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Low-Energy Theorems in Higgs Physics

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Abstract

We present low-energy theorems for the calculation of loop amplitudes with external scalar or pseudoscalar Higgs bosons which are light compared to the loop particles. Starting from existing lowest-order versions of these theorems, we show how their applicability may be extended to the two-loop level. To illustrate the usefulness of these theorems, we discuss a number of applications to Higgs production and decay at and beyond the one-loop order.

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1 Introduction

The search for the scalar Higgs boson of the Standard Model ($\mathcal{SM}$) is one of the most important tasks to be performed at present and future high–energy experiments. The only unknown independent parameter of this particle is its mass, M_H. From the direct search with the CERN Large Electron Positron Collider (LEP1) and the SLAC Linear Collider (SLC) via the process $e^+e^- \to Z \to Z^*H$, a lower limit on M_H of 63.9 GeV has been obtained at the 95% confidence level [1]. There are general theoretical restrictions on the possible range of M_H. Unitarity arguments lead to an upper bound of ~ 700 GeV, if the $\mathcal{SM}$ is weakly interacting up to scale ~ 1 TeV; this value comes down to ~ 200 GeV, if the $\mathcal{SM}$ is assumed to be valid up to the GUT scale $\sim 10^{15}$ GeV [2]. On the other hand, the requirement that the $\mathcal{SM}$ vacuum be stable sets a lower bound on M_H. Assuming the $\mathcal{SM}$ to be valid up to scale $\Lambda \sim 1$ TeV and using $m_t = (176 \pm 13)$ GeV [3] for the top–quark mass, this lower bound amounts to about 55 GeV, whereas for $\Lambda \sim 10^{15}$ GeV it is shifted to ~ 130 GeV [4]. Recently, it has been pointed out that this M_H lower bound is significantly decreased by taking into account the possibility that the physical minimum of the effective $\mathcal{SM}$ potential is metastable [5].

It is attractive to study the minimal supersymmetric extension of the $\mathcal{SM}$ ($\mathcal{MSSM}$). It predicts five physical Higgs bosons: two neutral ($\mathcal{CP}$–even) scalars (h and H), one neutral ($\mathcal{CP}$–odd) pseudoscalar (A), and two charged scalars ($H^\pm$). The mass of the lightest scalar (h) is restricted to be below ~ 140 GeV [6], whereas those of the heavy scalars and the pseudoscalar will be typically of the order of the Fermi scale, $v = 246$ GeV. The direct search at LEP1 has excluded scalar–Higgs–boson masses below ~ 45 GeV and pseudoscalar–Higgs–boson masses below ~ 25 GeV [7].

If the Higgs bosons are lighter than the top quark and the Z and W bosons, the latter may be integrated out. In this way, the original Lagrangians describing the interactions of the Higgs bosons with these heavy particles get replaced by effective Lagrangians. These effective Lagrangians provide useful approximations for the interactions of Higgs bosons in the low and intermediate mass range, below $\sim 2M_Z$, where at least one of the $\mathcal{SM}$ or $\mathcal{MSSM}$ Higgs bosons should be found. The derivation of these effective Lagrangians can be simplified by using low–energy theorems (LETs) appropriate to external Higgs bosons with vanishing momentum. This is the topic of the present article.

This paper is organized as follows. In Section 2, the LETs for scalar and pseudoscalar Higgs bosons that are lighter than the loop particles will be formulated at the lowest order of the perturbative expansion. These will then be generalized to higher orders, appropriate to the application to multi–loop contributions. In Section 3, we shall present applications of the theorems to Higgs–boson production and decay processes within the $\mathcal{SM}$ and the $\mathcal{MSSM}$ at the one–loop level. These examples will then be extended in Section 4 so as to include two–loop corrections. Section 5 will summarize our main results.

2 Low–Energy Theorems

In this section, LETs for any type of neutral Higgs boson, generically denoted ϕ, will be derived in the limit of vanishing four–momentum p_ϕ. In this case, the Higgs boson acts as a constant field, since $[\mathcal{P}_\mu, \phi] = i\partial_\mu \phi = 0$, with $\mathcal{P}_\mu$ being the four–momentum operator. As a consequence, the kinetic terms of the Higgs Lagrangian vanish in this limit.

2.1 Scalar Higgs Bosons

In the $\mathcal{SM}$ and $\mathcal{MSSM}$, the Lagrangian for the interaction of the neutral scalar Higgs boson(s) with the massive fermions and intermediate bosons, having masses m_i $(i = f, V)$, may be generated by the substitution [8–11]

$$m_i \to m_i \left(1 + \sum_\phi g_i^\phi \frac{\phi}{v}\right), \tag{1}$$

where $v = 246$ GeV is the Higgs vacuum expectation value in the $\mathcal{SM}$ and g_i^ϕ are real numbers, which are listed for the neutral scalar Higgs bosons of the $\mathcal{SM}$ and the $\mathcal{MSSM}$ in Table 1. As usual, α is the mixing angle between the original neutral scalar Higgs fields of definite weak hypercharge and the mass eigenstates, h and H, and $\tan\beta$ is the ratio of the vacuum expectation values of the two Higgs doublets in the $\mathcal{MSSM}$. In the $\mathcal{SM}$, the sum in Eq. (1) collapses to one item, with g_i^ϕ being equal to unity.

ϕ		t	b	$V = W, Z$
$\mathcal{SM}$	H	1	1	1
$\mathcal{MSSM}$	h	$\cos\alpha/\sin\beta$	$-\sin\alpha/\cos\beta$	$\sin(\beta-\alpha)$
	H	$\sin\alpha/\sin\beta$	$\cos\alpha/\cos\beta$	$\cos(\beta-\alpha)$

Table 1: *Values of g_i^ϕ in Eq. (1) for the neutral scalar Higgs bosons of the $\mathcal{SM}$ and the $\mathcal{MSSM}$.*

In higher orders of the perturbative expansion, the masses m_i, the Higgs fields ϕ, the couplings g_i^ϕ, and the vacuum expectation value v have to be replaced by their bare counterparts, which we shall label with the superscript 0. This leads to the following LET for neutral scalar Higgs bosons [12–14]:

$$\lim_{p_\phi \to 0} \mathcal{M}(X\phi) = \sum_{i=f,V} \frac{g_i^{\phi 0}}{v^0} \frac{m_i^0 \partial}{\partial m_i^0} \mathcal{M}(X), \tag{2}$$

where the symbol $\mathcal{M}(X)$ denotes the matrix element of any particle configuration X, expressed in terms of bare quantities, and $\mathcal{M}(X\phi)$ is the corresponding one with a neutral scalar Higgs boson ϕ attached as an external particle in all possible ways. The renormalization of the bare quantities is performed after evaluating the right-hand side of Eq. (2). It is important to notice that the differentiation in Eq. (2) only acts on the bare masses appearing in the propagators of the massive particles, while bare mass-dependent couplings must be treated as constants. The reason is that such couplings may be considered as being generated by a substitution similar to Eq. (1), so that further application of Eq. (1) would introduce tree-level vertices between the Higgs bosons and the massive particles which are absent in the $\mathcal{SM}$ and the $\mathcal{MSSM}$.

