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Dependence of Inflationary Reconstruction upon Cosmological Parameters

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ABSTRACT

The inflationary potential and its derivatives determine the spectrum of scalar and tensor metric perturbations that arise from quantum fluctuations during inflation. The CBR anisotropy offers a promising means of determining the spectra of metric perturbations and thereby a means of constraining the inflationary potential. The relation between the metric perturbations and CBR anisotropy depends upon cosmological parameters – most notably the possibility of a cosmological constant. Motivated by some observational evidence for a cosmological constant (large-scale structure, cluster-baryon fraction, measurements of the Hubble constant and age of the Universe) we derive the reconstruction equations and consistency relation to second order in the presence of a cosmological constant. We also clarify previous notation and discuss alternative schemes for reconstruction.



1 Introduction

Inflation gives rise to nearly scale-invariant scalar (or density) and tensor (or gravity-wave) metric perturbations which are excited by quantum fluctuations in the inflaton field and in the metric itself [1] and are determined by the inflationary potential and its derivatives [2]. Measurements of the scalar and tensor metric perturbations permit partial reconstruction of the inflationary potential [3]. Both the scalar and tensor perturbations give rise to temperature anisotropies in the cosmic background radiation (CBR) [4], and precise measurements of CBR anisotropy on angular scales from 0.1° to 100° (multipole numbers $\ell = 2 - 1000$) probably offer the most promising means of determining the metric perturbations.

Copeland and his collaborators have emphasized the underlying relationship that exists between the inflationary potential and the power spectra describing the metric perturbations [5] (k -space reconstruction). These relations are independent of present cosmological parameters (e.g., Hubble constant, baryon density, cosmological constant, and ionization history of the Universe). On the other hand, Turner has emphasized that realizing reconstruction in practice requires connecting the potential and its first few derivatives to a handful of observables, e.g., the scalar and tensor contributions to the quadrupole CBR anisotropy (S and T) and the power-law spectral indices that characterize the scalar and tensor power spectra (n and n_T) (ℓ -space reconstruction). However, relating the power spectrum to CBR anisotropy necessarily brings in these cosmological parameters [6].

As we shall discuss, the only significant dependence that arises in going from k -space to ℓ -space is from a possible cosmological constant, and the purpose of this paper is to quantify that dependence. At the moment there is some motivation for considering a cosmological constant: the cold dark matter model with a cosmological constant provides a good fit to all the observational data (large-scale structure, measurements of the Hubble constant and cluster baryon fraction, and age of the Universe) [7]. In addition to calculating the dependence of the reconstruction and consistency equations upon the value of the cosmological constant and showing that the dependence upon other cosmological quantities is insignificant, we will clarify previous notation and normalization conventions and discuss alternative reconstruction strategies.

2 Perturbative Reconstruction

The program of perturbative reconstruction is spelled out in Refs. [8]. The basic idea is to express a handful of observables – e.g., S , T , n , n_T , and $dn/d \ln k$ – in terms of the derivatives of the inflationary potential, evaluated at some convenient point (here denoted by ‘*’). The perturbation expansion is in the deviation from exactly exponential inflation and exact scale invariance, quantified by the derivatives of the potential, or more conveniently in terms of the spectral indices n_T and $\tilde{n} \equiv n - 1$. In the scale-invariant limit $n_T = \tilde{n} = 0$ and all the derivatives of the potential vanish.

2.1 Notation and lowest-order reconstruction

Let's begin at lowest order. To lowest order (in n_T and \tilde{n}) the power spectra of scalar and tensor perturbations are described by

$$\tilde{n} = -\frac{3}{8\pi} \left(\frac{m_{\text{Pl}} V_*'}{V_*} \right)^2 + \frac{1}{4\pi} \left(\frac{m_{\text{Pl}}^2 V_*''}{V_*} \right) \quad (1)$$

$$n_T = -\frac{1}{8\pi} \left(\frac{m_{\text{Pl}} V_*'}{V_*} \right)^2 \quad (2)$$

$$P(k) = \frac{1024\pi^3}{75} \frac{k}{H_0^4} \left(\frac{g(\Omega_0)}{\Omega_0} \right)^2 \frac{V_*^3}{m_{\text{Pl}}^6 V_*'^2} [k/k_*]^{\tilde{n}} T^2(k) \quad (3)$$

$$P_T(k) = \frac{8}{3\pi} \frac{V_*}{m_{\text{Pl}}^4} [k/k_*]^{n_T} T_T^2(k) \quad (4)$$

where $V(\phi)$ is the inflationary potential, prime denotes $d/d\phi$, and $V_* = V(\phi_*)$, etc.

