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Scheme and Scale Dependences of Leading Electroweak Corrections

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Abstract

The scheme and scale dependences of leading M_t-dependent contributions to $\Delta\rho$, Δr, and τ, which arise because of the truncation of the perturbative series, are investigated by comparing expressions in the on-shell and $\overline{\mathrm{MS}}$ schemes of renormalization, and studying their scale variations. Starting from the conventional on-shell formulae, we find rather large scheme and scale dependences. We then propose a simple, physically motivated modification of the conventional expressions and show that it leads to a sharp reduction in the scheme and scale dependences. Implications for electroweak physics are discussed.

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1 Introduction

Recent developments in electroweak radiative corrections have focussed attention on leading effects that occur at two and, sometimes, higher-order loops. For example, the complete $O(\alpha\alpha_s)$ contributions to the electroweak corrections Δr [1] and $\Delta\hat{r}$ [2, 3] have been evaluated [4, 5]. Defining

$$X_t = \left(G_\mu M_t^2/8\pi^2\sqrt{2}\right), \tag{1}$$

where M_t is the top-quark pole mass, several analyses have investigated the M_t-dependent corrections of $O(X_t^2)$ [6, 7], $O(X_t\alpha_s)$ [8], and $O(X_t\alpha_s^2)$ [9] to $\Delta\rho$. There are also estimates of the contributions of $O(X_t\alpha_s^3)$ and higher to this amplitude, based on optimization methods and renormalon calculations [10]. In turn, these leading X_t contributions are simply related to those present in Δr and $\Delta\hat{r}$. In the case of τ, the parameter occurring in the $Z \to b\bar{b}$ amplitude, the M_t-dependent corrections have been carried out up to $O(X_t^2)$ [6, 7] and $O(X_t\alpha_s)$ [11]. Subleading terms of $O(\alpha X_t)$ to $\Delta\rho$ [12] and a subclass of subleading contributions of $O(\alpha\alpha_s^2)$ to Δr associated with the (t,b) isodoublet [13] have also been considered. In these order-of-magnitude characterizations, α should be roughly interpreted as $[\hat{\alpha}(M_Z)/\pi\hat{s}^2(M_Z)] \approx 0.01$ (the carets represent running $\overline{\mathrm{MS}}$ couplings), α_s as $\alpha_s(M_Z)/\pi \approx 0.04$, and $X_t \approx 0.0034$ (corresponding to $M_t \approx 180$ GeV).

In the present paper, we examine the scheme and scale dependences of the leading M_t-dependent corrections, which arise because of the truncation of the perturbative series. Therefore, we focus our attention on the contributions of $O(X_t^2, X_t\alpha_s, X_t\alpha_s^2)$ to $\Delta\rho$ and Δr, and those of $O(X_t^2, X_t\alpha_s)$ to τ. In Section 2, we give the basic expressions of these corrections in the conventional on-shell (OS) scheme, the relation between the Yukawa coupling of the top quark and OS parameters, and the corresponding expressions for $\Delta\rho$, Δr, and τ in the $\overline{\mathrm{MS}}$ formulation. In all applications, we use $M_t = 180$ GeV, the current average value [14].

The scheme dependence is studied in Section 3 by comparing the leading contributions to $\Delta\rho$, Δr, and τ evaluated in the OS and $\overline{\mathrm{MS}}$ schemes of renormalization, at the scale $\mu = M_t$ or, alternatively, at their respective minima. An analogous analysis in the case of $\Delta\rho$ was carried out in the interesting paper by Bochkarev and Willey [15], who reported a scheme dependence which is very sensitive to the Higgs-boson mass, M_H, and becomes large, approximately 2×10^{-4} in $\Delta\rho$, for $M_H \approx 1$ TeV. Starting with the usual OS formulation of $\Delta\rho$, we obtain $\overline{\mathrm{MS}}$ expressions that agree with those of Ref. [15]. However, as explained in Section 3, we apply those expressions in a very different manner and find a scheme dependence that is considerably less sensitive to M_H, but still rather large.

The scale dependence is studied in Section 4, for a fixed value $M_H = 300$ GeV, mainly by examining the scale variation in the $\overline{\mathrm{MS}}$ scheme, which is more sensitive, at current levels of precision, than the OS formulation. Using the same conventional expressions as in Section 2, we find a rather large scale dependence. We also briefly discuss the scale variation when two different parameters, ξ_w and ξ_{QCD}, associated with the electroweak and QCD corrections, are employed.

In Section 5, we examine a simple, physically motivated modification of the conventional formula for $\Delta\rho$ in the OS scheme, as well as in the expression relating the top-quark Yukawa coupling with X_t, and show that it sharply decreases the scale and scheme dependences of $\Delta\rho$ and Δr.

Section 6 presents a summary of the main results of the paper and discusses their implications.

2 Basic Expressions in the Conventional Formulation

The conventional expression for the leading M_t-dependent contributions to $\Delta\rho$ is given, in the OS scheme, by

$$\Delta\rho = 3X_t\left[1 + X_t\rho^{(2)}(r) + \delta_1 a(\mu_{QCD}) + \delta_2(\mu_{QCD})a^2(\mu_{QCD})\right], \tag{2}$$

where X_t is defined in Eq. (1), $a(\mu) = \alpha_s(\mu)/\pi$, and $r = M_H/M_t$. The function $\rho^{(2)}(r)$ is given in closed form in Eq. (12) of Ref. [7], and by useful interpolating formulae in Eq. (34) of Ref. [5]. Representative values of $\rho^{(2)}(r)$ are shown in Table 1. The constant δ_1 is the familiar coefficient [8]

$$\delta_1 = -\frac{2}{3}\left(\frac{\pi^2}{3} + 1\right) = -2.85991, \tag{3}$$

and, to good approximation, we have [9]

$$\delta_2(\mu) = -14.59403 - 5.00485L, \tag{4}$$

where

$$L = \ln(\mu^2/M_t^2), \tag{5}$$

and henceforth all $\overline{\mathrm{MS}}$ running couplings are assumed to evolve with six active flavors.

The relation between the $\overline{\mathrm{MS}}$ Yukawa coupling y_t of the top quark, G_μ, and M_t is given by

$$y_t(\mu_w, \mu_{QCD}) = 2^{3/4}G_\mu^{1/2}M_t\left[1 + X_t\Delta_t^w(\mu_w)/2 + \delta_t^{QCD}(\mu_{QCD})\right], \tag{6}$$

where

$$\Delta_t^w(\mu) = 9L + 11 - \frac{r^2}{2} + r^2(r^2-6)\ln r + \frac{r^2-4}{2}rg(r), \tag{7}$$

$$g(r) = \begin{cases} 2\sqrt{4-r^2}\cos^{-1}\dfrac{r}{2}, & \text{if } 0 \le r \le 2, \\ -2\sqrt{r^2-4}\cosh^{-1}\dfrac{r}{2}, & \text{if } r \ge 2, \end{cases} \tag{8}$$

$$\delta_t^{QCD}(\mu) = \frac{\hat{m}_t(\mu)}{M_t} - 1 = -a(\mu)\left(L + \frac{4}{3}\right) - a^2(\mu)\left(\frac{3}{8}L^2 + \frac{35}{8}L + 9.12545\right), \tag{9}$$

Table 1: Representative values of $\rho^{(2)}(r)$, $\tau^{(2)}(r)$, and $\Delta_t^w(M_t)$ as functions of M_H for $M_t = 180$ GeV.

M_H [GeV]	r	$-\rho^{(2)}(r)$	$\tau^{(2)}(r)$	$\Delta_t^w(M_t)$
60	0.33333	3.69253	3.13525	8.07587
150	0.83333	6.31751	1.68121	5.61002
300	1.66667	8.82919	1.23854	3.71993
450	2.50000	10.23518	1.59534	3.45827
600	3.33333	11.05048	2.21713	4.37506
1000	5.55556	11.76739	4.18305	11.42232

and $\hat{m}_t(\mu)$ is the $\overline{\text{MS}}$ running mass in QCD. It is interesting to observe that $\Delta_t^w(\mu)$, and therefore y_t, has a minimum at $M_H = 2.243 M_t$ (representative values are shown in Table 1). Through terms of $O(X_t\alpha_s)$, Eq. (6) can be gleaned from Ref. [16]. To the same accuracy, an equivalent expression was obtained in Ref. [15]. An independent derivation of Eq. (6), in the framework of Ref. [17], is given in the Appendix, where the terms of $O(X_t\alpha_s^2)$ in Eq. (9) are also obtained. As pointed out in Refs. [15, 16], the tadpole contributions cancel in Eq. (6). Furthermore, as shown in the Appendix, Eq. (6) is gauge independent. Defining $x_t = (y_t^2/32\pi^2)$, Eq. (6) leads to

$$x_t\left(\mu_w, \mu_{QCD}\right) = X_t\left[1 + X_t\Delta_t^w(\mu_w)/2 + \delta_t^{QCD}(\mu_{QCD})\right]^2. \tag{10}$$

Through terms of $O(X_t^2, X_t\alpha_s, X_t\alpha_s^2)$, the expression for $\Delta\rho$ in terms of $\overline{\text{MS}}$ parameters is obtained by writing X_t in terms of x_t via Eq. (10), inserting the result in Eq. (2), and truncating the answer at the corresponding orders. One readily finds

$$\Delta\rho = 3x_t\left\{1 + x_t\left[\rho^{(2)}(r) - \Delta_t^w(\mu_w)\right] + \bar{\delta}_1(\mu_{QCD})a(\mu_{QCD}) + \bar{\delta}_2(\mu_{QCD})a^2(\mu_{QCD})\right\}, \tag{11}$$

where

$$\bar{\delta}_1(\mu) = 2L - 0.19325, \tag{12}$$

$$\bar{\delta}_2(\mu) = \frac{15}{4}L^2 + 6.02533L + 1.36377. \tag{13}$$

In order to facilitate later discussions, in Eqs. (2), (6), (10), and (11), we have employed two distinct scales, μ_w and μ_{QCD}, associated with electroweak and QCD corrections, respectively. Because the QCD correction to the $\overline{\text{MS}}$ parameter involves $\hat{m}_t(\mu)/M_t - 1$ and the top quark does not decouple in the evaluation of $\hat{m}_t(\mu)$,[1] for the purposes of this paper we employ

[1] We are indebted to K.G. Chetyrkin for this observation.

six active quark flavors in the evolution of the $\overline{\text{MS}}$ parameters. It is worth emphasizing, however, that if one is interested in simply applying the OS formula for $\Delta\rho$ [Eq. (2)], it is also consistent and, in fact, simpler to evolve $a(\mu)$ for $\mu < M_t$ with five active flavors [18], in which case the coefficient of L in Eq. (4) must be changed to -5.48150.

