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Chiral Fermions Coupled to Lattice Gauge Fields

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ABSTRACT

Lattice fermions have well-known difficulties with chiral symmetry. To evade them it is possible to couple *continuum* fermions to *lattice* gauge fields, by introducing an interpolation of the latter. Following this line of thinking, this paper presents two Euclidean formulations of the effective action that appears after functional integration over fermion fields, one for vector-like and the other for chiral couplings. With suitable finite-mode regulators both effective actions can be evaluated in a finite computation. The prescriptions provided here contain some details not found in previous work marrying continuum fermions to the lattice via an interpolation. For example, the counter-terms needed to maintain chiral gauge invariance are explicitly given. By construction coupling-constant renormalization, anomaly structure, and (in the chiral gauge theory) fermion nonconserving amplitudes all satisfy one's expectations from perturbative and semi-classical analyses.



1 Introduction

A long-standing problem in quantum field theory is a nonperturbative formulation of chiral fermions. Our only general nonperturbative formulation of quantum field theory is the renormalization-group limit of functional integrals defined on a lattice. But when chiral symmetry is an issue, there are notorious problems [1, 2]. Briefly, one must either sacrifice locality or positivity, tolerate additional states (doubling or mirror states), or break chiral symmetry explicitly. When the coupling of fermions to gauge fields is vector-like, the standard formulations [3, 4] are adequate, if imperfect. On the other hand, when fermions couple to chiral gauge fields, it has been difficult to prove a conceptually clean theory; see ref. [5] for a review. An encouraging proposal replaces the functional integral over fermions with an auxiliary quantum-mechanical system [6], inspired by domain-wall [7] and lattice Pauli-Villars [8] methods.

This paper offers constructions of vector-like and chiral gauge theories, coupling continuum fermions to lattice gauge fields by introducing an interpolation of the latter. Ideas of this type were first discussed by Flume and Wyler for the Schwinger model [9], and recently 't Hooft advocated a similar approach for four-dimensional gauge theories [10]. The appeal stems from the nontrivial (instanton) topology of continuum gauge fields, because the Atiyah-Singer index theorem [11] implies an intimate relation between chiral properties of fermions and the topology of the gauge field.

In 1987–1988 there was some discussion about topology and fermions in lattice gauge theory [12, 13]. Except for a conference reports [14], however, none of the applications to chiral gauge theories have been published. Spurred by ref. [10], I would like to present my variation on the theme.

Ref. [10] regulates the gauge field with the lattice and the fermions with a standard Pauli-Villars scheme. The number of fermionic degrees of freedom (per unit volume) remains infinite—in the words of Smit, the method is desperate [13]. In particular, a numerical evaluation of the effective action would require infinite computation, even at fixed cutoff. This paper, on the other hand, examines a sharp cutoff on determinants, which was first studied in ref. [15]. The number of fermionic degrees of freedom is now finite, so the numerical computation of the effective action is finite too.

Another difference between this paper's proposal and the one in ref. [10] is the strategy for removing the cutoffs. Let a denote the lattice spacing and M the ultraviolet cutoff on fermions. In the formulation of ref. [10] one takes $M \rightarrow \infty$ for a fixed, and afterwards $a \rightarrow 0$. As stressed in sect. 7.3 the cutoff in ref. [10] maintains the gauge symmetry of the chiral theory only in the $M \rightarrow \infty$ limit. With the sharp cutoff formulated below, however, it is permissible and natural to take $a \rightarrow 0$, $M \rightarrow \infty$ with Ma constant. The latter approach is far superior in a numerical computation. If M must vary at fixed a , it will be extremely difficult to generate useful ensembles of lattice gauge fields, because to obtain a renormalized theory the bare gauge coupling must depend on M .

Sect. 2 begins with a discussion of vector-like theories. The analysis starts with

anomalies, because they are such a stumbling block for lattice formulations [1]. The functional integral over fermion fields is defined in the manner of Fujikawa [16], and a specific cutoff procedure is formulated and justified in sect. 3. These sections define an effective action for a continuous background gauge field. Sect. 4 summarizes the essential features of an interpolation from the lattice field of parallel transporters to a connection. Sect. 5 derives a heuristic relation between the fermion measure of this proposal and the usual measure of lattice fermions. While inessential to the main line of argument, the derivation suggests a rationale for relating the cutoff on determinants to the lattice spacing. Sect. 6 briefly considers numerical aspects. As usual, the generalization to chiral gauge theories is not immediate, but sect. 7 produces a satisfactory definition of the chiral effective action, including fermion nonconservation. Finally, sect. 8 remarks on some of the loose ends, and compares the status of this formulation with ref. [6].