The factor $[g(\Omega_0)/\Omega_0]^2$ in Eq. (3) takes into account the growth of the perturbations and the relation between the density and curvature perturbations, with $g(\Omega_0)$ well fit by [9]

$$g(\Omega) = \frac{5}{2} \Omega \left[\frac{1}{70} + \frac{209\Omega}{140} - \frac{\Omega^2}{140} + \Omega^{4/7} \right]^{-1} \quad (5)$$

where Ω_0 is the matter density (cold dark matter + baryons). The functions $T(k)$ and $T_T(k)$ are the “transfer functions” which describe the cosmological evolution of the modes that arises due to the transition of the Universe from an early radiation-dominated epoch to a matter-dominated epoch, and are defined so that $T(k), T_T(k) \rightarrow 1$ for $k \rightarrow 0$. The transfer functions together with the growth factor for scalar perturbations take “primordial” spectra to presently “observed” spectra. The scalar transfer function can be fit by [10, 11]

$$T(k) = \left[1 + (ak + (bk)^{3/2} + (ck)^2)^\nu \right]^{-1/\nu} \quad (6)$$

with $a = (6.4/\Gamma)\text{Mpc}$, $b = (3.0/\Gamma)\text{Mpc}$, $c = (1.7/\Gamma)\text{Mpc}$ and $\nu = 1.13$. Here $\Gamma \simeq \Omega_0 h$ is a measure of the size of the horizon at matter-radiation equality. In the absence of a cosmological constant, the gravity-wave transfer function can be written as [12]

$$T_T(k) = \frac{3j_1(k\tau_0)}{k\tau_0} \mathcal{T}(k) \quad (7)$$

where $j_1(x)$ is the spherical bessel function of the first order and $\mathcal{T}(k)$ is a factor analogous to $T(k)$ and is given in Ref. [12]. For the modes of most interest, those that enter the horizon during matter domination, $\mathcal{T}(k) \approx 1$.

It is convenient to rewrite the scalar and tensor spectra in terms of quantities $A_S^2(k)$ and $A_T^2(k)$ whose only k -dependence arises from a deviation from scale invariance¹ and which have a simple physical interpretation,

$$P(k) \equiv \frac{2\pi^2}{H_0^4} k A_S^2(k) \left(\frac{g(\Omega_0)}{\Omega_0} \right)^2 T^2(k) \quad (8)$$

$$P_T(k) \equiv \frac{25}{4\pi} A_T^2(k) T_T^2(k) \quad (9)$$

where to lowest order in \tilde{n} , n_T

$$A_S^2(k) = \frac{512\pi}{75} [k/k_*]^{\tilde{n}} \frac{V_*^3}{m_{\text{Pl}}^6 V_*'^2} \quad (10)$$

$$A_T^2(k) = \frac{32}{75} [k/k_*]^{n_T} \frac{V_*}{m_{\text{Pl}}^4} \quad (11)$$

For the scalar perturbations, $A_S^2(k = H_0)$ is the present contribution of this mode to the *rms* mass fluctuation per logarithmic interval in k (in the absence of a cosmological constant):

$$A_S^2(k = H_0) = \Delta^2(H_0) \equiv \frac{k^3}{2\pi^2} P(k) \Big|_{k=H_0} \quad (12)$$

The quantity $A_T^2(k = H_0)$ is related to the present energy density in long-wavelength, inflation-produced gravitational waves (in the absence of a cosmological constant):

$$\frac{d \log \Omega_{\text{GW}}}{d \ln k} = \frac{75}{32} A_T^2(k) \left(\frac{k}{H_0} \right)^{-2}, \quad (13)$$

valid for $k \ll k_{\text{EQ}} \sim 200H_0$.

In Eqs. (3,4) the expansion point ϕ_* is the point about which the power-law indices and all the derivatives of the potential are evaluated. It is defined by the fact that the comoving scale k_* crossed outside the horizon during inflation when $\phi = \phi_*$; given the details of inflation (reheat temperature and so on) it is straightforward to relate ϕ_* to the number of e-foldings before the end of inflation and/or to k_* . For lowest-order reconstruction ϕ_* is irrelevant as any dependence upon it involves higher-order corrections in n_T and \tilde{n} . For second-order reconstruction, the choice of k_* is important. It will prove very convenient to choose $k_* = H_0$; later we will discuss the dependence of the reconstruction equations upon k_* .

In order to practically implement reconstruction the power spectra must be related to observables. At lowest order the natural set of observables is S , T , \tilde{n} , and n_T . Since \tilde{n} and

¹Copeland et al. have introduced several definitions of A_S^2 and A_T^2 (also denoted as A_G^2); we shall use the definitions given in Ref. [13]. We note that the relationship between A_i^2 and the power spectra given in the Appendix of Ref. [15] has errant factors of k_{50}^{n-1} and $k_{50}^{n_T}$ as well as not using the current definitions of the A_i^2 . Copeland et al. have not explicitly discussed the definitions of A_i^2 for $\Omega_\Lambda \neq 0$; we define without the Ω_0 dependence that arises from the growth of perturbations and relating density and curvature perturbations.

n_T are already given in terms of the potential, it only remains to relate S and T to $A_S^2(k_*)$ and $A_T^2(k_*)$. The variance of the multipole moments of the expansion of the CBR temperature field are integrals of the power spectra times kernels which depend on the cosmological parameters (h , $\Omega_B h^2$, Ω_Λ) and the ionization history of the Universe. These integrals introduce the dependence of the reconstruction equations upon cosmological parameters. The dependence upon all of these except Ω_Λ is very weak (less than 1% for sensible variations in $\Omega_B h^2$ and 4% for sensible variations in h).