In order to evaluate $\alpha_s^{(6)}(\mu)$, we proceed as follows. Using the world average of $\alpha_s^{(5)}(M_Z) = 0.118$, $M_Z = 91.1887$ GeV [14], and a three-loop expression, we find $\alpha_s^{(5)}(M_t) = 0.10703$ for $M_t =$180 GeV, and $\alpha_s^{(6)}(M_t) = \alpha_s^{(5)}(M_t)\left\{1 + (7/24)\left[a^{(5)}(M_t)\right]^2\right\} = 0.10707$, where we apply the matching condition of Ref. [19]. As we have only included terms of $O(X_t\alpha_s, X_t\alpha_s^2)$ in our expressions, for the purposes of this paper we evolve $\alpha_s^{(6)}(\mu)$ with a two-loop formula,

$$\alpha_s^{(6)}(\mu) = \frac{\pi}{\beta_0^{(6)}\ln(\mu^2/\Lambda^2)}\left[1 - \frac{\beta_1^{(6)}\ln\ln(\mu^2/\Lambda^2)}{\left(\beta_0^{(6)}\right)^2\ln(\mu^2/\Lambda^2)}\right], \tag{14}$$

where $\beta_0^{(6)} = 11/4 - n_f/6 = 7/4$ [20] and $\beta_1^{(6)} = 51/8 - 19n_f/24 = 13/8$ [21]. We employ $\Lambda = 91.332$ MeV, adjusted to reproduce $\alpha_s^{(6)}(M_t)$. The evaluation of $\alpha_s^{(6)}(\mu)$ in the τ case is discussed later on.

Recalling the basic relation [1]

$$s^2c^2 = \frac{\pi\alpha}{\sqrt{2}G_\mu M_Z^2(1-\Delta r)}, \tag{15}$$

where $s^2 = 1 - c^2$ is an abbreviation for $\sin^2\theta_W = 1 - M_W^2/M_Z^2$, and that, to leading orders in X_t [22],

$$1 - \Delta r = \left[1 + (c^2/s^2)\Delta\rho\right](1-\Delta\alpha) + \cdots, \tag{16}$$

we have

$$c^2 = (\rho/2)\left[1 + \sqrt{1 - 4\bar{A}/\rho}\right], \tag{17}$$

where $\rho = (1-\Delta\rho)^{-1}$, $\bar{A} = \left(\pi\bar{\alpha}/\sqrt{2}G_\mu M_Z^2\right)$, and $\bar{\alpha} = \alpha/(1-\Delta\alpha)$ is the value of $\alpha(\mu^2)$ at $\mu^2 = M_Z^2$ in the OS scheme. From Eq. (16) we also obtain

$$\Delta r = \Delta\alpha - (c^2/s^2)\Delta\rho(1-\Delta\alpha). \tag{18}$$

Equation (18) is our leading-order expression for Δr in the OS scheme. It is understood that here $\Delta\rho$ is computed from Eq. (2), and c^2 and s^2 from Eq. (17). To obtain the corresponding expression in the $\overline{\text{MS}}$ scheme, we note that, to leading order in X_t,

$$\hat{s}^2(\mu) = s^2\left[1 + (c^2/s^2)\Delta\rho(\mu)\right], \tag{19}$$

an expression that can be inferred from Eq. (18) of Ref. [3]. It follows that $c^2 = \hat{c}^2(\mu)\rho(\mu)$ [in these expressions, $\hat{s}^2(\mu) = 1 - \hat{c}^2(\mu)$ is an abbreviation for the $\overline{\mathrm{MS}}$ parameter $\sin^2\hat{\theta}_W(\mu)$]. We thus obtain $c^2/s^2 = \hat{c}^2/(\hat{s}^2 - \Delta\rho)$ and

$$\Delta r = \Delta\alpha - \frac{\hat{c}^2\Delta\rho}{\hat{s}^2 - \Delta\rho}(1 - \Delta\alpha), \tag{20}$$

which is our leading-X_t expression for Δr the $\overline{\mathrm{MS}}$ scheme. In Eq. (20), it is understood that $\Delta\rho$ is evaluated from Eq. (11) and $\hat{s}^2$ from

$$\hat{s}^2(\mu) = \frac{1}{2}\left[1 - \sqrt{1 - 4\bar{A}/\rho(\mu)}\right]. \tag{21}$$

A few words of clarification are required at this point. In the $\overline{\mathrm{MS}}$ scheme, it is natural to employ the coupling $\hat{\alpha}(M_Z) = \alpha/(1 - \alpha\Delta_\gamma/\pi)$ [5] rather than $\bar{\alpha}(M_Z) = \alpha/(1 - \Delta\alpha)$. This would require the substitution $\Delta\alpha \to \alpha\Delta_\gamma/\pi$ in Eq. (16) and $\bar{\alpha} \to \hat{\alpha}$ in Eq. (21). However, $\alpha\Delta_\gamma/\pi$ and $\Delta\alpha$ differ by sizeable contributions of $O(\alpha)$, namely $\alpha\Delta_\gamma/\pi - \Delta\alpha \approx 7.1 \times 10^{-3}$. In a complete calculation, this $O(\alpha)$ difference is cancelled when the subleading contributions [the dots in Eq. (16)] are taken into account. In a truncated version, in which only leading contributions are considered, it is important to compare OS and $\overline{\mathrm{MS}}$ expressions that coincide at the one-loop level. Therefore, for the purposes of this paper, we employ the same $\Delta\alpha$ correction and $\bar{A}$ parameter in the OS and $\overline{\mathrm{MS}}$ formulations.

For the τ correction in the $Z \to b\bar{b}$ amplitude, we have, in the OS scheme,

$$\tau = -2X_t\left[1 + X_t\tau^{(2)}(r) - (\pi^2/3)a(\mu_{QCD})\right], \tag{22}$$

where $\tau^{(2)}(r)$ is given in closed form in Eq. (20) of Ref. [7]. Representative values are shown in Table 1. In the $\overline{\mathrm{MS}}$ scheme, one obtains

$$\tau = -2x_t\left\{1 + x_t\left[\tau^{(2)}(r) - \Delta_t^w(\mu_w)\right] + \bar{\delta}_\tau(\mu_{QCD})a(\mu_{QCD})\right\}, \tag{23}$$

where

$$\bar{\delta}_\tau(\mu) = 2L - (\pi^2 - 8)/3. \tag{24}$$

In Eqs. (22)–(24), it is understood that the universal correction $1 + \alpha_s/\pi$ has been factored out in the expression for the $Z \to b\bar{b}$ width. Because Eqs. (22) and (23) include only $O(X_t\alpha_s)$ corrections, in their analysis we employ $\alpha_s^{(6)}(\mu)$ evaluated with the one-loop β function [20] [first term of Eq. (14) with $\Lambda = 41.148$ MeV, adjusted to reproduce the value of $\alpha_s^{(6)}(M_t)$].

Our basic expressions to study the scheme and scale dependence of $\Delta\rho$ are Eqs. (2), (10), and (11). In the case of Δr, the basic relations are Eq. (18) (OS) and Eq. (20) ($\overline{\mathrm{MS}}$). It is understood that in Eq. (18) one identifies $\Delta\rho$ with Eq. (2) and c^2 with Eq. (17), while in Eq. (20) one employs Eq. (11) for $\Delta\rho$ and Eq. (21) for $\hat{s}^2(\mu)$. For τ, the relevant expressions are Eqs. (10), (22), and (23). It should be emphasized that, in these OS and $\overline{\rm MS}$ expressions, the corrections $\Delta\rho$, Δr, and τ stand for the same quantities, expressed in two different renormalization frameworks. The scheme dependence arises because these expressions are truncated at $O(X_t^2, X_t\alpha_s, X_t\alpha_s^2)$.

An $\overline{\rm MS}$ formulation of $\Delta\rho$ and τ, in the presence of leading electroweak corrections, has also been proposed in the interesting paper of Ref. [7]. These authors use the simple prescription of dropping the $1/\epsilon$ singularities in the bare expression $\rho_B^{(2)}$ and refer to the resulting renormalized parameter as the $\overline{\rm MS}$ mass of the top quark. We note that, although simple and interesting in its own right, this procedure does not correspond, in the case at hand, to the usual $\overline{\rm MS}$ renormalization. The point is that the $\overline{\rm MS}$ mass counterterm contains $1/\epsilon$ singularities that not only cancel the corresponding ones in $\rho_B^{(2)}$, but also generate finite terms when combined with the $O(\epsilon)$ parts of the one-loop expression. Contributions of this nature were included in Ref. [7] in the derivation of the OS formulae, but not in the transition to the $\overline{\rm MS}$ formulation. A more serious difficulty is that the prescription of Ref. [7] cannot be easily generalized in a gauge-independent manner to the full theory. This can be understood as follows: From the relation $M_t - \delta M_t = \hat{m}_t - \delta\hat{m}_t$, where M_t and δM_t are the pole mass[2] and the associated counter term, and $\hat{m}_t$ and $\delta\hat{m}_t$ the corresponding $\overline{\rm MS}$ quantities, one sees that a gauge-independent definition of $\hat{m}_t$ requires that δM_t be a gauge independent. To leading order in X_t the conventional top-quark self-energy Σ is gauge independent, so that there is no immediate problem (we ignore here subtleties associated with the top-quark instability). However, this is not so in the full theory, where the tadpole contribution must be included to obtain a gauge-independent expression for δM_t. A potential difficulty is that tadpole diagrams not only contain gauge-dependent pieces necessary to cancel corresponding ones in Σ, but also large gauge-independent contributions proportional to $(M_t^4/m_Z^2 m_H^2)\ln(M_t/m_Z)$ [16, 17].