2.2 Pseudoscalar Higgs Bosons

The pseudoscalar Higgs boson A of the $\mathcal{MSSM}$ does not interact with the gauge bosons at tree level. The Lagrangian for its interaction with the massive fermions reads

$$\begin{aligned}\mathcal{L}_{Aff} &= -\sum_f m_f^0 \bar{f}_L^0 f_R^0 \left(1 + i g_f^{A0} \frac{A^0}{v^0}\right) + \text{h.c.} \\ &= -\sum_f m_f^0 \left(\bar{f}^0 f^0 + i g_f^{A0} \bar{f}^0 \gamma_5 f^0 \frac{A^0}{v^0}\right), \end{aligned} \tag{3}$$

where $g_b^{A0} = 1/g_t^{A0} = \tan\beta$. From Eq. (3) it is obvious that the tree-level interaction of A with the massive fermions may be generated from their mass terms by the substitution [15]

$$m_f^0 \to m_f^0 \left(1 + i g_f^{A0} \frac{A^0}{v^0}\right)$$

prior to adding the complex conjugate of the chiral mass operator $m_f^0 \bar{f}_L^0 f_R^0$. This can be achieved more systematically by introducing left- and right-handed masses, $m_{f\pm}^0$, and writing the bare fermion propagators as

$$S_F(p) = \frac{\not{p} + m_{f+}^0 \omega_+ + m_{f-}^0 \omega_-}{p^2 - m_{f+}^0 m_{f-}^0}, \tag{4}$$

where $\omega_\pm = (1 \pm \gamma_5)/2$ are the chiral projectors. Then, the Aff interaction may be generated by the substitutions

$$m_{f\pm}^0 \to m_{f\pm}^0 \left(1 \pm i g_f^{A0} \frac{A^0}{v^0}\right)\Bigg|_{m_{f\pm}^0 = m_f^0}. \tag{5}$$

This leads to the following LET for the Aff interaction:

$$\lim_{p_A \to 0} \mathcal{M}(XA) = \sum_f i \frac{g_f^{A0}}{v^0} m_f^0 \left(\frac{\partial}{\partial m_{f+}^0} - \frac{\partial}{\partial m_{f-}^0}\right) \mathcal{M}(X)\Bigg|_{m_{f\pm}^0 = m_f^0}, \tag{6}$$

where $\mathcal{M}(X)$ denotes the matrix element of any particle configuration X and $\mathcal{M}(XA)$ is the corresponding one with an external pseudoscalar Higgs boson A added in all possible ways. Again, the renormalization of the bare quantities is to be performed after the right–hand side of Eq. (6) has been evaluated.

However, substitution (5) does not yield the full effective Lagrangian. In the case of the interaction of a pseudoscalar particle with vector bosons, additional contributions may arise due to the Adler–Bell–Jackiw (ABJ) anomaly [16]. Such contributions appear if an odd number of external pseudoscalar Higgs bosons, which carry odd $\mathcal{CP}$ parity at vanishing momentum transfer, is coupled to a pair of vector bosons via a single fermion loop. Therefore, the LETs for odd numbers of external pseudoscalar Higgs particles differ from those for even numbers.

2.2.1 Odd Number of Pseudoscalars

In the case of an odd number of pseudoscalar Higgs bosons coupled to one heavy–fermion loop, a contribution related the ABJ anomaly has to be added to the effective Lagrangian of the model with the heavy fermion integrated out. This contribution may be derived by observing that, in addition to the pseudoscalar mass term, the divergence of the axial vector current, $j_5^\mu = \bar{f}\gamma^\mu\gamma_5 f$, receives a contribution from the ABJ anomaly [16],

$$\partial_\mu j_5^\mu = 2im_f \bar{f}\gamma_5 f + \sum_{V,V'=g,\gamma,W,Z} \frac{\alpha_{VV'}}{4\pi} V^{a\mu\nu}\tilde{V}'^a_{\mu\nu}, \tag{7}$$

where $V^{a\mu\nu}$ is the field–strength tensor of V and $\tilde{V}^a_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}V^{a\rho\sigma}$ is its dual. Here, the index a stems from the respective gauge group. The couplings $\alpha_{VV'}$ are listed for the $\mathcal{SM}$ (and the $\mathcal{MSSM}$) gauge bosons in Table 2. As usual, α_{em} is the fine–structure constant, G_F is the Fermi constant, $v_f = 2I_{3f} - 4e_f s_w^2$, $a_f = 2I_{3f}$, $c_w^2 = 1 - s_w^2 = M_W^2/M_Z^2$, and N_c, e_f, and I_{3f} are the number of colours, the fractional charge, and the third isospin component of the fermion f, respectively.

$\alpha_{VV'}$	g	γ	Z	W
g	α_s	0	0	0
γ	0	$N_c e_f^2 \alpha_{em}$	$N_c e_f v_f \sqrt{\frac{\alpha_{em} G_F M_Z^2}{8\sqrt{2}\pi}}$	0
Z	0	$N_c e_f v_f \sqrt{\frac{\alpha_{em} G_F M_Z^2}{8\sqrt{2}\pi}}$	$N_c \frac{G_F M_Z^2}{8\sqrt{2}\pi}\left(v_f^2 + \frac{a_f^2}{3}\right)$	0
W	0	0	0	$N_c \frac{G_F M_W^2}{2\sqrt{2}\pi}$

Table 2: *Values of $\alpha_{VV'}$ in Eq. (7) for the $\mathcal{SM}$ (and the $\mathcal{MSSM}$) gauge bosons V, V'.*

The Adler–Bardeen theorem [17] states that Eq. (7) is not modified by radiative corrections. On the other hand, we have [18]

$$\lim_{p_A \to 0} \langle A | \partial_\mu j_5^\mu A | VV' \rangle = 0. \tag{8}$$

This allows us to derive from Eq. (3) the anomalous part of the low-p_A effective interaction Lagrangian of A [19–21],

$$\mathcal{L}_{ABJ} = g_f^A \sum_{V,V'} \frac{\alpha_{VV'}}{8\pi} V^{a\mu\nu} \tilde{V}'^a_{\mu\nu} \frac{A}{v}, \tag{9}$$

which is valid to all orders. This contribution has to be added to the part of the effective Lagrangian which is generated by LET (6).

2.2.2 Even Number of Pseudoscalars

An even number of pseudoscalar Higgs bosons carry positive $\mathcal{CP}$ parity at vanishing momentum transfer. Hence, the ABJ anomaly does not contribute to the effective Lagrangian for an even number of pseudoscalars coupled to a single heavy-fermion loop. Substitution (5) leads us to the following LET:

$$\lim_{p_A \to 0} \mathcal{M}(XA^{2n}) = \sum_f \left(i \frac{g_f^{A0}}{v^0} \right)^{2n} (m_f^0)^{2n} \left(\frac{\partial}{\partial m_{f+}^0} - \frac{\partial}{\partial m_{f-}^0} \right)^{2n} \mathcal{M}(X) \Big|_{m_{f\pm}^0 = m_f^0}. \tag{10}$$

As in the scalar case, the renormalization of the bare quantities has to be performed after taking the derivative on the right-hand side of Eq. (10), and mass-dependent couplings must be kept fixed with respect to mass differentiation. Notice that we may also use Eq. (10) for odd numbers of A bosons, if we take X to implicitly include one of them.

3 Applications at One Loop

In the following, we shall consider generic neutral scalar and pseudoscalar Higgs bosons, H and A, with $g_t^H = g_V^H = g_t^A = 1$ and $g_V^A = 0$ ($V = W, Z$). For simplicity, we shall neglect the masses of all loop fermions, except for the top quark. Our results can easily be generalized to arbitrary couplings by means of Eqs. (1) and (5).

3.1 Higgs Couplings to Two Photons and Two Gluons

3.1.1 Scalar Higgs Bosons

In order to calculate the effective coupling of the neutral scalar Higgs boson H to two photons, we have to evaluate the contributions from the charged massive particles, *i.e.*, in

our case the top quark and the W boson, to the on–shell photon self–energy. The result may be formulated as the following effective Lagrangian:

$$\mathcal{L}_{\gamma\gamma} = -\frac{1}{4}F^{0\mu\nu}F^0_{\mu\nu}\left[1 + \Pi^t_{\gamma\gamma}(0) + \Pi^W_{\gamma\gamma}(0)\right], \tag{11}$$

where $F^0_{\mu\nu}$ is the bare electromagnetic field–strength tensor and

$$\Pi^t_{\gamma\gamma}(0) = N_c e_t^2 \frac{\alpha_{em}}{3\pi}\left(\frac{4\pi\mu^2}{m_t^2}\right)^{\epsilon}\Gamma(1+\epsilon)\frac{1}{\epsilon}, \tag{12}$$

$$\Pi^W_{\gamma\gamma}(0) = -\frac{\alpha_{em}}{4\pi}\left(\frac{4\pi\mu^2}{M_W^2}\right)^{\epsilon}\Gamma(1+\epsilon)\left[\frac{7}{\epsilon} + \frac{2}{3} + \mathcal{O}(\epsilon)\right] \tag{13}$$

are the lowest–order expressions of the top–quark and W–boson contributions to the (dimensionless) photon vacuum–polarization function at zero momentum transfer, respectively. Notice that Eq. (13) has been calculated using the pinch technique [22] and is thus manifestly gauge independent. Application of LET (2) leads to [8, 9]

$$\mathcal{L}_{H\gamma\gamma} = \frac{\alpha_{em}}{2\pi}F^{\mu\nu}F_{\mu\nu}\frac{H}{v}\left(N_c\frac{e_t^2}{3} - \frac{7}{4}\right), \tag{14}$$

which is valid for $M_H \ll M_W, m_t$. This expression is in agreement with the leading term of the full one–loop result [8, 9].[1] We may upgrade the range of validity of Eq. (14) to be $M_H \ll m_t$ by replacing the second term contained within the parentheses with the full M_W–dependent result [9].