The ‘‘Rosetta Stone’’ relations for lowest-order reconstruction, which take k -space equations to ℓ -space equations, are:

$$S \equiv \frac{5C_2^S}{4\pi} = 0.10 f_S^{(0)}(\Omega_\Lambda) A_S^2(k_*) \quad (14)$$

$$T \equiv \frac{5C_2^T}{4\pi} = 1.4 f_T^{(0)}(\Omega_\Lambda) A_T^2(k_*) \quad (15)$$

where we have followed conventional practice and expanded the two-point function of the CBR temperature perturbations in Legendre polynomials

$$\left\langle \frac{\Delta T}{T}(\hat{x}_1) \frac{\Delta T}{T}(\hat{x}_2) \right\rangle \equiv \frac{1}{4\pi} \sum_\ell (2\ell + 1) C_\ell P_\ell(\hat{x}_1 \cdot \hat{x}_2) \quad (16)$$

where brackets denote the average over the sky. The functions $f_S^{(0)}(\Omega_\Lambda)$ and $f_T^{(0)}(\Omega_\Lambda)$ quantify the dependence of reconstruction upon the cosmological parameter $\Omega_\Lambda = 1 - \Omega_0$. We have evaluated them numerically (with $h = 0.75$ and $\Omega_B h^2 = 0.0125$) and normalized them such that all expressions have their familiar values with $f_i^{(0)} \simeq 1$. The functions and their ratio are shown in Fig. 1; they are well fit by quadratics over the range $0.0 \leq \Omega_\Lambda < 0.8$:

$$f_S^{(0)} = 1.04 - 0.82\Omega_\Lambda + 2\Omega_\Lambda^2 \quad (17)$$

$$f_T^{(0)} = 1.0 - 0.03\Omega_\Lambda - 0.1\Omega_\Lambda^2 \quad (18)$$

The correction to the familiar scalar relation in the $\Omega_\Lambda = 0$ limit (i.e., $f_S^{(0)}(0) \neq 1$) arises from including the integrated Sachs-Wolfe effect, due to the decay of the potentials near matter-radiation equality (see also Fig. 2).

Using these relations in place of the scalar and tensor power spectra, the lowest-order reconstruction equations and consistency relation follow directly:

$$\frac{V_*}{m_{\text{Pl}}^4} = 1.65 T / f_T^{(0)} \quad (19)$$

$$\frac{V'_*}{m_{\text{Pl}}^3} = \pm 8.3 \sqrt{-n_T} T / f_T^{(0)} \quad (20)$$

$$\frac{V''_*}{m_{\text{Pl}}^2} = 21 (\tilde{n} - 3n_T) T / f_T^{(0)} \quad (21)$$

$$n_T = -\frac{1}{7} \frac{f_S^{(0)} T}{f_T^{(0)} S} \quad (22)$$

where the sign of V'_* is indeterminate as it can be changed by taking ϕ to $-\phi$. The final expression is the consistency relation that arises since the four observables can be expressed in terms of three properties of the potential. The familiar factor of $1/7$ is modified by ratio of $f_S^{(0)}/f_T^{(0)}$, introduced in Ref. [14]. In practice n_T is likely to be difficult to measure, and so the consistency relation can be used to eliminate n_T in the expressions for V'_* and V''_* .

2.2 Second-order reconstruction

Including the Ω_Λ dependence in second-order reconstruction is in principle as easy as it was in lowest-order reconstruction. (Second-order refers to including the order \tilde{n} and n_T corrections to the reconstruction and consistency equations.) However the strategy is slightly different because while there are second-order expressions for the power spectra, cf. Ref. [15],² similar explicit expressions for the spectral indices do not exist. In addition, another observable is needed; the plausible candidate is the “running” of the scalar spectral index [16], $dn/d\ln k$, which is $\mathcal{O}(\tilde{n}^2, n_T^2)$.