As pointed out by Bochkarev and Willey [15], these potential problems are neatly circumvented by employing the $\overline{\rm MS}$ Yukawa coupling of the top quark, in which case the tadpole diagrams cancel due to the combination of the top-quark and W-boson self-energies. This is, in fact, the strategy followed in the present paper.

3 Scheme Dependence in the Conventional Formulation

We study the scheme dependence by comparing the OS and $\overline{\rm MS}$ expressions of $\Delta\rho$, Δr, and τ for $\mu = \mu_w = \mu_{QCD} = M_t$. Alternatively, one may compare the extrema of these

[2]The frequently employed expression *pole mass* is, rigorously speaking, a misnomer. Because the gluon is massless, M_t represents the start of a branch cut. We are indebted to W. Zimmermann for reminding us of this fact.

corrections, regarded as functions of μ. The relevant expressions are given in Section 2. We fix $M_t = 180$ GeV and allow M_H to vary. Using Eq. (10), we determine x_t as a function of M_H. Inserting this value of x_t in Eq. (11) leads to the $\overline{\mathrm{MS}}$ evaluation of $\Delta\rho$, which is then compared with that from the OS formulation of Eq. (2). For Δr, we simply insert the $\overline{\mathrm{MS}}$ value for $\Delta\rho$ in Eqs. (20) and (21), and compare the result with Eq. (18), where $\Delta\rho$ is given by Eq. (2) and c^2 is determined from Eq. (17). For definitiveness, we employ $\Delta\alpha = 0.0595$, obtained from Ref. [23] by appending the two-loop QED correction to the leptonic contribution [5]. This corresponds to $\bar{\alpha}(M_Z)^{-1} = 128.882$. Alternative evaluations are given in Ref. [24]. For τ, we insert the value of x_t in Eq. (23) and compare the result with Eq. (22).

Table 2: Scheme dependence in the conventional formulation. The OS and $\overline{\mathrm{MS}}$ values of $\Delta\rho$, Δr, and $-\tau$ are compared for $\mu_w = \mu_{QCD} = M_t = 180$ GeV. The relevant formulae are given in Section 2. The columns labelled *Diff.* give the differences between the respective OS and $\overline{\mathrm{MS}}$ values.

M_H	$10^3\Delta\rho$			$10^2\Delta r$			$-10^3\tau$		
[GeV]	OS	$\overline{\mathrm{MS}}$	Diff.	OS	$\overline{\mathrm{MS}}$	Diff.	OS	$\overline{\mathrm{MS}}$	Diff.
60	8.865	8.924	−0.060	3.0163	2.9954	0.021	6.082	6.111	−0.030
300	8.688	8.771	−0.083	3.0780	3.0491	0.029	6.038	6.068	−0.030
600	8.612	8.709	−0.098	3.1046	3.0705	0.034	6.061	6.088	−0.028
1000	8.587	8.688	−0.101	3.1132	3.0779	0.035	6.106	6.134	−0.029

The results for $\mu_w = \mu_{QCD} = M_t$ are given in Table 2. We see that the $\overline{\mathrm{MS}}$ evaluation of $\Delta\rho$ is larger than its OS counterpart by 6×10^{-5} for $M_H = 60$ GeV, by 8×10^{-5} for $M_H = 300$ GeV, and by 1.0×10^{-4} for $M_H = 1$ TeV. Through terms of $O(X_t^2, X_t\alpha_s)$, Eqs. (2), (10), and (11) are consistent with those of Ref. [15], a welcome check. However, the scheme dependence of $\Delta\rho$ obtained in that work shows a sharp dependence on M_H, reaches about 2×10^{-4} for $M_H = 1$ TeV, and is generally very different from that in Table 2; for instance, for large M_H it has the opposite sign. The reason is easy to understand: In the evaluation of x_t from X_t [Eq. (10)] carried out in Ref. [15], M_H is fixed to be 300 GeV, while in Eq. (11) it is allowed to vary. This procedure breaks the equivalence of the OS and $\overline{\mathrm{MS}}$ formulations to $O(X_t^2)$, *i.e.*, at the order of validity of the expansions, and induces a large but artificial M_H dependence.

It is instructive to estimate the order of magnitude of the higher-order terms that may possibly affect the scheme-dependence. Since in Eq. (2) for $\mu = M_t$ the order of magnitude of the QCD corrections is approximately 12%, $X_t \approx 3.4 \times 10^{-3}$, and, for a significant range of M_H, $\left|\rho^{(2)}\right| \approx 10$, we may reasonably expect $O(X_t^2\alpha_s)$ contributions in $\Delta\rho$ of magnitude $10^{-2} \times (3.4 \times 10^{-3}) \times 10 \times 0.12 \approx 4 \times 10^{-5}$. If these corrections were to affect the OS and $\overline{\text{MS}}$ calculation differently, one would expect similar scheme-dependent effects. In fact, Table 2 shows a scheme dependence of that magnitude, albeit larger by roughly a factor of 2–2.5 for $M_H \geq 300$ GeV. The corresponding effects for Δr in Table 2 are 2.1×10^{-4} for $M_H = 60$ GeV, 2.9×10^{-4} for $M_H = 300$ GeV, and 3.5×10^{-4} for $M_H = 1$ TeV, *i.e.*, a factor of about 3.5 larger than in $\Delta\rho$, as roughly expected from the c^2/s^2 enhancement factor in Eq. (18). By comparison, the error in Δr induced by the present uncertainty in $\Delta\alpha$ is about 7×10^{-4} [23]. We would characterize the scheme dependence for $\Delta\rho$ and Δr shown in Table 2 as rather large. In fact, recalling that it arises from three-loop contributions and beyond $[O(X_t^2\alpha_s, X_t^3)]$, its magnitude is, at first hand, surprising. On the other hand, in the case of τ, we see a small, nearly M_H-independent effect of about -3×10^{-5}.

Another way to study the scheme dependence is to compare the OS and $\overline{\text{MS}}$ evaluations, not at a common scale $\mu = M_t$, but at their respective minima. For example, as shown in Section 4, for $M_H = 300$ GeV the OS and $\overline{\text{MS}}$ evaluations of $\Delta\rho$ take the minimum values 8.642×10^{-3} at $\mu = 0.204\,M_t$ and 8.708×10^{-3} at $\mu = 0.520\,M_t$, respectively. This shows a difference of -6.5×10^{-5}, which is similar, albeit somewhat smaller than the effect in Table 2.

4 Scale Dependence in the Conventional Formulation

In order to study the scale variation, we use the expressions of Section 2 to evaluate $\Delta\rho$, Δr, and τ as functions of $\mu = \mu_w = \mu_{QCD} = \xi M_t$, over a wide range of scales. For definitiveness, the value $M_H = 300$ GeV is used. The results are shown in Table 3 for the OS and $\overline{\text{MS}}$ frameworks. The last three rows indicate the maximum variations over the intervals $0.5 \leq \xi \leq 2$, $0.25 \leq \xi \leq 4$, and $0.125 \leq \xi \leq 8$. The scale variation of $\Delta\rho$ in the OS and $\overline{\text{MS}}$ formulations of Section 2 is also displayed in Figs. 1, 2, and 3 for $M_H = 300$, 60, and 1000 GeV, respectively (in all cases, we employ $M_t = 180$ GeV).

As expected, the $\overline{\text{MS}}$ evaluations are more sensitive than their OS counterparts, although this effect is not pronounced in the τ case. The scale variations shown in Table 3 are rather large. For instance, over the relatively conservative interval $0.5 \leq \xi \leq 2$, we have an $\overline{\text{MS}}$ scale variation of 7.8×10^{-4} in Δr, which is slightly larger than the current uncertainty in $\Delta\alpha$ [23], and a scale variation of 2.2×10^{-4} in $\Delta\rho$, which is comparable to the $O(X_t^2)$ contribution to Eq. (2) (the latter amounts to 3.0×10^{-4}). In other words, over this interval, the scale variation due to neglected three-loop effects is about 70% as large as the leading two-loop electroweak correction. If one considers the larger intervals $0.25 \leq \xi \leq 4$ and $0.125 \leq \xi \leq 8$, the variations become very large and encompass values of $\Delta\rho$, Δr, and τ very different from the accepted ones.

The optimization points of $\Delta\rho$ in the OS formulation cluster around the lower end of the table:

1. The BLM [25] scale is $\xi = 0.154$, where $\Delta\rho = 8.645 \times 10^{-3}$;

appear to admit a simple interpretation because leading neglected corrections include mixed contributions of $O(X_t^2\alpha_s)$.