The effective Lagrangian $\mathcal{L}_{H\gamma\gamma}$ of Eq. (14) fixes the photonic Higgs decay width $\Gamma(H \to \gamma\gamma)$ as well as the cross section of Higgs production via photon fusion $\sigma(\gamma\gamma \to H)$. For $M_H \lesssim 140$ GeV, the decay $H \to \gamma\gamma$ has a branching ratio of order 10^{-3} and will play an important rôle for the search for the Higgs boson in this mass range at the LHC [23]. On the other hand, $\gamma\gamma \to H$ will be the relevant Higgs–boson–production mechanism at future photon colliders [24].

Similarly to the $H\gamma\gamma$ case, the derivation of the effective Hgg Lagrangian starts from

$$\mathcal{L}_{gg} = -\frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu}\left[1 + \Pi^t_{gg}(0)\right], \tag{15}$$

where

$$\Pi^t_{gg}(0) = \frac{\alpha_s}{6\pi}\left[\frac{4\pi\mu^2}{(m_t^0)^2}\right]^{\epsilon}\Gamma(1+\epsilon)\frac{1}{\epsilon} \tag{16}$$

is the top–quark contribution to the (dimensionless) gluon self–energy, and yields [8]

$$\mathcal{L}_{Hgg} = \frac{\alpha_s}{12\pi}G^{a\mu\nu}G^a_{\mu\nu}\frac{H}{v}, \tag{17}$$

[1] If we use an ultraviolet cut–off, Λ_{UV}, instead of dimensional regularization, then Eq. (12) assumes the form $\Pi^t_{\gamma\gamma}(0) = N_c e_t^2(\alpha_{em}/3\pi)\log(\Lambda^2_{UV}/m_t^2)$ [11], which also leads to the first term of Eq. (14). This nicely demonstrates that this term is independent of the regularization scheme.

which is valid for $M_H \ll m_t$. This Lagrangian determines the gluonic decay width $\Gamma(H \to gg)$, which, for $M_H \lesssim 150$ GeV, has a branching ratio of a few percent and should be observable at future e^+e^- colliders [25]. Furthermore, it controls the production of a light Higgs boson via gluon fusion $gg \to H$, which will be the dominant production mechanism of this particle at the LHC [23, 26].

3.1.2 Pseudoscalar Higgs Bosons

Due to the absence of a tree–level AWW vertex and the fact that Eq. (6) annihilates the top–quark contributions to the photon and gluon self–energies, the effective $A\gamma\gamma$ and Agg Lagrangians only consist of the ABJ parts of Eq. (9) [19, 20],

$$\mathcal{L}_{A\gamma\gamma} = N_c e_t^2 \frac{\alpha_{em}}{8\pi} F^{\mu\nu}\tilde{F}_{\mu\nu}\frac{A}{v}, \tag{18}$$

$$\mathcal{L}_{Agg} = \frac{\alpha_s}{8\pi} G^{a\mu\nu}\tilde{G}^a_{\mu\nu}\frac{A}{v}. \tag{19}$$

We recall that these effective Lagrangians do not receive radiative corrections. They determine the photonic and gluonic decays of a pseudoscalar Higgs boson A, with $M_A \ll m_t$, as well as its single production via photon and gluon fusion, which will be the dominant production mechanisms at future photon colliders and the LHC [23, 26], respectively.

3.2 Multi–Higgs Couplings to Two Photons and Two Gluons

3.2.1 Scalar Higgs Bosons

We may derive an effective Lagrangian describing the coupling of any number of neutral scalar Higgs bosons H to two photons by iteratively applying LET (2) to Eq. (14), which we rewrite as

$$\mathcal{L}_{H^1\gamma\gamma} = \frac{\alpha_{em}}{2\pi} F^{\mu\nu}F_{\mu\nu}H\left(N_c\frac{e_t^2}{3}\frac{g_t}{m_t} - \frac{7}{4}\frac{g_W}{M_W}\right), \tag{20}$$

since the coupling constants of the Higgs boson to the top–quark and the W–boson, $g_t = m_t/v$ and $g_W = M_W/v$, have to be kept fixed with respect to the mass differentiation. The effective Lagrangian for the interactions of n H bosons with two photons is then given by

$$\begin{aligned}\mathcal{L}_{H^n\gamma\gamma} &= F^{\mu\nu}F_{\mu\nu}\frac{H^n}{n!}\frac{\alpha_{em}}{2\pi}\left(N_c\frac{e_t^2}{3}g_t^n\frac{\partial^{n-1}}{\partial m_t^{n-1}}\frac{1}{m_t} - \frac{7}{4}g_W^n\frac{\partial^{n-1}}{\partial M_W^{n-1}}\frac{1}{M_W}\right)\\ &= -F^{\mu\nu}F_{\mu\nu}\frac{1}{n}\left(-\frac{H}{v}\right)^n\frac{\alpha_{em}}{2\pi}\left(N_c\frac{e_t^2}{3} - \frac{7}{4}\right).\end{aligned} \tag{21}$$

Summing up all these Lagrangians, we find [9]

$$\mathcal{L}_{H\gamma\gamma} = \sum_{n=1}^{\infty}\mathcal{L}_{H^n\gamma\gamma} = \frac{\alpha_{em}}{2\pi}\left(N_c\frac{e_t^2}{3} - \frac{7}{4}\right)F^{\mu\nu}F_{\mu\nu}\log\left(1+\frac{H}{v}\right). \tag{22}$$

The analogous calculation for the gluon case yields [27]

$$\mathcal{L}_{Hgg} = \frac{\alpha_s}{12\pi} G^{a\mu\nu} G^a_{\mu\nu} \log\left(1 + \frac{H}{v}\right). \tag{23}$$

These Lagrangians govern the cross sections for multi–Higgs production via photon and gluon fusion in the limit $M_H \ll m_t$, where Higgs self–interactions are suppressed.

3.2.2 Pseudoscalar Higgs Bosons

Similarly to the scalar case, the effective Lagrangian describing the coupling of any number of pseudoscalar Higgs bosons A to two photons (gluons) can be deduced by iterative application of LET (10). For even numbers of pseudoscalar Higgs bosons, LET (10) has to be applied to the photon (gluon) self–energy, while for odd numbers it has to be applied to Lagrangian (18) [(19)]. In this way, we obtain

$$\begin{aligned}
\mathcal{L}_{A^{2n}\gamma\gamma} &= -N_c e_t^2 \frac{\alpha_{em}}{12\pi} F^{\mu\nu} F_{\mu\nu} \frac{A^{2n}}{(2n)!} \left[i \frac{m_t}{v} \left(\frac{\partial}{\partial m_{t+}} - \frac{\partial}{\partial m_{t-}} \right) \right]^{2n} \log(m_{t+} m_{t-}) \Big|_{m_{t\pm}=m_t}, \\
\mathcal{L}_{A^{2n+1}\gamma\gamma} &= -iN_c e_t^2 \frac{\alpha_{em}}{16\pi} F^{\mu\nu} \tilde{F}_{\mu\nu} \frac{A^{2n+1}}{(2n+1)!} \left[i \frac{m_t}{v} \left(\frac{\partial}{\partial m_{t+}} - \frac{\partial}{\partial m_{t-}} \right) \right]^{2n+1} \log \frac{m_{t+}}{m_{t-}} \Big|_{m_{t\pm}=m_t},
\end{aligned} \tag{24}$$