Reconstruction proceeds from k -space expressions relating the inflationary potential and its derivatives at ϕ_* to $A_T^2(k_*)$ and $A_S^2(k_*)$, and follows the $\Omega_\Lambda = 0$ case done in Ref. [15]. The key k -space equations are

$$\frac{V_*}{m_{\text{Pl}}^4} = \frac{75}{32} A_T^2 \left[1 + 0.21 \frac{A_T^2}{A_S^2} \right] \quad (23)$$

$$\frac{V'_*}{m_{\text{Pl}}^3} = -\frac{75\sqrt{\pi}}{8} \frac{A_T^3}{A_S} \left[1 - 0.85 \frac{A_T^2}{A_S^2} - 0.53\tilde{n} \right] \quad (24)$$

$$\frac{V''_*}{m_{\text{Pl}}^2} = \frac{25\pi}{4} A_T^2 \left[\tilde{n} + 6 \frac{A_T^2}{A_S^2} - 16 \frac{A_T^4}{A_S^4} - \frac{\tilde{n}^2}{6} - 9.8\tilde{n} \frac{A_T^2}{A_S^2} + 1.1 \frac{d\tilde{n}}{d\ln k} \right] \quad (25)$$

$$\frac{V'''_*}{m_{\text{Pl}}} = \pm 4\pi \sqrt{-8\pi n_T} \left[\frac{d\tilde{n}/d\ln k}{n_T} - 6n_T + 4\tilde{n} \right] \frac{V_*}{m_{\text{Pl}}^4} \quad (26)$$

$$\frac{A_T^2}{A_S^2} = -\frac{n_T}{2} \left[1 - \frac{n_T}{2} + \tilde{n} \right] \quad (27)$$

where for simplicity the arguments of $A_S(k_*)$ and $A_T(k_*)$ have been omitted. Note too that the spectral indices and the derivative of the scalar spectral index are also evaluated at k_* . These equations are Eqns. (3.4), (3.6), and (3.15) of Ref. [17] and Eqns. (39) and (46) of Ref. [15], as modified to be consistent with the definitions of A_S^2 and A_T^2 in Ref. [13]. The last equation is the second-order consistency equation. It can be used to eliminate the factors of A_T^2/A_S^2 in the first three equations.

Once again, the key to going from k -space equations to ℓ -space equations is relating $A_T^2(k_*)$ and $A_S^2(k_*)$ to the CBR observables T and S . At second order, the order \tilde{n} and n_T corrections must be taken into account. The second-order “Rosetta Stone” equations are

²We will not need these expressions; in any case, the second-order corrections are multiplicative factors of $1 + 7n_T/6 + (-7/3 + \ln 2 + \gamma)\tilde{n}$ to A_S^2 and of $1 + (-7/6 + \ln 2 + \gamma)n_T$ to A_T^2 .

given by

$$S = 0.10 f_S^{(0)}(\Omega_\Lambda) [1 + f_S^{(1)}(\Omega_\Lambda) \tilde{n}] A_S^2(k_*) \quad (28)$$

$$T = 1.4 f_T^{(0)}(\Omega_\Lambda) [1 + f_T^{(1)}(\Omega_\Lambda) n_T] A_T^2(k_*) \quad (29)$$

where $f_i^{(0)}(\Omega_\Lambda)$ are the same functions as in the previous Section and second-order expressions for $A_i(k_*)$ must be used. The functions $f_i^{(1)}(\Omega_\Lambda)$ quantify the Ω_Λ dependence of the second-order corrections that arise from relating the A_i^2 to S and T . They depend upon the ‘‘pivot point’’ k_* and for $k_* = H_0$, they can be accurately fit by:

$$f_S^{(1)} = 0.45 - 0.51\Omega_\Lambda + 1.04\Omega_\Lambda^2 - 0.14\Omega_\Lambda^3 \quad (30)$$

$$f_T^{(1)} = 0.58 - 0.50\Omega_\Lambda + 0.31\Omega_\Lambda^2 - 0.88\Omega_\Lambda^3 \quad (31)$$

Concerning the pivot-point dependence of $f_i^{(1)}(\Omega_\Lambda)$; using the fact that $A_S^2(k) \propto [k/k_*]^{\tilde{n}}$ and $A_T^2(k) \propto [k/k_*]^{n_T}$ it is simple to show that under the change $k_* \rightarrow k'_*$, $f_i^{(1)} \rightarrow f_i^{(1)} + \ln(k_*/k'_*)$. We note that changing the pivot point does not affect the form of higher-order corrections, i.e., the values of $f_i^{(j)}$ for $j = 2, \dots$.