5 Proposed Modification

We have seen that the expressions of Section 2, based on the conventional OS formulation of $\Delta\rho$ [Eq. (2)], lead to rather large scheme and scale dependences. In this section, we propose a simple modification of Eq. (2), *i.e.*, the conventional OS expression for $\Delta\rho$, and Eq. (10), *i.e.*, the relation between the Yukawa coupling of the top quark and the OS parameters, and show that it leads to a sharply smaller scale variation, as well as a much reduced scheme dependence. The modified expressions agree with the previous ones at the $O(X_t^2, X_t\alpha_s, X_t\alpha_s^2)$ level, where complete calculations have been carried out. The physical motivation is the observation that, to $O(X_t)$, large QCD corrections, of order 12%, occur at $\mu_{QCD} = M_t$ if the pole mass M_t of the top quark is involved, as in Eqs. (2) and (10), while they are very small if the $\overline{\mathrm{MS}}$ Yukawa coupling is the relevant expansion parameter, as in Eq. (11). As a working hypothesis, supported by observations based on optimization and renormalon considerations [10], we assume that similarly large effects are induced by the presence of M_t also at the $O(X_t^2)$ level. Defining

$$\tilde{X}_t = X_t\left[1 + \delta_1 a(\mu_{QCD}) + \delta_2(\mu_{QCD})a^2(\mu_{QCD})\right], \tag{25}$$

this suggests the modification

$$\Delta\rho = 3\tilde{X}_t\left[1 + \tilde{X}_t\rho^{(2)}(r)\right], \tag{26}$$

which replaces Eq.(2).

Through term of $O(X_t^2, X_t\alpha_s, X_t\alpha_s^2)$, Eq. (10) can be expressed as

$$x_t(\mu_w, \mu_{QCD}) = X_t\left[1 + X_t\Delta_t^w(\mu_w) + \epsilon_1(\mu_{QCD})a(\mu_{QCD}) + \epsilon_2(\mu_{QCD})a^2(\mu_{QCD})\right], \tag{27}$$

where

$$\epsilon_1(\mu) = -2L - \frac{8}{3}, \tag{28}$$

$$\epsilon_2(\mu) = \frac{L^2}{4} - \frac{73}{12}L - 16.47312. \tag{29}$$

As X_t involves M_t, in analogy with Eq. (26), we modify Eq. (27) to read

$$x_t(\mu_w, \mu_{QCD}) = X_t\left(1 + \epsilon_1 a + \epsilon_2 a^2\right)\left[1 + \Delta_t^w(\mu_w)X_t\left(1 + \epsilon_1 a + \epsilon_2 a^2\right)\right]. \tag{30}$$

As Eq. (11) does not involve M_t, except in L, where it plays the rôle of a unit of mass in the scale definition, the $\overline{\mathrm{MS}}$ expression is not altered. In the case of τ, the OS expression becomes

$$\tau = -2X_t \left[1 - (\pi^2/3)a(\mu_{QCD})\right] \left\{1 + \tau^{(2)}(r) X_t \left[1 - (\pi^2/3)a(\mu_{QCD})\right]\right\}, \tag{31}$$

Eq. (23) does not change, and x_t is given by Eq. (30) with ϵ_2 put to zero. In summary, the proposed modification is to replace Eq. (2) by Eq. (26), Eq. (10) by Eq. (30), and Eq. (22) by Eq. (31). For clarity, we emphasize that these expressions do not include the complete three- and higher-loop effects of $O(X_t^3, X_t^2\alpha_s, \ldots)$, as the latter are currently unknown.

Table 4: Scheme dependence in the modified formulation. The relevant formulae are given in Section 5. Otherwise, the meaning of the table is as in Table 2.

M_H	$10^3\Delta\rho$			$10^2\Delta r$			$-10^3\tau$		
[GeV]	OS	$\overline{\mathrm{MS}}$	Diff.	OS	$\overline{\mathrm{MS}}$	Diff.	OS	$\overline{\mathrm{MS}}$	Diff.
60	8.892	8.874	0.018	3.0067	3.0131	−0.0064	6.067	6.074	−0.008
300	8.754	8.743	0.011	3.0551	3.0590	−0.0039	6.032	6.044	−0.012
600	8.694	8.679	0.015	3.0760	3.0813	−0.0053	6.050	6.062	−0.012
1000	8.674	8.625	0.049	3.0828	3.1000	−0.0172	6.085	6.088	−0.002

The scheme dependence obtained in this new framework is shown in Table 4. Again, the OS and $\overline{\mathrm{MS}}$ evaluations are compared at $\mu_w = \mu_{QCD} = M_t = 180$ GeV. Recalling Table 2, we see that the scheme dependence is much reduced when the proposed modification is employed. For example, in the $\Delta\rho$ case, the difference between the OS and $\overline{\mathrm{MS}}$ evaluations is decreased in magnitude relative to Table 2 by factors of 3.3 for $M_H = 60$ GeV, 7.5 for $M_H = 300$ GeV, 6.5 for $M_H = 600$ GeV, and 2.1 for $M_H = 1$ TeV. Similar improvements are manifest in the Δr and τ columns.

The scale variation in the modified version is shown in Table 5. In the $\Delta\rho$ case, it is also displayed in Figs. 1–3, where it is compared with the conventional approach of Section 2. Again, the OS and $\overline{\mathrm{MS}}$ expressions are evaluated as functions of $\mu_w = \mu_{QCD} = \xi M_t$ for $M_H = 300$ GeV. Comparison with Table 3 shows that the proposed modifications indeed lead to a sharp reduction in the scale dependence of the $\overline{\mathrm{MS}}$ evaluations of $\Delta\rho$ and Δr. For example, the scale variation of Δr is reduced from 7.8×10^{-4} to 1.8×10^{-4} for $0.5 \leq \xi \leq 2$, from 1.5×10^{-3} to 2.4×10^{-4} for $0.25 \leq \xi \leq 4$, and from 2.4×10^{-3} to 3.4×10^{-4} for $0.125 \leq \xi \leq 8$. The scale variation of $\Delta\rho$ over $0.5 \leq \xi \leq 2$ now is 5.2×10^{-5}, which is about 6 times smaller than the leading $O(X_t^2)$ contribution. We also note that this modified framework leads to $\overline{\mathrm{MS}}$ scale variations which are similar and frequently smaller than their OS counterparts, in sharp contrast with the conventional expressions in Section 2.

Table 5: Scale dependence in the modified formulation (Section 5). Otherwise, the meaning of the table is as in Table 3.

ξ	$10^3\Delta\rho$		$10^2\Delta r$		$-10^3\tau$	
	OS	$\overline{\rm MS}$	OS	$\overline{\rm MS}$	OS	$\overline{\rm MS}$
0.125	8.718	8.792	3.0676	3.0418	5.780	5.259
0.25	8.711	8.697	3.0699	3.0750	5.881	5.748
0.5	8.727	8.708	3.0645	3.0709	5.963	5.979
1	8.754	8.743	3.0551	3.0590	6.032	6.044
2	8.786	8.761	3.0438	3.0525	6.091	6.004
4	8.821	8.751	3.0317	3.0561	6.141	5.899
8	8.856	8.709	3.0194	3.0708	6.184	5.753
0.5–2	0.059	0.052	0.021	0.018	0.13	0.065
0.25–4	0.11	0.068	0.038	0.024	0.26	0.30
0.125–8	0.15	0.098	0.062	0.034	0.40	0.79

In the case of τ, we see a significant reduction in the scheme dependence and in the $\overline{\rm MS}$ scale variation in the interval $0.5 \le \xi \le 2$ (from 1.6×10^{-4} to 6.5×10^{-5}). On the other hand, the variations over the wider ranges $0.25 \le \xi \le 4$ and $0.125 \le \xi \le 8$ are similar to those based on Section 2. It is important to emphasize, however, that our analysis of τ is at a considerably lower level of precision than those involving $\Delta\rho$ and Δr. Because the correction of $O(X_t\alpha_s^2)$ has not been evaluated, our study of τ has been restricted to leading-order QCD effects. Under these conditions, the QCD corrections induce a large scale dependence even in the OS framework, and the evaluation of τ becomes much less reliable for values of ξ very different from unity. To check this point, we have repeated our analysis of τ, employing two-loop formulae, in an hypothetical scenario in which the QCD corrections to τ in the OS framework are assumed to be $-(\pi^2/3)a(M_t) - 15a^2(M_t) + \cdots$. In this scenario, the QCD corrections to τ are very similar to those occurring in $\Delta\rho$. We find again the same pattern we encountered in $\Delta\rho$ and Δr. For $M_H = 300$ GeV, the modified three-loop expressions reduce the scheme dependence from -1.4×10^{-5} to -1.4×10^{-6}. The scale dependence in the $\overline{\rm MS}$ scheme decreases from 1.1×10^{-4} to 1.6×10^{-5} for $0.5\le\xi\le2$, from 2.4×10^{-4} to 7.2×10^{-5} for $0.25\le\xi\le4$, and from 5.7×10^{-4} to 2.3×10^{-4} for $0.125\le\xi\le8$.

Equal-level curves for the $\overline{\rm MS}$ evaluation of $\Delta\rho$, based on the modified framework [Eqs. (11) and (30)] in the $(\log_2\xi_w, \log_2\xi_{QCD})$ plane are shown in Fig. 5 for $M_t = 180$ GeV and $M_H = 300$ GeV. The crosses correspond to a maximum at $(\xi_w, \xi_{QCD}) = (1.199, 2.387)$, where $\Delta\rho = 8.780\times10^{-3}$, and a saddle point at $(\xi_w, \xi_{QCD}) = (0.304, 0.300)$, where $\Delta\rho = 8.694\times10^{-3}$ [this is a local maximum (minimum) in the ξ_w (ξ_{QCD}) direction]. For $\xi = \xi_w = \xi_{QCD}$, the small circles indicate the locations of a maximum at $\xi = 2.247$, where $\Delta\rho = 8.762\times10^{-3}$, and a minimum at $\xi = 0.298$, where $\Delta\rho = 8.694 \times 10^{-3}$. The latter nearly coincides with the saddle point in the two-dimensional analysis. The figure shows a large, relatively flat plateau enclosing the extrema. This feature and the greater spacing of the equal-level contours relative to Fig. 4 illustrates the reduced scale variation of the modified approach.