and similarly for gluons. Summing separately over even and odd numbers of pseudoscalar Higgs bosons, we obtain

$$\mathcal{L}^{(even)}_{A\gamma\gamma} = N_c e_t^2 \frac{\alpha_{em}}{12\pi} F^{\mu\nu} F_{\mu\nu} \log\left(1 + \frac{A^2}{v^2}\right), \tag{25}$$

$$\mathcal{L}^{(even)}_{Agg} = \frac{\alpha_s}{24\pi} G^{a\mu\nu} G^a_{\mu\nu} \log\left(1 + \frac{A^2}{v^2}\right) \tag{26}$$

for even numbers and

$$\begin{aligned}
\mathcal{L}^{(odd)}_{A\gamma\gamma} &= -iN_c e_t^2 \frac{\alpha_{em}}{16\pi} F^{\mu\nu} \tilde{F}_{\mu\nu} \left[\log\left(1 + i\frac{A}{v}\right) - \log\left(1 - i\frac{A}{v}\right) \right], \\
\mathcal{L}^{(odd)}_{Agg} &= -i \frac{\alpha_s}{16\pi} G^{a\mu\nu} \tilde{G}^a_{\mu\nu} \left[\log\left(1 + i\frac{A}{v}\right) - \log\left(1 - i\frac{A}{v}\right) \right]
\end{aligned} \tag{27}$$

for odd numbers. Lagrangian (26) agrees with the pseudoscalar χ^3 part of Ref. [27]. These Lagrangians describe the production of many pseudoscalar Higgs bosons by photon and gluon fusion, which may be relevant at future photon colliders and the LHC, respectively.

3.3 Higgs Couplings to One Z Boson and One Photon

3.3.1 Scalar Higgs Bosons

In Section 3.1.1, we have seen how the $H\gamma\gamma$ coupling is related to the photon self-energy. In a similar fashion, the $HZ\gamma$ coupling may be derived from the γ–Z transition amplitude. In order for the $HZ\gamma$ amplitude to be gauge independent, all three external particles must be on their mass shells [28]. Then, however, it is unjustified to integrate out the virtual W boson. In the following, we shall therefore concentrate on the top–quark loop. Similarly to Eq. (11), we may write

$$\mathcal{L}_{Z\gamma} = -\frac{1}{4} F^{0\mu\nu} Z^0_{\mu\nu} \Pi^t_{Z\gamma}(0), \tag{28}$$

where $Z^0_{\mu\nu}$ is the bare Z–boson field–strength tensor and

$$\Pi^t_{Z\gamma}(0) = \frac{N_c e_t v_t}{3\pi} \sqrt{\frac{\alpha_{em} G_F M_Z^2}{8\sqrt{2}\pi}} \left(\frac{4\pi\mu^2}{m_t^2}\right)^{\epsilon} \Gamma(1+\epsilon) \frac{1}{\epsilon} \tag{29}$$

is the top–quark contribution to the (dimensionless) Z–γ mixing amplitude at zero momentum transfer. Here, we have used the notation introduced below Eq. (7). Differentiating this expressions with respect to m_t, we end up with the effective $HZ\gamma$ Lagrangian,

$$\mathcal{L}_{HZ\gamma} = \frac{N_c e_t v_t}{6\pi} \sqrt{\frac{\alpha_{em} G_F M_Z^2}{8\sqrt{2}\pi}} F^{\mu\nu} Z_{\mu\nu} \frac{H}{v}, \tag{30}$$

appropriate to the limit where $M_H, M_Z \ll m_t$. Lagrangian (30) may also be derived by directly expanding the corresponding one–loop diagram [28, 29]. The W–boson contribution may be found in Ref. [28]. It is significant and must be included in order to obtain a satisfactory description [28, 29]. The full $HZ\gamma$ Lagrangian determines the width of the rare Z–boson decay $Z \to H\gamma$.

3.3.2 Pseudoscalar Higgs Bosons

Similarly to the $A\gamma\gamma$ case discussed in Section 3.1.2, the coupling of the pseudoscalar Higgs boson A to a Z boson and a photon is controlled by Lagrangian (9). Specifically, we have

$$\mathcal{L}_{AZ\gamma} = \frac{N_c e_t v_t}{4\pi} \sqrt{\frac{\alpha_{em} G_F M_Z^2}{8\sqrt{2}\pi}} Z^{\mu\nu} \tilde{F}_{\mu\nu} \frac{A}{v}. \tag{31}$$

This Lagrangian fixes the width of the rare $Z \to A\gamma$ decay for $M_A, M_Z \ll m_t$.

3.4 $H \to b\bar{b}$

By means of LET (2), we may also extract the $\mathcal{O}(G_F m_t^2)$ correction to the $b\bar{b}$ decay rate of the neutral scalar Higgs bosons H in the limit where $M_H \ll m_t$ [13]. This just requires knowledge of the $\mathcal{O}(G_F m_t^2)$ contribution to the bottom–quark self–energy. To

compute this contribution, we must put the bottom quark on mass shell and neglect its mass, except for one overall power. Furthermore, it is sufficient to take into account the longitudinal component of the W boson, $w^{\pm}$, which we may take to be massless, too. The bare amplitude describing the propagation of the b quark can be cast into the form

$$\mathcal{M}(b \to b) = \bar{b}^0 \left\{ m_b^0 [-1 + \Sigma_S(0)] + \not{p} [\Sigma_V(0) + \gamma_5 \Sigma_A(0)] \right\} b^0. \tag{32}$$

The Yukawa couplings of $w^{\pm}$ to the bottom and top quarks must be kept fixed with respect to the mass differentiation. Therefore, we call them $g_q^0 = m_q^0/v^0$ $(q = t, b)$. The various amplitudes in Eq. (32) are, at $\mathcal{O}(G_F m_t^2)$, given by [30]

$$m_b^0 \Sigma_S(0) = -\frac{g_b^0 g_t^0}{(4\pi)^2} m_t^0 \left[\frac{4\pi\mu^2}{(m_t^0)^2} \right]^{\epsilon} \Gamma(1+\epsilon) \left[\frac{2}{\epsilon} + 2 + 2\epsilon + \mathcal{O}(\epsilon^2) \right], \tag{33}$$

$$\Sigma_V(0) = \frac{(g_t^0)^2}{(4\pi)^2} \left[\frac{4\pi\mu^2}{(m_t^0)^2} \right]^{\epsilon} \Gamma(1+\epsilon) \left[\frac{1}{2\epsilon} + \frac{3}{4} + \frac{7}{8}\epsilon + \mathcal{O}(\epsilon^2) \right], \tag{34}$$

$$\Sigma_A(0) = -\Sigma_V(0). \tag{35}$$

Applying LET (2) to Eq. (32),

$$\lim_{p_H \to 0} \mathcal{M}(b \to bH) = \frac{1}{v^0} \left(\frac{m_b^0 \partial}{\partial m_b^0} + \frac{m_t^0 \partial}{\partial m_t^0} \right) \mathcal{M}(b \to b), \tag{36}$$

and then using the Dirac equation, we find the following effective Lagrangian:

$$\mathcal{L}_{Hb\bar{b}} = -m_b^0 \bar{b}^0 b^0 \frac{H^0}{v^0} (1 + \delta_{Hb\bar{b}}^0), \tag{37}$$

where

$$\begin{aligned} \delta_{Hb\bar{b}}^0 &= -m_t^0 \frac{\partial [\Sigma_S(0) + \Sigma_V(0)]}{\partial m_t^0} \\ &= -\Sigma_S(0) + 2\epsilon [\Sigma_S(0) + \Sigma_V(0)]. \end{aligned} \tag{38}$$