The ℓ -space reconstruction and consistency equations now follow from the k -space equation through use of the ‘‘Rosetta Stone’’ equations:

$$\frac{V_*}{m_{\text{Pl}}^4} = 1.65 [1 - (f_T^{(1)} + 0.1)n_T] T/f_T^{(0)} \quad (32)$$

$$\frac{V'_*}{m_{\text{Pl}}^3} = \pm 8.3\sqrt{-n_T} [1 - (f_T^{(1)} - 0.18)n_T - 0.03\tilde{n}] T/f_T^{(0)} \quad (33)$$

$$\frac{V''_*}{m_{\text{Pl}}^2} = 21 \left[(\tilde{n} - 3n_T) + (3f_T^{(1)} - 2.6)n_T^2 + (1.9 - f_T^{(1)})n_T\tilde{n} \right. \quad (34)$$

$$\left. - 0.2\tilde{n}^2 + 1.1 \frac{d\tilde{n}}{d \ln k} \right] T/f_T^{(0)} \quad (35)$$

$$\frac{V'''_*}{m_{\text{Pl}}} = \pm 104\sqrt{-n_T} \left[\frac{d\tilde{n}/d \ln k}{n_T} - 6n_T + 4\tilde{n} \right] T/f_T^{(0)} \quad (36)$$

$$n_T = -\frac{1}{7} \frac{f_S^{(0)}}{f_T^{(0)}} \frac{T}{S} \left[1 + \frac{1}{7}(f_T^{(1)} - \frac{1}{2})(f_S^{(0)}/f_T^{(0)}) \frac{T}{S} + (f_S^{(1)} - 1)\tilde{n} \right] \quad (37)$$

While the signs of V'_* and V'''_* are arbitrary, the relative sign is not. By using the consistency equation the factors of n_T (which is likely to be very difficult to measure) can be eliminated in favor of T/S .

Finally, we note that the previous results for $\Omega_\Lambda = 0$ in Ref. [15] can be recovered by substituting $f_T^{(0)} = f_S^{(0)} = 1$, $f_T^{(1)} = 1.3$, and $f_S^{(1)} = 1.15$. (In Ref. [15] the pivot point $k_* = H_0/2$, so that $\ln 2$ must be added to the $f_i^{(1)}$ defined above.)

2.3 Alternative schemes

The goal of perturbative reconstruction is to use data, most likely measurements of CBR anisotropy, to infer the value of the inflationary potential and its first few derivatives at ϕ_* . To achieve this goal, one needs to pick a set of observables and then relate the power spectra to these observables. At second order, the pivot point k_* also comes into play.

The spectral indices \tilde{n} , n_T , and $dn/d\ln k$ (for second order) are obvious choices for the observables (though in practice one will probably wish to use the consistency equation to eliminate n_T). The quantities S and T are sensible choices as they: (i) serve to normalize the scalar and tensor contributions to CBR anisotropy; (ii) are easy to extract from CBR measurements; and (iii) are relatively insensitive to all the cosmological parameters except Ω_Λ . If one uses S and T then it is also sensible to select the pivot point $k_* = H_0$, which minimizes the dependence of S and T upon n_T and \tilde{n} since the dominant contribution to S and T comes from modes with $k \sim H_0$.

On the other hand, since the multipoles that will have the most leverage in determining \tilde{n} (and $dn/d\ln k$) are $\ell \sim 30 - 300$, it might be more useful to choose $k_* \sim (30 - 100)H_0/2$ (recall, \tilde{n} and $dn/d\ln k$ are evaluated at $k = k_*$). However, the higher multipoles are more sensitive to the cosmological parameters (e.g., h and $\Omega_B h^2$).

In any case, it is a simple matter to substitute other multipoles for S and T . For example, consider

$$S_{30} = \frac{61C_{30}^S}{4\pi} \quad (38)$$

$$T_{30} = \frac{61C_{30}^T}{4\pi} \quad (39)$$

Writing the ‘‘Rosetta Stone’’ equations in precisely the same form as before,

$$S_{30} = 0.10 f_S^{(0)}(\Omega_\Lambda) [1 + f_S^{(1)}(\Omega_\Lambda) \tilde{n}] A_S^2(k_*) \quad (40)$$

$$T_{30} = 1.4 f_T^{(0)}(\Omega_\Lambda) [1 + f_T^{(1)}(\Omega_\Lambda) n_T] A_T^2(k_*) \quad (41)$$

the form of the ℓ -space reconstruction equations and consistency relation are unchanged (except $T \rightarrow T_{30}$ and $S \rightarrow S_{30}$). It must of course be remembered that $f_i^{(0)}(\Omega_\Lambda)$ and $f_i^{(1)}(\Omega_\Lambda)$ are completely different functions which also have significant dependence upon other cosmological parameters. Taking $k_* = 20H_0$, $\Omega_B h^2 = 0.0125$ and $h = 0.75$, the Ω_Λ dependence can be fit by,

$$f_S^{(0)}(\Omega_\Lambda) = 0.11 - 0.02\Omega_\Lambda + 0.07\Omega_\Lambda^2 \quad (42)$$

$$f_T^{(0)}(\Omega_\Lambda) = 0.08 - 0.00\Omega_\Lambda + 0.01\Omega_\Lambda^2 \quad (43)$$

$$f_S^{(1)}(\Omega_\Lambda) = 0.25 - 0.35\Omega_\Lambda + 0.07\Omega_\Lambda^2 - 0.52\Omega_\Lambda^3 - \ln(k_*/20H_0) \quad (44)$$

$$f_T^{(1)}(\Omega_\Lambda) = -0.11 - 0.54\Omega_\Lambda + 0.37\Omega_\Lambda^2 - 0.93\Omega_\Lambda^3 - \ln(k_*/20H_0) \quad (45)$$