In the OS formulation, $\Delta\rho$ has a minimum at $\xi = 0.204$, where $\Delta\rho = 8.710 \times 10^{-3}$. The difference with the minimum in the $\overline{\mathrm{MS}}$ formulation is 1.6×10^{-5}, which is larger than the variation between the OS and $\overline{\mathrm{MS}}$ results at $\mu_w = \mu_{QCD} = M_t$, but still very small.

6 Conclusions

Starting from the conventional on-shell expression for the leading M_t-dependent contributions to $\Delta\rho$ [Eq. (2)], and employing the relation between the Yukawa coupling y_t of the top-quark, G_μ, and M_t [Eq. (10)], we obtained the associated $\overline{\mathrm{MS}}$ formula [Eq. (11)]. The corresponding expressions for Δr and τ are also given in Section 2.

For $M_t = 180$ GeV and 60 GeV $\leq M_H \leq 1$ TeV, the comparison of the OS and $\overline{\mathrm{MS}}$ expressions at $\mu_w = \mu_{QCD} = M_t$ shows differences ranging from -6.0×10^{-5} to -1.0×10^{-4} in $\Delta\rho$ and from 2.1×10^{-4} to 3.5×10^{-4} in Δr (Section 3). Similar, albeit somewhat smaller variations occur if the two expressions are compared at their respective extrema. For Δr, these scheme-dependent variations are somewhat less than half the current uncertainty of approximately 7×10^{-4} induced by the analysis of $\Delta\alpha$ [23]. For $M_H \geq 300$ GeV, they are, however, larger by factors of 2–2.5 than the naïve estimate of the $O(X_t^2 \alpha_s)$ effects, namely $(c^2/s^2) \times (4 \times 10^{-5}) \approx 1.4 \times 10^{-4}$. We have characterized this scheme dependence, which arises from three-loop effects of $O(X_t^2 \alpha_s, X_t^3)$ as rather large.

For $\mu_w = \mu_{QCD} = \xi M_t$, $M_t = 180$ GeV, and $M_H = 300$ GeV, the scale variation of the $\overline{\mathrm{MS}}$ expressions in the $0.5 \leq \xi \leq 2$ interval is about 2.2×10^{-4} for $\Delta\rho$ and 7.8×10^{-4} for Δr. We also have characterized these variations as large: for Δr, the variation is larger than the current uncertainty induced by $\Delta\alpha$; for $\Delta\rho$, it amounts to about 70% of the $O(X_t^2)$ contribution. If one considers larger intervals, $0.25 \leq \xi \leq 4$ and $0.125 \leq \xi \leq 8$, the variations in Δr reach 1.5×10^{-3} and 2.4×10^{-3}, respectively, which are much larger than current estimations of the theoretical error. It could be argued that these two intervals are too large and that one should not expect a small scale variation from an expression truncated at low orders. However, the fact that a rather large scale variation occurs already in the conservative range $0.5 \leq \xi \leq 2$ is cause for some concern.

Motivated by the above considerations, we have proposed in Section 5 a simple modification of the conventional expression for $\Delta\rho$ [Eq. (26)] and of the relation between the Yukawa coupling y_t of the top quark, G_μ, and M_t [Eq. (30)]. These modifications are based on a heuristic argument explained in Section 5. Because, for values of ξ very different from unity, the electroweak and QCD corrections contain large logarithmic contributions, it is not a priori clear that the new framework ameliorates the scale and scheme dependences. Fortunately, the detailed numerical analysis in Section 5 shows that this is indeed the case. For instance, as mentioned in Section 5, in the crucial Δr case, the scale variation of the $\overline{\mathrm{MS}}$ expressions is reduced from 7.8×10^{-4} to 1.8×10^{-4} for $0.5 \leq \xi \leq 2$, from 1.5×10^{-3} to 2.4×10^{-4} for $0.25 \leq \xi \leq 4$, and from 2.4×10^{-3} to 3.4×10^{-4} for $0.125 \leq \xi \leq 8$. The scheme dependence of Δr is decreased by factors of 3.3, 7.4, 6.4, and 2.0 for $M_H = (60, 300, 600, 1000)$ GeV, respectively. Analogous improvements occur in $\Delta\rho$. In the τ case, there are significant improvements in the scheme dependence (which is already small in the conventional formulation) and in the $\overline{\mathrm{MS}}$ scale variation over the interval $0.5 \leq \xi \leq 2$. However, the τ analysis is at a different level of precision than the $\Delta\rho$ and Δr studies because the $O(X_t\alpha_s^2)$ corrections have not been evaluated in this case. A more detailed discussion is given in Section 5.

For precise calculations, simplified formulae such as Eqs. (16) and (17) are not sufficient, as there are other important one-loop contributions not included in these *leading* M_t-dependent formulae. On the other hand, the leading M_t-dependent contributions to $\Delta\rho$ are integral part of basic corrections such as Δr and $\Delta\hat{r}$. As it has become customary to express $\Delta\rho$ in terms of the pole mass M_t, it is perhaps simplest to focus our attention on the on-shell expressions. In this regard, our proposal amounts to replacing Eq. (2) by the equally simple Eq. (26). The incorporation of QCD corrections in the $\tilde{X}_t^2\rho^{(2)}$ term results in a slight *anti-screening* effect. As a consequence, as is clear from Tables 2–5, the OS $\Delta\rho$ becomes slightly larger and approaches its $\overline{\mathrm{MS}}$ counterpart. In the OS evaluation of $\Delta\rho$, there remains the question of the choice of the QCD scale. A number of authors simply employ $\mu_{QCD} = M_t$, although there is no particularly clear reason for this selection when the pole mass M_t is involved. From Table 5, we see that this determination differs from the minimum value, 8.710×10^{-3}, by 4.4×10^{-5}. In fact, some recent discussions, based on optimization methods and renormalon considerations, favour central values close to the minimum. A specific evaluation of the QCD corrections along these lines, including an estimate of the theoretical error, is given in Ref. [10].

For the purposes of this paper, it is interesting to inquire what effect the proposed modifications have on the electroweak observables. For $M_t = 180$ GeV and $M_H = 300$ GeV, using the results of Sections 4 and 5 in the OS framework, we see that in the new approach $\Delta\rho$ is increased, relative to the conventional framework, by 6.6×10^{-5} at $\mu_{QCD} = M_t$ and by 6.7×10^{-5} if the two evaluations are carried out at their respective minima. For $M_W = 80.32$ GeV, employing Eq. (37a) of Ref. [5], we find that the theoretical prediction of M_W is increased by

$$\delta M_W = \frac{M_W}{2} \frac{c^2}{c^2 - s^2 - 6c^2 X_t} \delta(\Delta\rho) \approx 3.9 \text{ MeV}. \tag{32}$$

Similarly, from Eq. (37b) of Ref. [5] we have

$$\delta\hat{s}^2 = -\frac{\hat{c}^2\hat{s}^2}{\hat{c}^2 - \hat{s}^2} \rho^2 \delta(\Delta\rho) \approx -2 \times 10^{-5}, \tag{33}$$

with an equivalent change in $\sin^2\theta_{eff}^{lept}$. Finally, for the M_t prediction we find the approximate shift $\delta M_t = (M_t/2)\delta(\Delta\rho)/\Delta\rho \approx 0.7$ GeV. These are small changes, but they are of the same order of magnitude as recently evaluated higher-order effects [13]. An interesting feature is the anti-screening character of this modification, which is of opposite sign to most of the higher-order corrections. From the theoretical standpoint, it is reassuring that the simple and easily understandable modifications mentioned above sharply reduce the somewhat worrisome scheme and scale dependences encountered in Sections 3 and 4.

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A Appendix

As mentioned in Section 2, through terms of $O(X_t\alpha_s)$ Eq. (6) can be gleaned from Ref. [16] and an equivalent expression given in Ref. [15]. For completeness, we give an independent derivation based on the framework of Ref. [17], and evaluate the terms of $O(X_t\alpha_s^2)$.

Starting from the relation between bare parameters, $M_t^0 = y_t^0 v^0/\sqrt{2}$, where M_t and y_t are the mass and Yukawa coupling of the top quark, respectively, and v is the vacuum expectation value of the Higgs field, and writing $y_t^0 = (\mu^2 e^\gamma/4\pi)^{\epsilon/2}(y_t - \delta y_t)$, $v^0 = (\mu^2 e^\gamma/4\pi)^{-\epsilon/2}(v - \delta v)$, where $\epsilon = 2 - D/2$ and D is the dimension of space time, we have

$$M_t^0 = y_t v/\sqrt{2} - [\delta y_t(v - \delta v) + y_t\delta v]/\sqrt{2}. \tag{34}$$

We adjust the counterterms in such a manner that $M_t = y_t v/\sqrt{2}$ can be identified with the pole mass of the top quark. Defining $\delta M_t = [\delta y_t(v - \delta v) + y_t\delta v]/\sqrt{2}$, this is implemented by choosing

$$\delta M_t = \Sigma(\not p = M_t) + M_t\tilde{T}/(vM_H^2), \tag{35}$$

where Σ is the top-quark self-energy and $i\tilde{T}$ represents the sum of the tadpole diagrams and the tadpole counterterm. For the sake of generality, the latter is left unspecified. Solving for the δy_t, we find

$$\delta y_t = \left(\sqrt{2}\delta M_t - y_t\delta v\right)/(v - \delta v). \tag{36}$$

In order to relate $v - \delta v$ to G_μ, we recall that

$$\frac{G_\mu}{\sqrt{2}} = \frac{(g-\delta g)^2}{8(m_W^0)^2}(1+\mathcal{D})^{-1}, \tag{37}$$

$$\mathcal{D} = \frac{A_{WW}(0)}{(m_W^0)^2} + \frac{2\tilde{T}}{vM_H^2} - E, \tag{38}$$

where g is the SU(2) coupling, δg is the corresponding counterterm, m_W^0 is the bare W-boson mass, $A_{WW}(0)$ is the W-boson self-energy at $q^2 = 0$, and $-E$ stands for the remaining radiative corrections in the muon-decay amplitude [1, 17]. From the relation $m_W^0 = g^0 v^0/2$, we have $(g-\delta g)^2/(m_W^0)^2 = 4/(v-\delta v)^2$, and Eq. (37) becomes

$$v - \delta v = \left(\sqrt{2}G_\mu\right)^{-1/2}(1+\mathcal{D})^{-1}. \tag{39}$$