It should be noted that the axial part $\Sigma_A(0)$ is eliminated by the Dirac equation. Next, we have to renormalize the bottom-quark mass and wave function. This may be achieved by substituting

$$m_b^0 \bar{b}^0 b^0 = m_b \bar{b} b \frac{1}{1 - \Sigma_S(0)}. \tag{39}$$

In this way, we obtain the non-universal correction to the effective $Hb\bar{b}$ Lagrangian [13, 30],

$$\mathcal{L}_{Hb\bar{b}} = -m_b \bar{b} b \frac{H^0}{v^0} (1 + \delta_{Hb\bar{b},nu}), \tag{40}$$

with

$$\delta_{Hb\bar{b},nu} = 2\epsilon [\Sigma_S(0) + \Sigma_V(0)] = -3x_t^0, \tag{41}$$

where $x_t^0 = G_F(m_t^0)^2/(8\sqrt{2}\pi^2)$. Using also the universal relation [13, 30]

$$\frac{H^0}{v^0} = 2^{1/4} G_F^{1/2} H(1+\delta_u), \tag{42}$$

with

$$\delta_u = \frac{7}{6} N_c x_t, \tag{43}$$

we find

$$\mathcal{L}_{Hb\bar{b}} = -2^{1/4} G_F^{1/2} m_b \bar{b} b H(1+\delta_{Hb\bar{b}}), \tag{44}$$

with

$$\delta_{Hb\bar{b}} = \delta_{Hb\bar{b},\,nu} + \delta_u = \left(\frac{7}{6} N_c - 3\right) x_t, \tag{45}$$

where we have used that m_t^0 and m_t coincide in lowest order. The $\mathcal{O}(G_F m_t^2)$ correction to $\Gamma(H \to b\bar{b})$ is then $2\delta_{Hb\bar{b}}$, which agrees with the explicit calculation [30]. For $m_t = 176$ GeV, this term enhances $\Gamma(H \to b\bar{b})$ by approximately 0.3%, but it does not yet dominate the full weak correction at one loop. For example, at $M_H = 70$ GeV, the latter amounts to approximately -0.4%.

3.5 HZZ coupling and $e^+e^- \to ZH$

Another useful application of LET (2) is to derive the leading top-mass dependent correction of $\mathcal{O}(G_F m_t^2)$ to the coupling of the neutral scalar boson H to a pair of Z bosons. The starting point is the amplitude describing the propagation of an on-shell Z boson interacting with virtual top quarks,

$$\mathcal{M}(Z \to Z) = \frac{1}{2} Z^{0\mu} Z_\mu^0 \left[(M_Z^0)^2 - \Pi_{ZZ}(0)\right], \tag{46}$$

where [31, 32]

$$\Pi_{ZZ}(0) = -2N_c (g_Z^0 v^0)^2 x_t^0 \left[\frac{4\pi\mu^2}{(m_t^0)^2}\right]^\epsilon \Gamma(1+\epsilon)\frac{1}{\epsilon} \tag{47}$$

is the top-quark contribution to the Z-boson self-energy. Here, $g_Z^0 = M_Z^0/v^0$, x_t^0 is defined below Eq. (41), and we have used $M_Z \ll m_t$ in the loop amplitude. In the case at hand, LET (2) takes the form

$$\lim_{p_H \to 0} \mathcal{M}(Z \to ZH) = \frac{1}{v^0}\left(\frac{m_t^0 \partial}{\partial m_t^0} + \frac{M_Z^0 \partial}{\partial M_Z^0}\right) \mathcal{M}(Z \to Z), \tag{48}$$

where g_Z^0 has to be kept fixed. After evaluating the right-hand side of Eq. (48), we are in a position to write down the effective HZZ Lagrangian,

$$\mathcal{L}_{HZZ} = (M_Z^0)^2 Z^{0\mu} Z_\mu^0 \frac{H^0}{v^0}(1+\delta_{HZZ}^0), \tag{49}$$

where

$$\delta^0_{HZZ} = -(1-\epsilon)\frac{\Pi_{ZZ}(0)}{(M^0_Z)^2}. \tag{50}$$

Renormalizing the Z-boson mass and wave function,

$$\begin{aligned}(M^0_Z)^2 &= M^2_Z + \delta M^2_Z,\\ Z^0_\mu &= Z_\mu(1+\delta Z_Z)^{1/2},\end{aligned} \tag{51}$$

with the on-shell counter terms

$$\begin{aligned}\delta M^2_Z &= \Pi_{ZZ}(0),\\ \delta Z_Z &= -\Pi'_{ZZ}(0),\end{aligned} \tag{52}$$

we obtain the finite non-universal correction to the HZZ coupling,

$$\delta_{HZZ,nu} = \epsilon\frac{\Pi_{ZZ}(0)}{(M^0_Z)^2} = -2N_c x^0_t. \tag{53}$$

It should be noted that δZ_Z does not receive any contribution in $\mathcal{O}(G_F m^2_t)$ and thus does not contribute here. Replacing the bare Higgs field H^0 and the bare vacuum expectation value v^0 with their renormalized counterparts, we introduce the universal correction δ_u of Eq. (43). Consequently, the effective Lagrangian reads [32]

$$\mathcal{L}_{HZZ} = 2^{1/4}G^{1/2}_F M^2_Z Z^\mu Z_\mu H(1+\delta_{HZZ}), \tag{54}$$

with

$$\delta_{HZZ} = \delta_{HZZ,nu} + \delta_u = -\frac{5}{6}N_c x_t. \tag{55}$$

The decay width $\Gamma(H \to ZZ)$ is then corrected by the factor $(1+2\delta_{HZZ})$, which agrees with the expansion of the full one-loop correction [32]. This provides an approximation for $2M_Z < M_H \ll m_t$, which is not satisfied for the actual Z-boson and top-quark masses. However, this result may be used as a building block for the calculation of the $\mathcal{O}(G_F m^2_t)$ correction to the Higgs production mechanism $e^+e^- \to ZH$. In fact, by invoking the improved Born approximation [33] for the on-shell scheme formulated with G_F, we find that $\sigma(e^+e^- \to ZH)$ is corrected by the factor $(1+\delta_{HZe^+e^-})$, where [34]

$$\delta_{HZe^+e^-} = 2\delta_{HZZ} + \left(1 - 8\frac{c^2_w Q_e v_e}{v^2_e + a^2_e}\right)\Delta\rho \tag{56}$$

$$= -2N_c x_t\left(\frac{1}{3} + 4\frac{c^2_w Q_e v_e}{v^2_e + a^2_e}\right), \tag{57}$$

with $\Delta\rho = N_c x_t$ [35]. This agrees with the corresponding expansion of the full one-loop correction [34]. Numerically, we find $\delta_{HZe^+e^-} \approx -1\%$ for $m_t = 176$ GeV. This has to be compared with the full one-loop correction, which, for $M_H = 70$ GeV and LEP2 energy, amounts to approximately -3%.

3.6 HWW coupling and $e^+e^- \to \nu_e\bar{\nu}_e H$

In close analogy to the HZZ case, we can also derive the $\mathcal{O}(G_F m_t^2)$ correction to the HWW coupling by using LET (2). Starting from the amplitude describing the propagation of an on-shell W boson in the presence of virtual top and bottom quarks,

$$\mathcal{M}(W \to W) = (W^{+\mu})^0 (W_\mu^-)^0 \left[(M_W^0)^2 - \Pi_{WW}(0)\right], \tag{58}$$

where [31, 32]

$$\Pi_{WW}(0) = -N_c (g_W^0 v^0)^2 x_t^0 \left[\frac{4\pi\mu^2}{(m_t^0)^2}\right]^\epsilon \Gamma(1+\epsilon)\left[\frac{2}{\epsilon} + 1 + \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2)\right], \tag{59}$$

with $g_W^0 = M_W^0/v^0$, is the respective contribution to the W-boson self-energy, applying LET (2) in the form

$$\lim_{p_H \to 0} \mathcal{M}(W \to WH) = \frac{1}{v^0}\left(\frac{m_t^0 \partial}{\partial m_t^0} + \frac{M_W^0 \partial}{\partial M_W^0}\right) \mathcal{M}(W \to W), \tag{60}$$

where g_W^0 must be treated as a constant, and renormalizing the parameters according to the on-shell scheme, we end up with the effective HWW Lagrangian,

$$\mathcal{L}_{HWW} = 2^{5/4} G_F^{1/2} M_W^2 W^{+\mu} W_\mu^- H(1 + \delta_{HWW}), \tag{61}$$

with

$$\delta_{HWW} = -\frac{5}{6} N_c x_t, \tag{62}$$

which coincides with δ_{HZZ} of Eq. (55).

