitudes like (85) and (87). One obtains a constant term by dividing out the whole numerator. The s, t or u exchange diagrams are obtained by dividing out parts of the numerator. In this way one gets a handle on the form of the 3- and 4-vertices.

For equal particle scattering the structure of the amplitude we find is for odd spin

$$\mathcal{C} = \frac{N_1^s}{su}\ , \tag{88}$$

and for even spin

$$\mathcal{A} = \frac{N_2^{s/2}}{stu}\ . \tag{89}$$

Since both N_1/su and N_2/stu can already be split into a non-pole and pole part, also $\mathcal{C}$ and $\mathcal{A}$ can be split in such a way. Whether one can derive acceptable 3- and 4-vertices from this and whether the pole parts are related to only spin s exchange is to be seen. As a side remark, one can also understand in terms of field strengths how the difference between the A_1 and A_2 terms of eqs. (37), (38) and (42), (43) arise. For both sets of equations one needs a factor $N_1^{s_a}$. For the first set one then completes the matrix element by adding a factor $\{Tr(F_3^- \cdot F_4^-)\}^{s_b - s_a}$, for the second set one needs $\{K_1 \cdot F_2^+ \cdot F_4^- \cdot K_1\}^{s_b - s_a}$, since a non-vanishing expression without momenta is not possible. This can be seen from eqs. (77), (78). Moreover one thus obtains the correct spinorial factors for the A_1 and A_2 amplitudes.

So, we have derived the form of higher spin scattering amplitudes but do not know whether there exist a field theories that lead to them. Let us summarize what we know about the dimensions of the amplitudes.

If the three vertices for self interaction of spin s bosons as derived in the literature would lead to a gauge theory, the number of momenta in the product of 3-vertices in an

$ss \to ss$ scattering amplitude would be $2s$. For $s+1/2$ fermion–fermion scattering the same dimension would be found when spin s bosons are exchanged. For unequal spin boson scattering, $s_a s_a \to s_b s_b$, the pattern of table 1 would require $2s_b$ momenta. The same holds for fermion–boson scattering, $s_a + 1/2 s_a + 1/2 \to s_b s_b$ and $s_a s_a \to s_b + 1/2 s_b + 1/2$.

So if the 3-vertices were to be a part of a gauge theory involving a series of vertices with increasing number of particles, the dimension of the scattering amplitudes would be fixed. This dimension turns out to be different from the one found by a direct construction of scattering amplitudes on the basis of gauge invariance and pole structure. The latter method finds for $ss \to ss$ scattering and $s+1/2$ fermion–fermion scattering that $4s-2$ or $4s-4$ momenta are required in the product of 3-vertices in the amplitude for odd or even s. For unequal spin scattering, $s_a s_a \to s_b s_b$ and $s_a s_a \to s_b + 1/2 s_b + 1/2$, $4s_b - 2$ or $4s_b - 4$ (s_b odd/even) momenta are needed. This latter case naturally belongs to the case of equal spin scattering: all amplitudes $s_b s_b \to s_b s_b$ and $s_a s_a \to s_b s_b$ ($s_a < s_b$) have the same dimension.

Although the number of momenta in the product of 3-vertices in the amplitudes for equal spin scattering is higher than the ones suggested by table 1, it is lower than the $4s$ momenta one would get by using just field strengths in the vertices. The constructed scattering amplitudes for arbitrary spins reduce for $s \leq 2$ to the ones of known theories with the exception of the above-mentioned unequal spin amplitudes with $2s_a + 2s_b - 2$ (4) momenta.

For the construction of the amplitudes with massless particles the Weyl-van der Waerden formalism was again very convenient. It was also indicated that for decay amplitudes the general form can be easily derived within this formalism.

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