Similarly, identifying $M_W = gv/2$ with the W-boson mass, and recalling $G_\mu/\sqrt{2} = (g^2/8M_W^2)(1-\Delta r)^{-1}$ [1], we have

$$v = \left(\sqrt{2}G_\mu\right)^{-1/2}(1-\Delta r)^{-1/2}, \tag{40}$$

$$\delta v = \left(\sqrt{2}G_\mu\right)^{-1/2}\left[(1-\Delta r)^{-1/2} - (1+\mathcal{D})^{-1/2}\right]. \tag{41}$$

Inserting Eqs. (39) and (41) into Eq. (36), and using $y_t = \sqrt{2}M_t/v = \sqrt{2}M_t\left(\sqrt{2}G_\mu\right)^{1/2}(1-\Delta r)^{1/2}$, we find

$$\delta y_t = 2^{3/4}G_\mu^{1/2}M_t\left[(1-\Delta r)^{1/2} - (1+\mathcal{D})^{1/2}(1-\delta M_t/M_t)\right]. \tag{42}$$

The OS and $\overline{\text{MS}}$ expressions for y_t are related by $y_t - \delta y_t = \hat{y}_t - \delta\hat{y}_t$, where the carets denote $\overline{\text{MS}}$ quantities. Thus, $\hat{y}_t = y_t - (\delta y_t)_{\overline{MS}}$, where the $\overline{\text{MS}}$ subscript means that the $\overline{\text{MS}}$ renormalization has been carried out. Noting that the first term in Eq. (42) equals y_t, we have

$$\hat{y}_t = 2^{3/4}G_\mu^{1/2}M_t\left[(1+\mathcal{D})^{1/2}(1-\delta M_t/M_t)\right]_{\overline{MS}}. \tag{43}$$

Neglecting two-loop effects in the square brackets, their expression simplifies to $(1+\mathcal{D}/2-\delta M_t/M_t)_{\overline{MS}}$. Recalling Eqs. (35) and (38), we have

$$\hat{y}_t = 2^{3/4}G_\mu^{1/2}M_t\left[1 + \frac{A_{WW}(0)}{2M_W^2} - \frac{E}{2} - \frac{\Sigma(\not{p} = M_t)}{M_t}\right]_{\overline{MS}}. \tag{44}$$

It is important to note that the tadpole contributions have cancelled in Eq. (44)! The gauge independence can be checked by adding and subtracting $T/(vM_H^2)$, where iT stands for the sum of the tadpole diagrams, with the counterterm excluded. We note that $\Sigma(\not p = M_t)/M_t + T/(vM_H^2)$ is gauge independent, and so is $A_{WW}(0)/(2M_W^2) + T/(vM_H^2) - E/2$, as it represents (modulo a factor 1/2) the full one-loop contribution to muon decay, before the effect of the counterterms is taken into account. It is, therefore, sufficient to evaluate the separate parts of Eq. (44) in some particular gauge. For simplicity, we choose the 't Hooft–Feynman gauge. The term involving E does not contain M_t-dependent contributions, while

$$\left[\frac{A_{WW}(0)}{2M_W^2}\right]_{\overline{MS}} = X_t\left(3L + \frac{3}{2} + \frac{r^2}{4}\right) + \cdots, \tag{45}$$

where $r = M_H/M_t$, L is defined in Eq. (5), and the dots represent M_t-independent terms. The contribution from $-\Sigma(\not p = M_t)/M_t$ proportional to X_t is given in Eq. (35) of Ref. [7]. Combining these terms with Eq. (45), one obtains the electroweak term $X_t\Delta_t^w(\mu_w)/2$ in Eq. (6).

The QCD corrections are contained in Σ. Writing $M_t - \delta M_t^{QCD} = \hat{m}_t - \delta\hat{m}_t^{QCD}$ and recalling $\delta M_t^{QCD} = \Sigma^{QCD}(\not p = M_t, \mu)$, we have $-\left[\Sigma^{QCD}(\not p = M_t, \mu)\right]_{\overline{MS}}/M_t = \hat{m}_t(\mu)/M_t - 1$, where $\hat{m}_t(\mu)$ is the $\overline{\mathrm{MS}}$ running mass in QCD. To evaluate $\hat{m}_t(\mu)/M_t$, we consider $\hat{m}_t(\mu)/M_t = [\hat{m}_t(\mu)/\hat{m}_t(M_t)]\,[\hat{m}_t(M_t)/M_t]$. The first factor may be obtained from the QCD renormalization-group equation for $\hat{m}_t(\mu)$. Through terms of $O(\alpha_s^2)$, we have

$$\frac{\hat{m}_t(\mu)}{\hat{m}_t(M_t)} = 1 - a(M_t)\gamma_0 L + a^2(M_t)\left[\frac{\gamma_0}{2}(\beta_0 + \gamma_0)L^2 - \gamma_1 L\right], \tag{46}$$

where $\gamma_0 = 1$, $\gamma_1 = 101/24 - 5n_f/36$ [28], and β_0 is given below Eq. (14). For $n_f = 6$, the coefficient of $a^2(M_t)$ becomes $11L^2/8 - 27L/8$. Combining Eq. (46) with the expansion [29]

$$\frac{\hat{m}_t(M_t)}{M_t} = 1 - \frac{4}{3}a(M_t) - a^2(M_t)\left(10.90323 - \frac{16}{9}\right), \tag{47}$$

we obtain

$$\frac{\hat{m}_t(\mu)}{M_t} = 1 - a(M_t)\left(L + \frac{4}{3}\right) + a^2(M_t)\left(\frac{11}{8}L^2 - \frac{49}{24}L - 9.12545\right). \tag{48}$$

Expressing $a(M_t)$ in terms of $a(\mu)$, Eq. (48) leads to the $\delta_t^{QCD}(\mu)$ contribution in Eq. (6).

The asymptotic behaviours of $\Delta_w^t(M_t)$, $\rho^{(2)}(r)$, and $\tau^{(2)}(r)$ are given by [6, 16]

$$\Delta_w^t(M_t) = \begin{cases} 11 - 4\pi r + O(r^2\ln r), & \text{if } r \ll 1, \\ \dfrac{r^2}{2} - 6\ln r + \dfrac{13}{2} + O\left(\dfrac{\ln r}{r^2}\right), & \text{if } r \gg 1, \end{cases} \tag{49}$$

$$\rho^{(2)}(r) = \begin{cases} -12\zeta(2) + 19 - 4\pi r + O(r^2\ln r), & \text{if } r \ll 1, \\ 6\ln^2 r - 27\ln r + 6\zeta(2) + \dfrac{49}{4} + O\left(\dfrac{\ln^2 r}{r^2}\right), & \text{if } r \gg 1, \end{cases} \tag{50}$$

$$\tau^{(2)}(r) = \begin{cases} -2\zeta(2) + 9 - 4\pi r + O(r^2 \ln r), & \text{if } r \ll 1, \\ \frac{5}{2}\ln^2 r - \frac{47}{12}\ln r + \zeta(2) + \frac{311}{144} + O\left(\frac{\ln^2 r}{r^2}\right), & \text{if } r \gg 1. \end{cases} \tag{51}$$

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FIGURE CAPTIONS

Figure 1: $\Delta\rho$ evaluated in the OS and $\overline{\text{MS}}$ schemes with $\mu_w = \mu_{QCD} = \xi M_t$ as a function of ξ assuming $M_t = 180$ GeV and $M_H = 300$ GeV. The analysis is carried out in the conventional (Section 2) and modified (Section 5) formulations.

Figure 2: Same as Fig. 1, but for $M_H = 60$ GeV.

Figure 3: Same as Fig. 1, but for $M_H = 1$ TeV.

Figure 4: Scale dependence of the conventional $\overline{\text{MS}}$ evaluation of $\Delta\rho$ (Section 2) with $\mu_w = \xi_w M_t$ and $\mu_{QCD} = \xi_{QCD} M_t$ assuming $M_t = 180$ GeV and $M_H = 300$ GeV. The contours of constant $\Delta\rho$ are shown in the $(\log_2 \xi_w, \log_2 \xi_{QCD})$ plane. The maximum and saddle point in this plane are marked with "x;" the maximum and minimum on the diagonal $\xi_w = \xi_{QCD}$ are marked with "o."

Figure 5: Same as Fig. 2, but for the modified formulation (Section 5).