This contains all the information which is necessary to compute the $\mathcal{O}(G_F m_t^2)$ correction to Higgs-boson production via W-boson fusion, $e^+e^- \to H\nu_e\bar{\nu}_e$, in the G_F representation of the on-shell scheme. Since G_F is defined via a charged-current process, namely the muon decay, there are no additional $\mathcal{O}(G_F m_t^2)$ corrections from the W-boson propagators in this case. Thus, the correction factor for $\sigma(e^+e^- \to H\nu_e\bar{\nu}_e)$ is just $(1 + 2\delta_{HWW})$, which amounts to a reduction by about 2%.

4 Applications at Two Loops

4.1 Higgs Couplings to Two Photons

4.1.1 Scalar Higgs Bosons

In order to evaluate the two-loop QCD correction to the two-photon coupling of the neutral scalar Higgs boson H by means of the LET, we have to start from the top-quark contribution to the photon self-energy given in Eq. (12) and include its leading-order QCD correction,

$$\Pi_{\gamma\gamma}^{t(2)}(0) = N_c C_F e_t^2 \frac{\alpha_{em}\alpha_s}{8\pi^2}\left(\frac{4\pi\mu^2}{m_t^2}\right)^{2\epsilon} \Gamma^2(1+\epsilon)\left[\frac{1}{\epsilon} + \frac{15}{2} + \mathcal{O}(\epsilon)\right], \tag{63}$$

where $C_F = (N_c^2 - 1)/(2N_c) = 4/3$, m_t denotes the on–shell mass of the top quark, and it is understood that Eq. (12) is also written with m_t. LET (2) leads us to differentiate $\Pi^t_{\gamma\gamma}(0)$, *i.e.*, the sum of Eqs. (12) and (63), with respect to the bare mass m_t^0. When we express the differentiation with respect to m_t^0 in terms of m_t, we pick up an additional finite contribution involving the anomalous mass dimension γ_m,

$$\frac{m_t^0 \partial}{\partial m_t^0} = \frac{1}{1+\gamma_m} \frac{m_t \partial}{\partial m_t}. \tag{64}$$

On the other hand, differentiation of $\Pi^t_{\gamma\gamma}(0)$ with respect to m_t yields

$$\frac{m_t \partial}{\partial m_t} \Pi^t_{\gamma\gamma}(0) = -\frac{\beta^t_{\alpha_{em}}}{\alpha_{em}}, \tag{65}$$

where $\beta^t_{\alpha_{em}}$ is the top–quark part of the QED β function defined by $\mu \partial \alpha_{em}/\partial \mu = \beta_{\alpha_{em}}$. Combining these results with Eq. (14), we obtain the effective $H\gamma\gamma$ interaction Lagrangian to all orders in α_s [12, 36, 37],

$$\mathcal{L}_{H\gamma\gamma} = F^{\mu\nu} F_{\mu\nu} \frac{H}{v} \left(\frac{\beta^t_{\alpha_{em}}}{4\alpha_{em}} \frac{1}{1+\gamma_m} - \frac{7\alpha_{em}}{8\pi} \right). \tag{66}$$

The expansion of $\beta^t_{\alpha_{em}}$ up to $\mathcal{O}(\alpha_{em}\alpha_s)$ may be evaluated via Eq. (65) and reads

$$\frac{\beta^t_{\alpha_{em}}}{\alpha_{em}} = N_c e_t^2 \frac{2\alpha_{em}}{3\pi} \left[1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right]. \tag{67}$$

Furthermore, the mass anomalous dimension of QCD is given by

$$\gamma_m = \frac{3}{2} C_F \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2). \tag{68}$$

Thus, in next–to–leading order, Eq. (66) becomes [12, 36]

$$\mathcal{L}_{H\gamma\gamma} = \frac{\alpha_{em}}{2\pi} F^{\mu\nu} F_{\mu\nu} \frac{H}{v} \left[N_c \frac{e_t^2}{3} \left(1 - \frac{3}{4} C_F \frac{\alpha_s}{\pi} \right) - \frac{7}{4} \right]. \tag{69}$$

This result is in agreement with the high–m_t limit of the two–loop QCD correction to the $H\gamma\gamma$ coupling [12, 19, 38].

It is worthwhile to dwell on Eq. (66) for a little while. This result can also be obtained from a different type of LET, based on the trace anomaly of the energy–momentum tensor, $\Theta^{\mu\nu}$. It has been shown [39] that the effective form of $\Theta^{\mu\nu}$ including all orders of the contributing couplings is given by

$$\Theta^\mu_\mu = m_t^0 \bar{t}^0 t^0 (1+\gamma_m) + \frac{\beta_{\alpha_{em}}}{4\alpha_{em}} \mathcal{N}(F^{\mu\nu} F_{\mu\nu}), \tag{70}$$

where $\mathcal{N}(\cdots)$ denotes the normal product. Furthermore, it has been proven [40] that the matrix element $\langle 0|\Theta^\mu_\mu|\gamma\gamma\rangle$ vanishes at zero momentum transfer,

$$\lim_{Q^2 \to 0} \langle 0|\Theta^\mu_\mu|\gamma\gamma\rangle = 0. \tag{71}$$

Since a Higgs boson with vanishing momentum acts as a constant field, it hence follows that

$$\lim_{p_H \to 0} \langle H | \Theta^\mu_\mu H | \gamma\gamma \rangle = 0 \tag{72}$$

is fulfilled. If we then multiply Eq. (70) by H and substitute the result into Eq. (72), we obtain from $\mathcal{L}_{Ht\bar{t}} = -m_t^0 \bar{t}^0 t^0 H/v$ the effective $H\gamma\gamma$ Lagrangian as [12, 36, 37]

$$\mathcal{L}_{H\gamma\gamma} = F^{\mu\nu} F_{\mu\nu} \frac{H}{v} \frac{\beta^t_{\alpha_{em}}}{4\alpha_{em}} \frac{1}{1+\gamma_m}. \tag{73}$$

Here, we have exploited the facts that the operation in Eq. (72) projects out the top-quark contribution to $\beta_{\alpha_{em}}$ and that, in the low-energy limit, $\mathcal{N}(F^{\mu\nu}F_{\mu\nu})$ approaches the corresponding operator containing the renormalized free fields. This reproduces the first term of Eq. (66). The W-boson contribution may be derived in a similar way.

From Lagrangian (69) we can read off the QCD corrections to $\Gamma(H \to \gamma\gamma)$ and $\sigma(\gamma\gamma \to H)$, assuming that $M_H \ll M_W, m_t$. They are quite small, giving support to the notion that these processes are theoretically well under control.

4.1.2 Pseudoscalar Higgs Bosons

According to the Adler–Bardeen theorem [17], *i.e.*, the fact that the ABJ anomaly is not affected by renormalization, the effective Lagrangian for the $A\gamma\gamma$ interaction is fixed to all orders by Eq. (18). Consequently, the two-loop QCD correction to the $A\gamma\gamma$ coupling vanishes in the high-m_t limit, as may be also verified by explicit computation [12, 19].