2.4 Power-law inflation: an exact result

If the inflaton potential is an exponential,

$$V(\phi) = V_0 \exp[-\sqrt{16\pi/p} \phi/m_{\text{Pl}}], \quad (46)$$

the growth of the scale factor during inflation is precisely a power law, $R(t) \propto t^p$, and it is possible to solve for the perturbation spectra exactly [18]. In this case the only parameter to be determined is the Hubble constant during inflation ($= H_*$) when the mode k_* crossed outside the horizon.

For power-law inflation the solution of the equation of motion (the massless Klein-Gordon equation) for fluctuations in the inflaton and the gravitational fields is a Hankel function. For modes that are well outside the horizon at the end of inflation (all those of astrophysical interest are), matching values of the field and its first derivative at the end of inflation allows one to calculate the Bogoliubov coefficients relating the creation and annihilation operators describing the quantum field before and after the end of inflation (see e.g., Ref. [19] and references therein). From these one can calculate the two-point function of the (classical, random) field at the present, and $P(k)$ and $P_T(k)$. They are exact power laws with spectral indices $\tilde{n} = n_T = -2/(p-1)$, and³

$$A_S(k_*) = \frac{2}{5\sqrt{\pi}} \left(\frac{H_*}{m_{\text{Pl}}}\right) F\left(\frac{-n_T}{2}\right) \sqrt{\frac{3-n}{1-n}} \quad (47)$$

$$A_T(k_*) = \frac{2}{5\sqrt{\pi}} \left(\frac{H_*}{m_{\text{Pl}}}\right) F\left(\frac{-n_T}{2}\right) \quad (48)$$

and the term coming from the small-argument expansion of the Hankel function

$$F(x) = \frac{1+2x}{1+x} \frac{2^x \Gamma(\frac{1}{2}+x)}{\sqrt{\pi}} \quad (49)$$

$$= 1 + (1 - \gamma - \ln 2)x + \dots \quad (50)$$

$$\simeq 1 - 0.27x + \dots \quad (51)$$

where $\gamma = 0.577\dots$ is Euler's constant. Further, H_* is related to V_* by

$$H_*^2 = \frac{8\pi}{3} \frac{V_*}{1-1/3p} = \frac{8\pi V_*}{3} [1 - n_T/6 + \mathcal{O}(n_T^2)] \quad (52)$$

The consistency equation can be written as

$$T/S = -7 n_T f_{T/S}^{(\text{PLI})}(\Omega_\Lambda, n_T) \quad (53)$$

³Exponential inflation is also analyzed in Ref. [15]; the second-order correction to the power spectra, given in Eq. (52), is missing a factor of $1/\sqrt{1-1/3p}$. This improves significantly the accuracy of the reconstruction of an exponential potential.

where for $k_* = H_0$ the correction factor is well fit by

$$f_{T/S}^{(\text{PLI})}(\Omega_\Lambda, n_T) = 0.97 + 0.58n_T + 0.25\Omega_\Lambda - (1 + 1.1n_T + 0.28n_T^2)\Omega_\Lambda^2 \quad (54)$$

over the range of astrophysical interest: $0.8 \leq n < 1$ and $0 \leq \Omega_\Lambda < 0.8$.

Fitting the scalar + tensor CBR power spectrum to the *COBE* 2-year maps yields the normalization [20]

$$A_S = (2.25 \pm 0.2) \times 10^{-5} \Omega_0^{-0.775-0.04 \ln \Omega_0} [\Omega_0/g(\Omega_0)] \exp[0.76\tilde{n}] \quad (55)$$

valid over the same range of n and Ω_Λ as above. This allows us to calculate the one parameter to be determined, H_*/m_{Pl} , as a function of Ω_Λ and n ; the results are shown in Fig. 3.

3 Discussion

Inflation makes three generic predictions: a flat Universe with nearly scale-invariant spectra of scalar and tensor metric perturbations. The anisotropy of the CBR offers a means of testing all three: the positions of the peaks or damping tail of the CBR anisotropy spectrum can test the spatial flatness of the Universe [21] and measurements of the CBR power spectrum can determine the relative amplitudes of scalar and tensor perturbations and their spectral indices. (In the case of tensor perturbations, unless $T/S > 0.1$, only an upper limit can be obtained [22]; and realistically, n_T is likely to be difficult to measure [14].) The CBR anisotropy probably offers the best means of measuring the scalar and tensor metric perturbations and thereby constraining the properties of the inflaton potential.