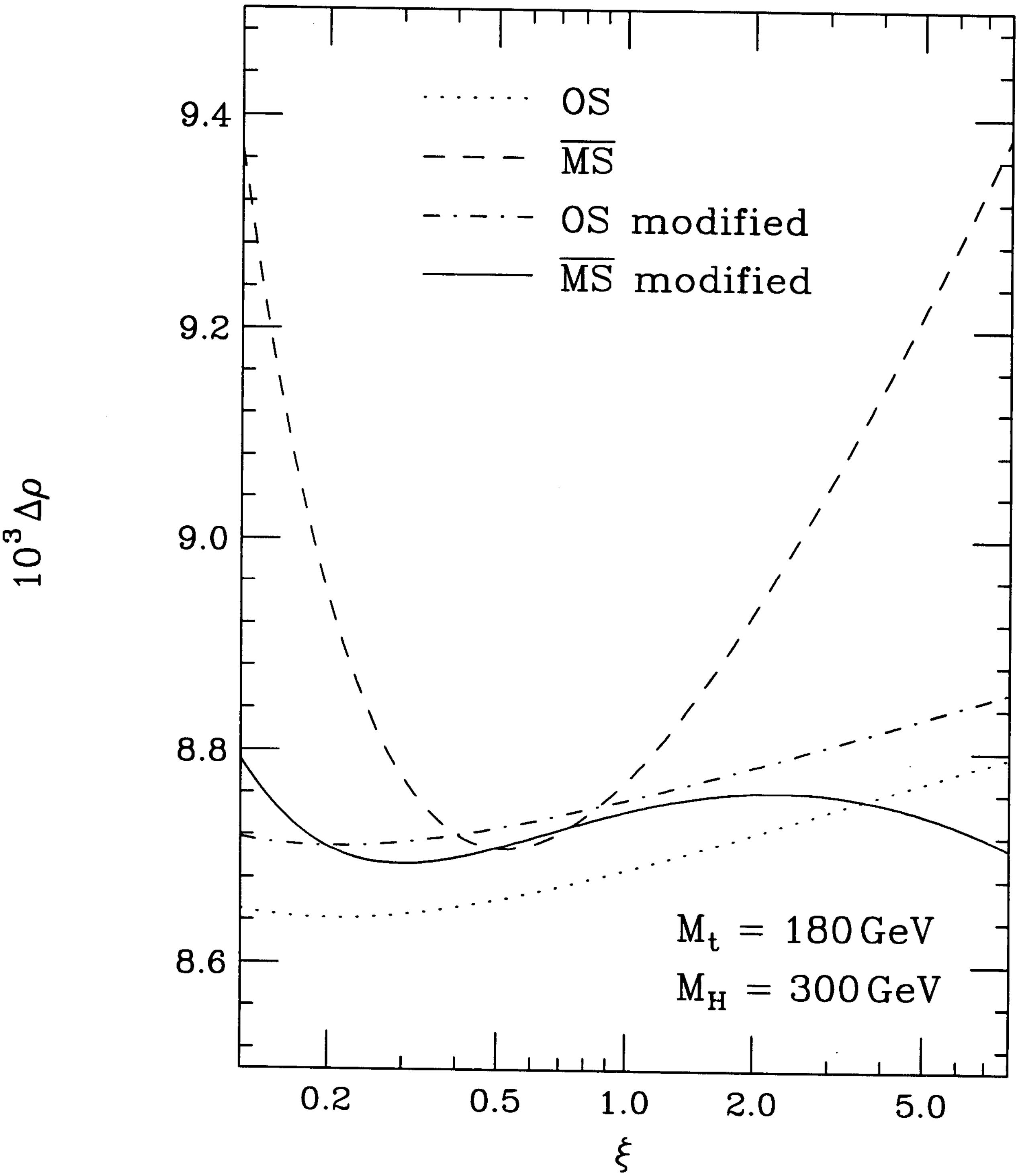

OS
$\overline{\mathrm{MS}}$
OS modified
$\overline{\mathrm{MS}}$ modified
$M_t = 180\,\mathrm{GeV}$
$M_H = 300\,\mathrm{GeV}$
9.4
9.2
9.0
8.8
8.6
$10^3\,\Delta\rho$
0.2
0.5
1.0
2.0
5.0
ξ

Fig. 1

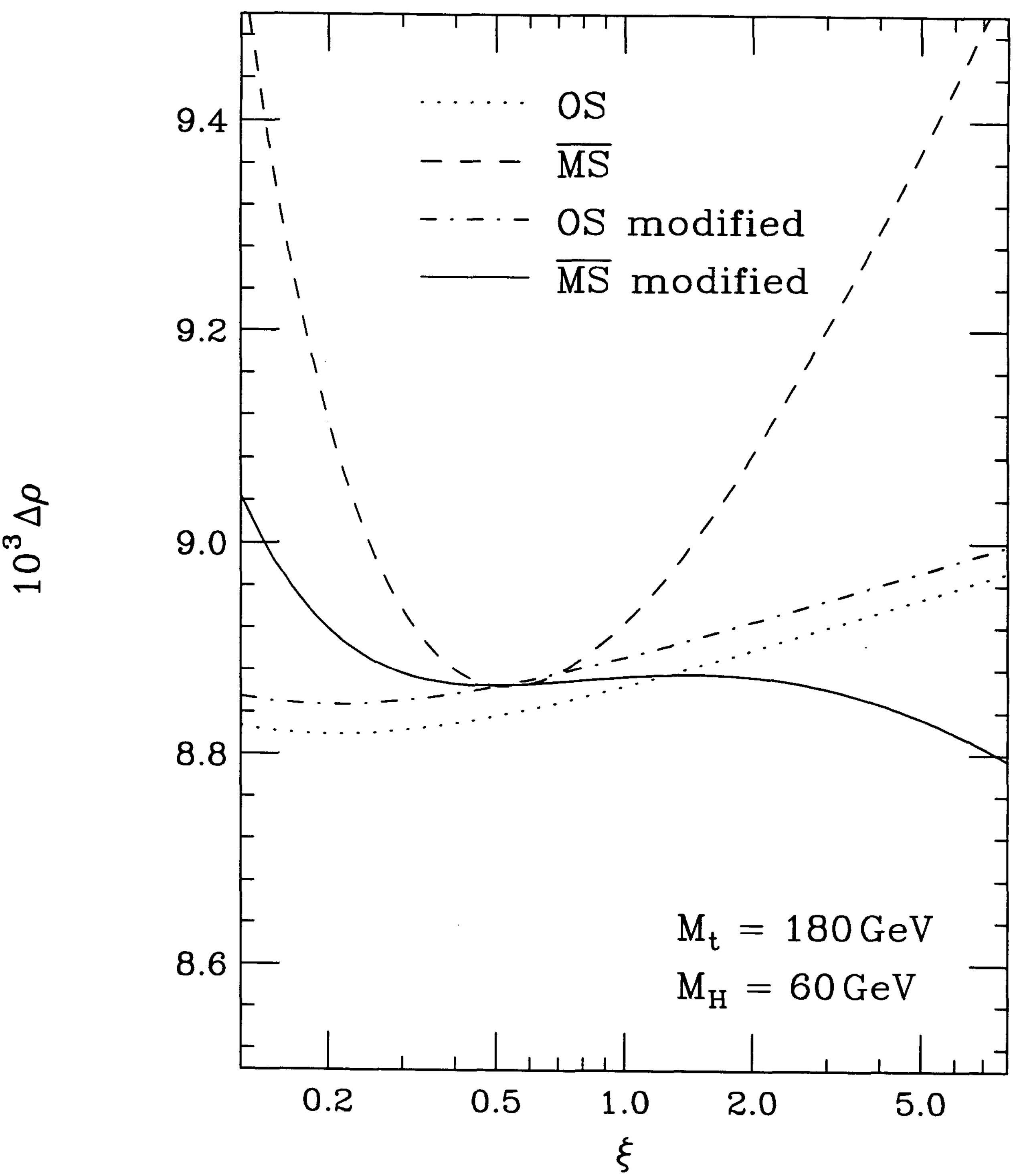

OS
$\overline{\mathrm{MS}}$
OS modified
$\overline{\mathrm{MS}}$ modified
$M_t = 180\,\mathrm{GeV}$
$M_H = 60\,\mathrm{GeV}$
9.4
9.2
9.0
8.8
8.6
$10^3\Delta\rho$
0.2
0.5
1.0
2.0
5.0
ξ