4.2 Higgs Couplings to Two Gluons

4.2.1 Scalar Higgs Bosons

QCD Corrections. In analogy to the top-quark-induced part of the effective $H\gamma\gamma$ Lagrangian (66), the QCD-corrected effective Hgg Lagrangian of the neutral scalar Higgs boson H is given by [12, 36, 37]

$$\mathcal{L}_{Hgg} = G^{a\mu\nu} G^a_{\mu\nu} \frac{H}{v} \frac{\beta^t_{\alpha_s}}{4\alpha_s} \frac{1}{1+\gamma_m}, \tag{74}$$

where γ_m is listed in Eq. (68) and $\beta^t_{\alpha_s}$ denotes the top-quark contribution to the QCD β function defined by $\mu \partial \alpha_s / \partial \mu = \beta_{\alpha_s}$. Up to next-to-leading order, we have

$$\frac{\beta^t_{\alpha_s}}{\alpha_s} = \frac{\alpha_s}{3\pi} \left[1 + \frac{\alpha_s}{4\pi} (5N_c + 3C_F) + \mathcal{O}(\alpha_s^2) \right]. \tag{75}$$

Thus, Eq. (74) becomes [12, 36, 37]

$$\mathcal{L}_{Hgg} = \frac{\alpha_s}{12\pi} G^{a\mu\nu} G^a_{\mu\nu} \frac{H}{v} \left[1 + \frac{\alpha_s}{4\pi} (5N_c - 3C_F) + \mathcal{O}(\alpha_s^2) \right]. \tag{76}$$

This Lagrangian characterizes the Hgg interaction in the theory where the top quark has been integrated out. For example, when we wish to compute the full two–loop QCD corrections to $\Gamma(H \to gg)$, we just need to consider this Lagrangian in connection with the usual Lagrangian of five–flavour QCD and calculate the one–loop virtual correction and the tree–level real correction. The ultraviolet divergence of the virtual correction is removed by renormalization, while the infrared and collinear singularities cancel when the virtual and real corrections are combined. The final result is [12, 36, 41]

$$\Gamma(H \to gg(g), gq\bar{q}) = \Gamma_{LO}(H \to gg)\left(1 + C_H \frac{\alpha_s}{\pi}\right), \tag{77}$$

where

$$\Gamma_{LO}(H \to gg) = \frac{N_c C_F \alpha_s^2 G_F M_H^3}{144\sqrt{2}\pi^3}, \tag{78}$$

$$C_H = \frac{103}{12} N_c - \frac{3}{2} C_F - \frac{7}{6} N_F + \frac{11 N_c - 2 N_F}{6} \log \frac{\mu^2}{m_H^2}, \tag{79}$$

with $N_F = 5$ being the number of active quark flavours. This agrees with the high–m_t limit of the two–loop calculation in six–flavour QCD [12, 36]. The correction is quite sizeable; it increases the two–gluon decay rate of the $\mathcal{SM}$ Higgs boson with intermediate mass by about 60%. This renders it more likely for this decay mode to be detected at future e^+e^- colliders. If we keep the full mass dependence of $\Gamma_{LO}(H \to gg)$ [26] in Eq. (77), we obtain an approximation which, in the intermediate Higgs–boson mass range, differs by at most 5% from the exact result [12].

Using Eq. (76), we can also calculate the QCD corrections to the cross section of Higgs production via gluon fusion [12, 36, 37, 42], which will be the primary source of Higgs bosons at the LHC. For the $\mathcal{SM}$ Higgs boson with intermediate mass, these corrections range between 50% and 80%. In this case, the high–m_t limit provides a good approximation [12, 42].

Electroweak Corrections. By virtue of LET (2), we can also conveniently extract the two–loop $\mathcal{O}(G_F m_t^2)$ electroweak correction to the Hgg coupling. Toward this end, we need to complement the one–loop top–quark contribution to the gluon self–energy given in Eq. (16) with its $\mathcal{O}(G_F m_t^2)$ correction. However, it turns out [43] that the latter is ultraviolet finite, provided that Eq. (16) is written with m_t^0, and thus does not contribute upon application of LET (2). Therefore, we just need to renormalize the top–quark mass appearing in the one–loop calculation to $\mathcal{O}(G_F m_t^2)$. In the on–shell scheme, this may be achieved by substituting

$$m_t^0 = m_t + \delta m_t, \tag{80}$$

with [30]

$$\frac{\delta m_t}{m_t} = x_t \left(\frac{4\pi\mu^2}{m_t^2}\right)^{\epsilon} \Gamma(1+\epsilon)\left[\frac{3}{2\epsilon} + 4 + \mathcal{O}(\epsilon)\right], \tag{81}$$

where x_t is defined below Eq. (41). In this way, we find the non–universal piece,

$$\delta_{Hgg,nu} = -3x_t. \tag{82}$$

Combining this with the universal part δ_u given in Eq. (43), we obtain the $\mathcal{O}(G_F m_t^2)$ term to be included within the square brackets of Eq. (76),

$$\delta_{Hgg,ew} = \delta_{Hgg,nu} + \delta_u = \left(\frac{7}{6}N_c - 3\right) x_t. \tag{83}$$

This is in agreement with Ref. [43]. The corresponding correction factor for $\Gamma(H \to gg)$ and $\sigma(gg \to H)$ is then $(1 + 2\delta_{Hgg,ew})$, which leads to an insignificant increase, by about three tenths of a percent.

4.2.2 Pseudoscalar Higgs Bosons

In Section 4.1.2, we have seen that, as a consequence of the Adler–Bardeen theorem [17], the $A\gamma\gamma$ coupling does not receive QCD corrections in the high–m_t limit. This also holds true for the Agg interaction. Thus, the two–loop QCD correction to $\Gamma(A \to gg)$ may be computed in five–flavour QCD by using the effective Lagrangian (19). The result is [12, 19]

$$\Gamma(A \to gg(g), gq\bar{q}) = \Gamma_{LO}(A \to gg)\left(1 + C_A\frac{\alpha_s}{\pi}\right), \tag{84}$$

where

$$\Gamma_{LO}(A \to gg) = \frac{N_c C_F \alpha_s^2 G_F M_A^3}{64\sqrt{2}\pi^3}, \tag{85}$$

$$C_A = \frac{97}{12}N_c - \frac{7}{6}N_F + \frac{11N_c - 2N_F}{6}\log\frac{\mu^2}{m_A^2}, \tag{86}$$

with $N_F = 5$. For an intermediate–mass A boson, this correction amounts to about 60%. One should bear in mind that, in the $\mathcal{MSSM}$, this result is only reliable for small values of $\tan\beta$, of order unity, where the top–quark contribution is dominant.

The QCD correction to $\sigma(gg \to A)$ in the high–m_t limit may be computed in a similar way. The $gg \to A$ mechanism will be the dominant source of A bosons at the LHC [23]. The QCD correction turns out to be 50–100% [12, 44].

4.3 Higgs Couplings to One Z Boson and One Photon

4.3.1 Scalar Higgs Bosons

In Section 4.1.1, we have derived the two–loop QCD correction to the top–quark–induced part of the $H\gamma\gamma$ coupling by applying LET (2) to the respective contribution to the photon self–energy. The corresponding result for the $HZ\gamma$ interaction follows by simply adjusting the coupling constants. The resulting QCD correction may be accommodated in Eq. (30) by multiplying the first term contained within the parentheses with the factor $[1 - 3C_F\alpha_s/(4\pi)]$. This agrees with the leading high–m_t term of the full two–loop calculation [29].

4.3.2 Pseudoscalar Higgs Bosons

From arguments similar to those in Sections 4.1.2 and 4.2.2 it follows on that the effective $AZ\gamma$ Lagrangian does not receive any QCD corrections in the high–m_t limit. At the two–loop order, this has been checked by an explicit analysis.

4.4 $H \to b\bar{b}$

The $\mathcal{O}(G_F m_t^2)$ analysis of Section 3.4 readily carries over to $\mathcal{O}(\alpha_s G_F m_t^2)$. We just need to evaluate the $\mathcal{O}(\alpha_s G_F m_t^2)$ corrections to Eqs. (33) and (34). These read [13]

$$\begin{aligned} m_b^0 \Sigma_S^{(2)}(0) &= C_F \frac{\alpha_s}{\pi} \frac{g_t^0}{(4\pi)^2} \left[\frac{4\pi\mu^2}{(m_t^0)^2}\right]^{2\epsilon} \Gamma^2(1+\epsilon) \left[-\frac{3}{2\epsilon^2} g_b^0 m_t^0 + \frac{1}{\epsilon}\left(\frac{3}{4} g_t^0 m_b^0 - 2 g_b^0 m_t^0\right) \right. \\ &\qquad \left. + \mathcal{O}(1)\right], \\ \Sigma_V^{(2)}(0) &= C_F \frac{\alpha_s}{\pi} \frac{(g_t^0)^2}{(4\pi)^2} \left[\frac{4\pi\mu^2}{(m_t^0)^2}\right]^{2\epsilon} \Gamma^2(1+\epsilon) \left[\frac{3}{8\epsilon^2} + \frac{1}{8\epsilon} + \mathcal{O}(1)\right]. \end{aligned} \tag{87}$$