2 Vector-like gauge theory

The formal expression for the Euclidean functional integral for fermions is

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S(A, \psi, \bar{\psi})}. \quad (2.1)$$

Let us assume a background gauge potential (or connection) A_μ . In the application to lattice field theory, this connection is determined from the lattice gauge field, cf. sect. 4. Staying momentarily “in the continuum,” the action is

$$S = \int d^4x \bar{\psi}(x)(\not{D} + m)\psi(x). \quad (2.2)$$

Formal integration over the fermion fields yields the Boltzmann factor

$$e^{-\Gamma(A)} = \text{Det}(\not{D} + m). \quad (2.3)$$

The objective is to give a rigorous meaning to the measure $\mathcal{D}\psi \mathcal{D}\bar{\psi}$, and/or to the determinant.

Consider the eigenfunctions and eigenvalues of the Dirac operator \not{D}

$$i\not{D}\varphi_n = \lambda_n\varphi_n. \quad (2.4)$$

Since \not{D} is anti-Hermitian, the λ_n are real. The Dirac operator transforms covariantly under the gauge group, so the λ_n are gauge invariant. The Dirac operator anti-commutes with γ_5 , i.e. $\gamma_5\not{D} = -\not{D}\gamma_5$. Hence, if φ_n is an eigenfunction with eigenvalue λ_n , then $\gamma_5\varphi_n$ is an eigenfunction with eigenvalue $-\lambda_n$. As usual, one imagines that the theory is defined on a compact space-time, and the infinite volume limit is taken at the end. Then the spectrum of \not{D} is discrete.

The Dirac operator can possess zero modes, $\lambda_n = 0$. In this subspace it is convenient to sort the eigenfunctions according to chirality, i.e. the eigenvalue of

γ_5 . Let n_{\pm} be the number of modes with $D\varphi_n = 0$ and $\gamma_5\varphi_n = \pm\varphi_n$, and let $n_0 = n_+ + n_-$. The determinant should then be

$$\text{Det}(D + m) = \prod_n (-i\lambda_n + m) = m^{n_0} \prod_{\lambda_n > 0} (\lambda_n^2 + m^2), \quad (2.5)$$

except that the infinite product still requires an ultraviolet regulator; this is postponed to sect. 3. (The second equality follows because nonzero eigenvalues come in pairs $\pm\lambda_n$.)

It is easy to see that the eigenvectors are orthonormal and form a complete set:

$$\sum_a \int d^4x \varphi_n^{a*}(x) \varphi_m^a(x) = \delta_{nm}, \quad (2.6)$$

$$\sum_n \varphi_n^i(x) \varphi_n^{j*}(y) = \delta(x - y) \delta^{ij}, \quad (2.7)$$

where i, j denote spinor and color indices. These properties of the eigenfunctions permit the expansions

$$\begin{aligned} \psi(x) &= \sum_n a_n \varphi_n(x), \\ \bar{\psi}(x) &= \sum_n \bar{a}_n \varphi_n^\dagger(x), \end{aligned} \quad (2.8)$$

where the coefficients a_n and \bar{a}_n are Grassman numbers. To obtain eq. (2.5) from eqs. (2.1) and (2.2) one takes the functional integral over fields given by eqs. (2.8), i.e. the fermion measure is *defined* to be

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} := \prod_n da_n d\bar{a}_n. \quad (2.9)$$

Fujikawa [16] makes a formal argument¹ to relate the right-hand side of eq. (2.9) to $\prod_x d\bar{\psi}(x) d\psi(x)$. Since the product over a continuous index is formal, it is logically cleaner to assert eq. (2.9) as a definition.

The Fujikawa measure is analogous to one based on Fourier modes,

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \sim \prod_k d\psi(k) d\bar{\psi}(k). \quad (2.10)$$

The momenta correspond to eigenvalues of \not{D} , and are also discrete (in a box). With a coupling to gauge fields, however, the momenta are not gauge invariant. Hence, the definition based on eigenfunctions of D is preferable.