Fig. 2

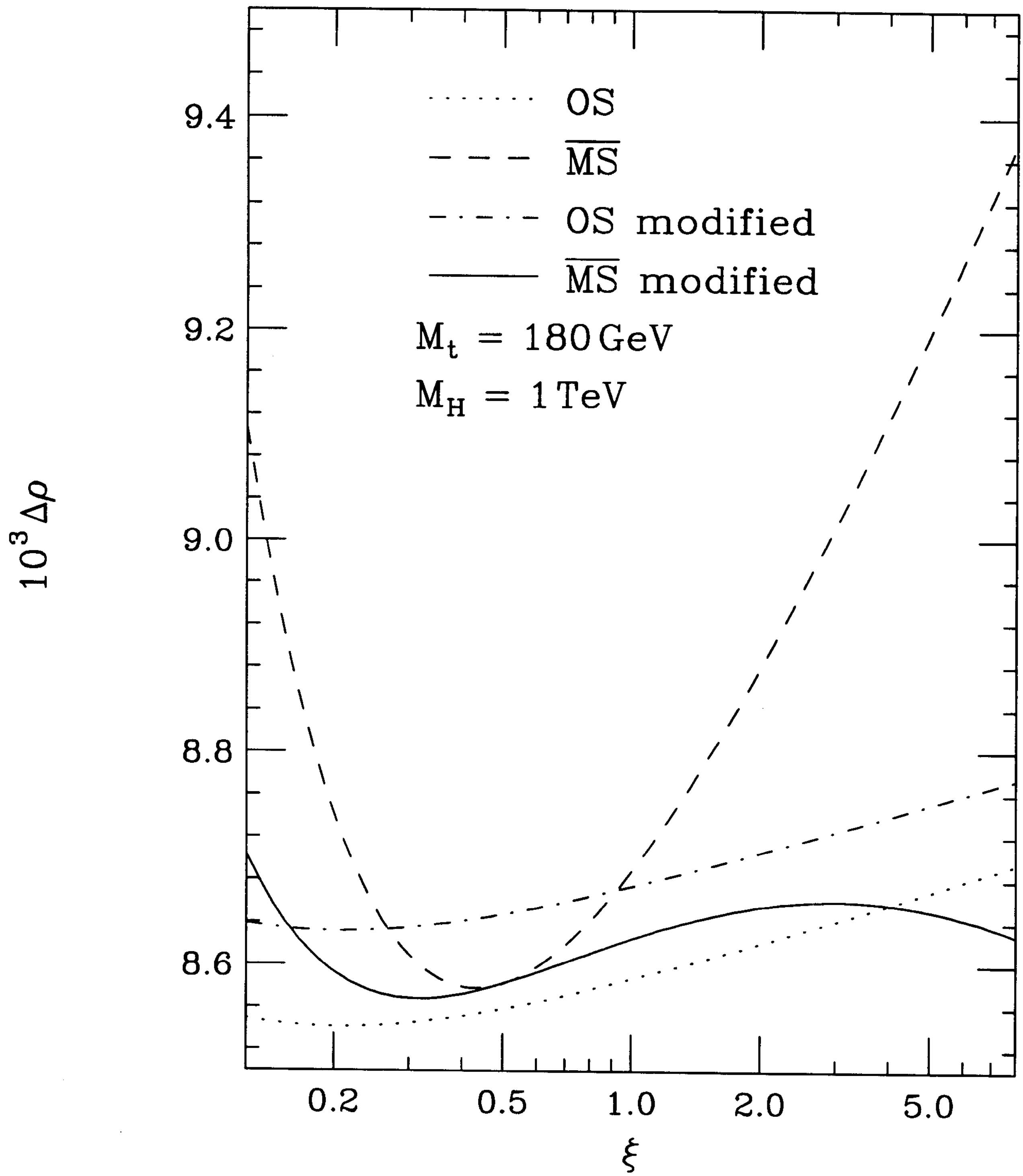
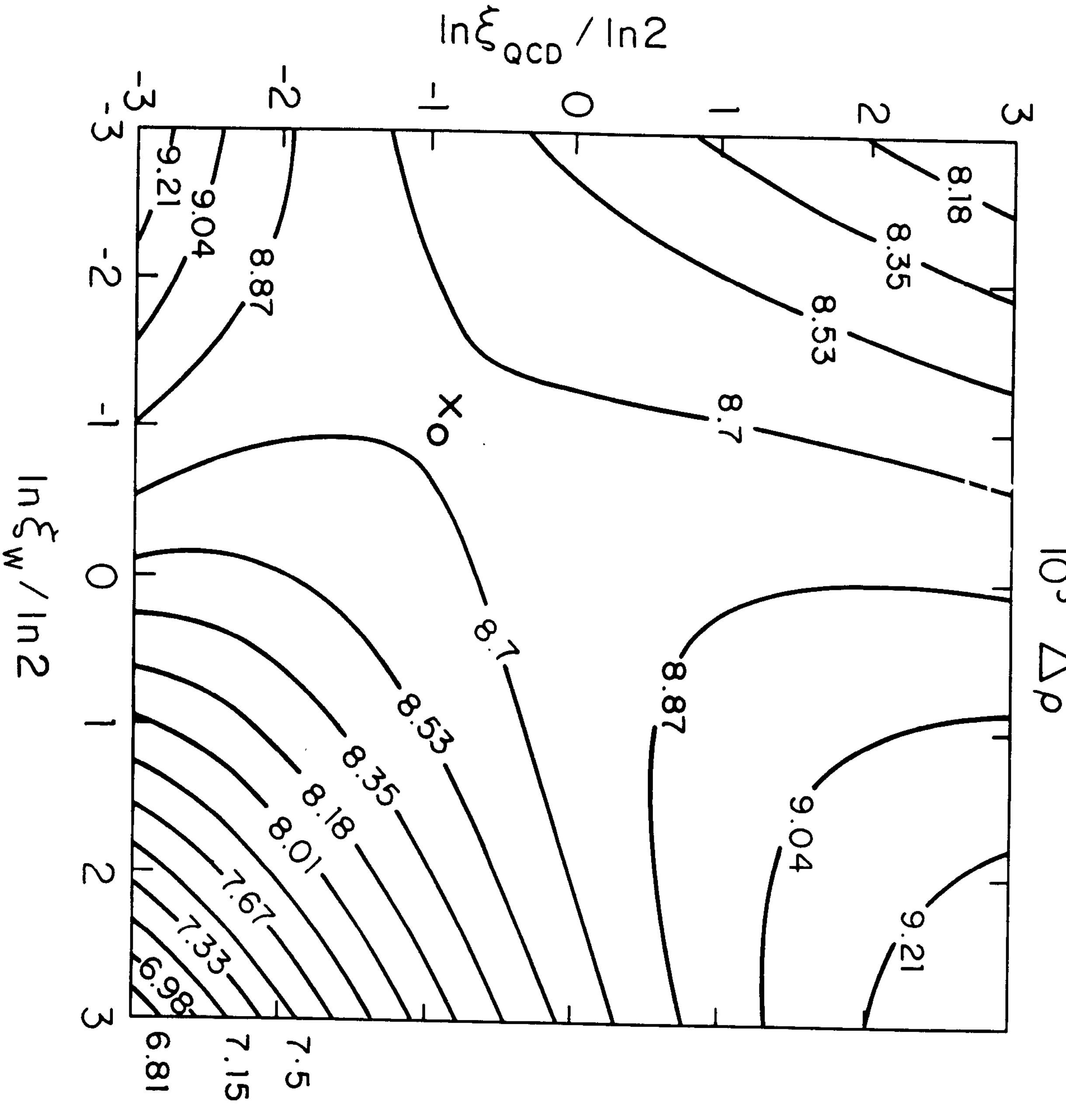
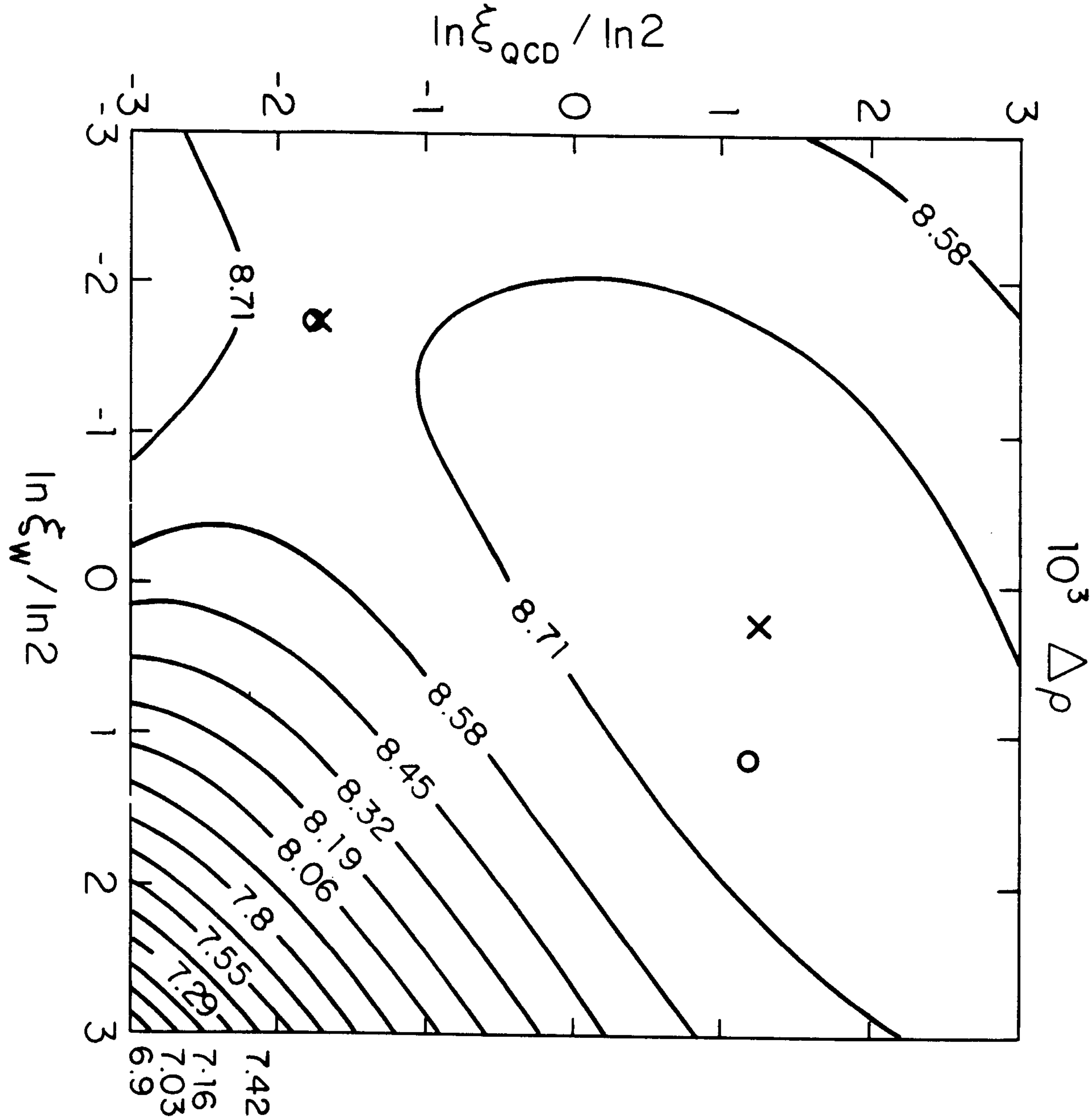

OS
$\overline{\mathrm{MS}}$
OS modified
$\overline{\mathrm{MS}}$ modified
$M_t = 180\,\mathrm{GeV}$
$M_H = 1\,\mathrm{TeV}$
9.4
9.2
9.0
8.8
8.6
$10^3 \Delta\rho$
0.2
0.5
1.0
2.0
5.0
ξ

Fig. 3



$\ln\xi_{QCD}/\ln 2$
$\ln\xi_W/\ln 2$
$10^3\ \Delta\rho$
-3
-2
-1
0
1
2
3
9.21
9.04
8.87
8.7
8.53
8.35
8.18
8.01
7.67
7.5
7.33
7.15
6.98
6.81

Fig. 4



ln ξ_QCD / ln2
ln ξ_W / ln2
10³ Δρ
8.58
8.71
8.45
8.32
8.19
8.06
7.8
7.55
7.29
7.42
7.16
7.03
6.9

Fig. 5