From the first line of Eq. (38) we then obtain the $\mathcal{O}(\alpha_s G_F m_t^2)$ term of $\delta_{Hb\bar{b}}^0$ in Eq. (37),

$$\delta_{Hb\bar{b}}^{0(2)} = -\Sigma_S^{(2)}(0) + 4\epsilon \left[\Sigma_S^{(2)}(0) + \Sigma_V^{(2)}(0)\right]. \tag{88}$$

Again, the $\Sigma_S(0)$ term is removed by the bottom–quark mass and wave–function renormalizations of Eq. (39). Since we are now working at next–to–leading order, we also need to renormalize the top–quark mass in the leading–order expressions, *i.e.*, we need to use Eq. (80) in connection with

$$\frac{\delta m_t}{m_t} = -C_F \frac{\alpha_s}{\pi} \left(\frac{4\pi\mu^2}{m_t^2}\right)^{\epsilon} \Gamma(1+\epsilon) \left[\frac{3}{4\epsilon} + 1 + 2\epsilon + \mathcal{O}(\epsilon^2)\right]. \tag{89}$$

This renders the non–universal $\mathcal{O}(\alpha_s G_F m_t^2)$ correction finite [13],

$$\delta_{Hb\bar{b},nu}^{(2)} = \frac{3}{4} C_F \frac{\alpha_s}{\pi} x_t. \tag{90}$$

We still need to renormalize the wave function and vacuum expectation value of the Higgs field in $\mathcal{O}(\alpha_s G_F m_t^2)$. This yields [45, 46]

$$\delta_u^{(2)} = -\frac{1}{2}\left[\zeta(2) + \frac{3}{2}\right] N_c C_F \frac{\alpha_s}{\pi} x_t. \tag{91}$$

Thus, the $\mathcal{O}(\alpha_s G_F m_t^2)$ term to be included in Eq. (44) comes out as [13]

$$\delta_{Hb\bar{b}}^{(2)} = \delta_{Hb\bar{b},nu}^{(2)} + \delta_u^{(2)} = \left\{\frac{3}{4} - \frac{1}{2}\left[\zeta(2) + \frac{3}{2}\right] N_c\right\} C_F \frac{\alpha_s}{\pi} x_t. \tag{92}$$

The $\mathcal{O}(\alpha_s G_F m_t^2)$ correction to $\Gamma(H \to b\bar{b})$ receives an additional contribution from the interference of the $\mathcal{O}(G_F m_t^2)$ term (45) and the well-known $\mathcal{O}(\alpha_s)$ correction [47]. In the limit $m_b \ll M_H$, the latter is given by [47]

$$\delta_{QCD} = \frac{3}{2} C_F \frac{\alpha_s}{\pi} \left(\frac{3}{2} - \log \frac{M_H^2}{m_b^2} \right), \tag{93}$$

where it is understood that the Born formula for $\Gamma(H \to b\bar{b})$ is written with the bottom-quark pole mass m_b. The combined correction is [13, 46] $(1+\delta_{Hb\bar{b}})^2(1+\delta_{QCD})$. Numerically, $\delta^{(2)}_{Hb\bar{b}}$ reduces the effect of $\delta^{(1)}_{Hb\bar{b}}$ by about 40%, so that the m_t dependence of $\Gamma(H \to b\bar{b})$ is weakened significantly. It is well known that the large logarithm of δ_{QCD} may be absorbed into the running bottom-quark mass evaluated at scale M_H [47].

4.5 HZZ coupling and $e^+e^- \to ZH$

Also the $\mathcal{O}(G_F m_t^2)$ analysis of Section 3.5 may be straightforwardly extended to $\mathcal{O}(\alpha_s G_F m_t^2)$. This only requires knowledge of the $\mathcal{O}(\alpha_s G_F m_t^2)$ term of $\Pi_{ZZ}(0)$, which may be extracted from Ref. [48],

$$\Pi^{(2)}_{ZZ}(0) = N_c C_F \frac{\alpha_s}{\pi} (g_Z^0 v^0)^2 x_t^0 \left[\frac{4\pi\mu^2}{(m_t^0)^2} \right]^{2\epsilon} \Gamma^2(1+\epsilon) \left[-\frac{3}{2\epsilon^2} + \frac{7}{4\epsilon} - \frac{1}{8} + \mathcal{O}(\epsilon) \right]. \tag{94}$$

Proceeding as in the one-loop case and renormalizing the top-quark mass in the $\mathcal{O}(G_F m_t^2)$ terms according to Eq. (89), we find the non-universal correction to be [14]

$$\delta^{(2)}_{HZZ,nu} = \frac{9}{2} N_c C_F \frac{\alpha_s}{\pi} x_t. \tag{95}$$

Combining this with Eq. (91), we get

$$\delta^{(2)}_{HZZ} = \delta^{(2)}_{HZZ,nu} + \delta^{(2)}_u = \frac{1}{2} \left[\frac{15}{2} - \zeta(2) \right] N_c C_F \frac{\alpha_s}{\pi} x_t, \tag{96}$$

which upgrades the effective HZZ Lagrangian (54) to $\mathcal{O}(\alpha_s G_F m_t^2)$.

Using Eq. (96) together with the well-known $\mathcal{O}(\alpha_s G_F m_t^2)$ correction to the ρ parameter [48],

$$\Delta\rho^{(2)} = -\left[\zeta(2) + \frac{1}{2} \right] N_c C_F \frac{\alpha_s}{\pi} x_t, \tag{97}$$

we obtain from Eq. (56) the corresponding correction to $\sigma(e^+e^- \to ZH)$ [14],

$$\delta^{(2)}_{HZe^+e^-} = N_c C_F \frac{\alpha_s}{\pi} x_t \left\{ 7 - 2\zeta(2) + 8 \frac{c_w^2 Q_e v_e}{v_e^2 + a_e^2} \left[\zeta(2) + \frac{1}{2} \right] \right\}. \tag{98}$$

The QCD correction screens the leading $\mathcal{O}(G_F m_t^2)$ term by about 20% and thus reduces the sensitivity to the top-quark mass.

4.6 HWW coupling and $e^+e^- \to \nu_e\bar{\nu}_e H$

The extension of the effective HWW Lagrangian (61) to $\mathcal{O}(\alpha_s G_F m_t^2)$ proceeds quite similarly to the HZZ case, and we merely list the starting point and the final result. The $\mathcal{O}(\alpha_s G_F m_t^2)$ contribution to $\Pi_{WW}(0)$ may be found in Ref. [48],

$$\Pi^{(2)}_{WW}(0) = N_c C_F \frac{\alpha_s}{\pi}(g_W^0 v^0)^2 x_t^0 \left[\frac{4\pi\mu^2}{(m_t^0)^2}\right]^{2\epsilon} \Gamma^2(1+\epsilon)\left[-\frac{3}{2\epsilon^2} + \frac{1}{4\epsilon} + \zeta(2) - \frac{7}{8} + \mathcal{O}(\epsilon)\right]. \quad (99)$$

The final result is new and reads

$$\delta^{(2)}_{HWW} = \frac{1}{2}\left[\frac{9}{2} - \zeta(2)\right] N_c C_F \frac{\alpha_s}{\pi} x_t. \quad (100)$$

This reduces the magnitude of the $\mathcal{O}(G_F m_t^2)$ term of Eq. (62) by about 8%. As we have seen in Section 3.6, in the on-shell scheme formulated with G_F, $\sigma(e^+e^- \to H\nu_e\bar{\nu}_e)$ is corrected by the factor $(1 + 2\delta_{HWW})$.

5 Conclusions

In this paper, we have reviewed low-energy theorems for the evaluation of one-loop amplitudes with light Higgs bosons as external particles. We have then shown how these theorems may be extended to the two-loop order. These theorems allow us to construct effective Lagrangians for the interactions of the Higgs bosons with other light particles by integrating out the heavy loop particles. We have demonstrated the usefulness of these theorems for practical calculations by presenting a variety of applications to Higgs-boson production and decay processes which will be of major phenomenological relevance at future colliding-beam experiments.

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