The scalar and tensor metric perturbations are both determined by the underlying inflationary potential, and so conversely, knowledge of the metric perturbations can be used to determine the potential and its first few derivatives (k -space reconstruction). To take advantage of this in practice, one must relate first the metric perturbations to CBR observables (ℓ -space reconstruction). However, doing this introduces dependence upon cosmological parameters not associated with inflation (the baryon density, the Hubble constant and a possible cosmological constant). As we have shown here, the most important of these is the cosmological constant.

For almost a decade, the advantages of a cosmological constant for inflationary cosmology have been touted – accommodating measurements of the matter density which fall short of the critical density, lessening the tension between measurements of the age and the Hubble constant, large-scale structure which is in better agreement with that measured by redshift surveys, and better agreement with the baryonic content of clusters [7].

In this paper we have carried out the program of perturbative reconstruction, allowing for the possibility of a cosmological constant. In particular, at lowest order we have derived the dependence upon a cosmological constant of the equations that relate the observables S , T , $(n-1)$, and n_T to the inflationary potential and its first two derivatives. We have done the same at second order, including the additional observable $dn/d \ln k$ and the third derivative of the potential. Likewise, we have also modified the consistency relation to allow

for a cosmological constant. In addition, we have clarified previous notation/conventions and generalized reconstruction to the use of other observables. Now all that is needed is a high-angular resolution map of the CBR sky! With NASA considering three proposals for a satellite mission in 1999 and ESA considering another proposal, that could happen within the next five years or so.

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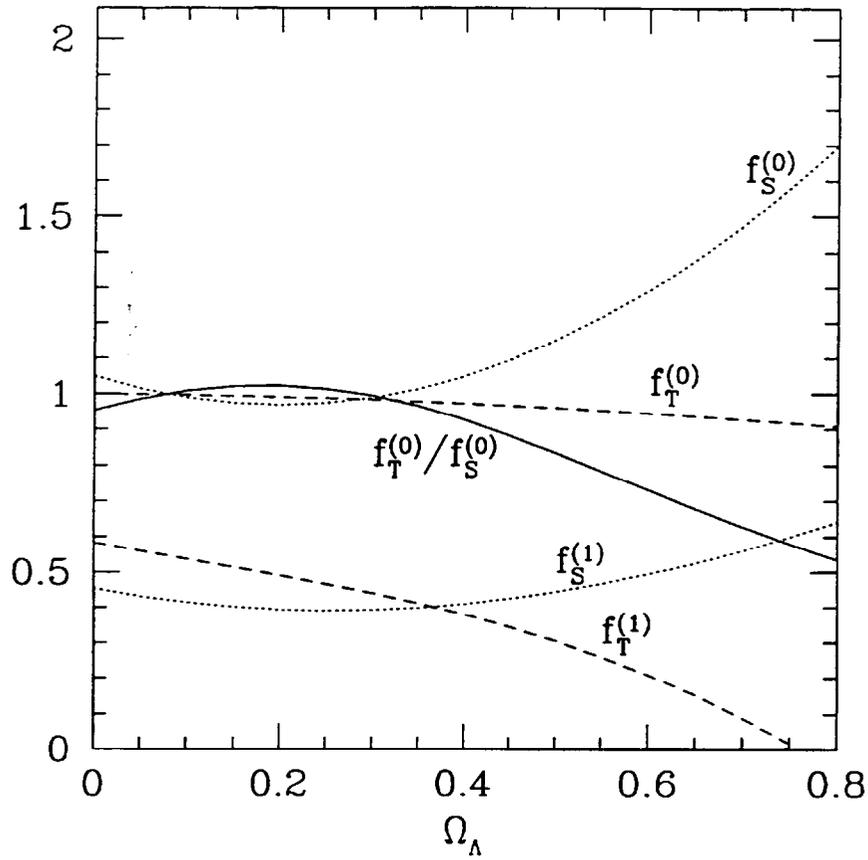


Figure 1: The functions $f_i^{(0)}$ and $f_i^{(1)}$ and the ratio $f_T^{(0)}/f_S^{(0)}$ as a function of Ω_Λ .

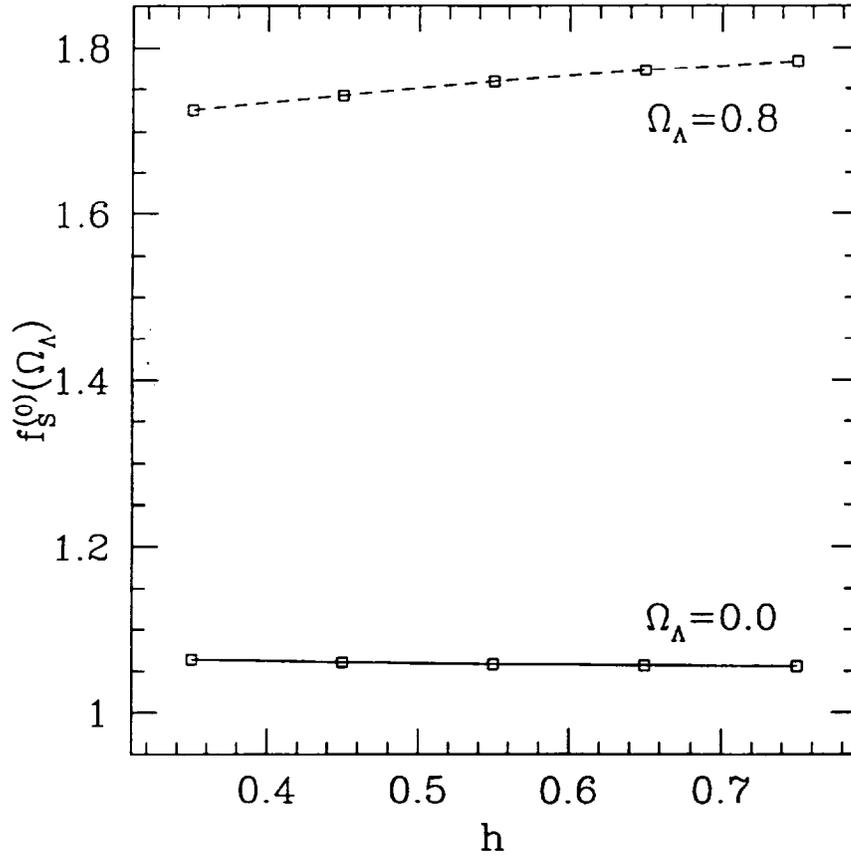


Figure 2: The dependence of $f_S^{(0)}$ on the Hubble constant, for $\Omega_\Lambda = 0$ and 0.8 . The dependence of $f_S^{(0)}$ on h is $\lesssim 4\%$, much less than the Ω_Λ dependence. Both dependences are due mostly to the evolution of the potentials from last-scattering till the present (i.e. the integrated Sachs-Wolfe effect).

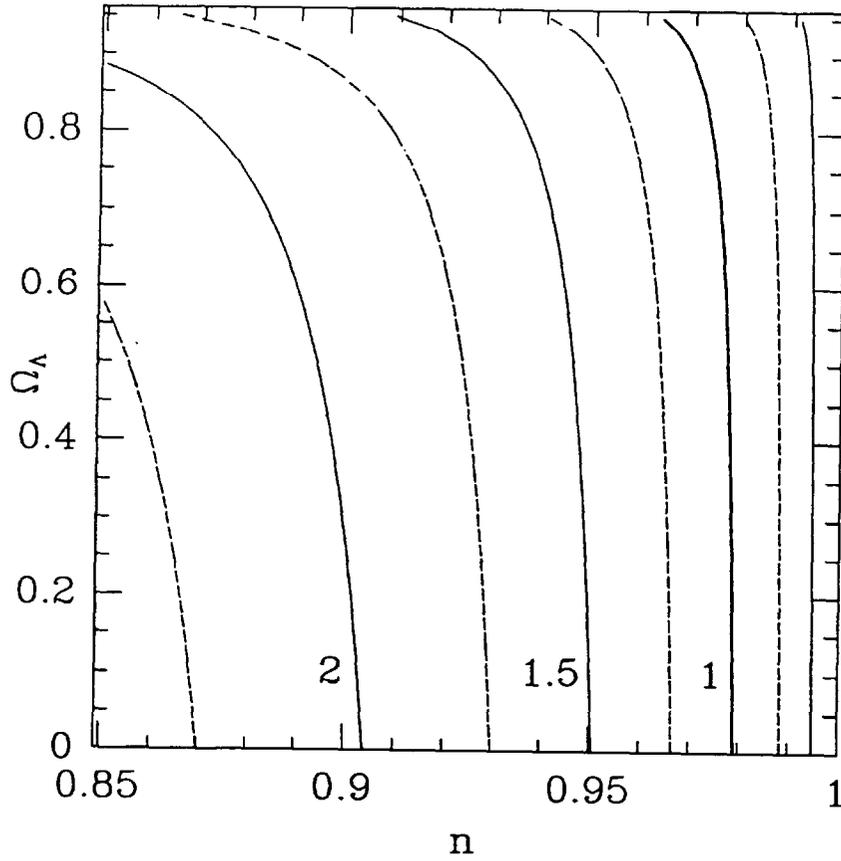


Figure 3: The scale of power-law inflation, quantified by $10^5(H_*/m_{Pl})$, as a function of Ω_Λ and n (H_* is the Hubble constant during inflation when the scale $k_* = H_0$ crossed outside the horizon). As described in Section 2.4, the *COBE* normalization, n and Ω_Λ fix H_*/m_{Pl} .