One must check that the formalism reproduces the axial anomaly. Consider space-time dependent chiral transformations

$$\begin{aligned} \psi(x) &= e^{-\alpha^a(x) T^a \gamma_5} \psi'(x), \\ \bar{\psi}(x) &= \bar{\psi}'(x) e^{-\alpha^a(x) T^a \gamma_5}, \end{aligned} \quad (2.11)$$

¹Sect. 5 pursues a similar, yet complementary, line of thought.

and $T^a = -(T^a)^\dagger$ is a generator of a global symmetry group. (For U(1) take $T = i$.) Under such transformations the measure of eq. (2.9) is not invariant, but

$$a'_n = \int d^4x \varphi_n^\dagger(x) e^{\alpha^a(x) T^a \gamma_5} \varphi_m(x) a_m =: C_{nm} a_m, \quad (2.12)$$

and similarly for \bar{a}'_n , with sums on m implied. The rules of Berezin integration imply

$$\prod_n da_n = \text{Det } C \prod_n da'_n. \quad (2.13)$$

The Jacobian determinants are then responsible for the anomaly. Using an eigenfunction expansion of $\alpha^a(x) T^a \gamma_5$ one can write $C_{nm} = \exp c_{nm}$ with

$$c_{nm} = \int d^4x \varphi_n^\dagger(x) \gamma_5 T^a \varphi_m(x) \alpha^a(x). \quad (2.14)$$

Then

$$\text{Det } C = e^{\text{Tr } c} = \exp \left(\int d^4x \mathcal{A}^a(x) \alpha^a(x) \right), \quad (2.15)$$

where the anomaly

$$\mathcal{A}^a(x) = \sum_n \varphi_n^\dagger(x) \gamma_5 T^a \varphi_n(x). \quad (2.16)$$

Collecting the anomaly from $\mathcal{D}\bar{\psi}$ and $\mathcal{D}\psi$, as well as terms from the chiral transformation of the action, yields the (anomalous) Ward-Takahashi identity

$$\partial_\mu \bar{\psi}(x) \gamma_\mu \gamma_5 T^a \psi(x) = \bar{\psi}(x) \{T^a, m\} \gamma_5 \psi(x) + 2\mathcal{A}^a(x). \quad (2.17)$$

3 Regulating the fermions

The preceding discussion skirts the need to regulate the determinants in the ultraviolet. Let us start with a kind of Pauli-Villars regulator. Eq. (2.16) becomes

$$\mathcal{A}_{\text{reg}}^a(x) = \sum_n f_{\epsilon_N} (\lambda_n^2 / M_N^2) \varphi_n^\dagger(x) \gamma_5 T^a \varphi_n(x). \quad (3.1)$$

Let the index n run over $\{1 - n_0, \dots, 0, 1, 2, \dots\}$ with the convention that $n > 0$ denotes nonzero modes; let us order the nonzero modes by λ_n^2 and take n odd (even) if λ_n is negative (positive). The regulating function $f_\epsilon(x)$ is chosen to look like the sketch in fig. 1. The cutoff M_N and smearing parameter ϵ_N should satisfy

$$|\lambda_N| < M_N < |\lambda_{N+1}|, \quad (3.2)$$

$$\epsilon_N \ll (\lambda_{N+1}^2 - \lambda_N^2) / M_N^2, \quad (3.3)$$

where N is even (so $\lambda_{N+1} \neq -\lambda_N$), and one takes $\epsilon_N \rightarrow 0$ with M_N fixed.² The Fermi function

$$f_\epsilon(x) = \frac{e^{-1/\epsilon} + 1}{e^{(x-1)/\epsilon} + 1} \quad (3.4)$$

²The notation ignores accidentally degenerate nonzero modes, but it is clear what to do.

changes rapidly from 1 to 0 as $\text{Re } x$ passes through 1. Since $f_\epsilon(0) = 1$ and f_ϵ and all its derivatives vanish at ∞ , it supports the standard analysis [16] that leads to

$$\mathcal{A}_{\text{reg}}^a = \frac{\text{tr } T^a}{16\pi^2} \text{tr}[F_{\mu\nu} * F_{\mu\nu}] \quad (3.5)$$

in four dimensions, where the two traces are over global and gauge-group indices. Details in Appendix A show that all higher dimension terms, e.g. $\text{tr}(F^{n+2})/M^{2n}$, are proportional to $\epsilon_N^{-n} e^{-1/\epsilon_N}$ as $\epsilon_N \rightarrow 0$.

To check that the regulator does not spoil the derivation of eq. (2.17), one restricts the chiral transformations $\alpha(x)$ to those for which

$$\int d^4x \varphi_n^\dagger(x) \alpha^a(x) T^a \gamma_5 \varphi_m(x) = 0, \quad (3.6)$$

if one of $n, m > N$. If both n and $m > N$, then the left-hand side of eq. (3.6) need not vanish. The Jacobian matrix C takes a block form, and the regulator decouples the block with large eigenvalues. Nevertheless, the transformation function $\alpha(x)$ is sufficiently arbitrary to derive the Ward-Takahashi identity.

The $\epsilon_N \rightarrow 0$ limit corresponds to keeping the first N modes of D completely and eliminating the rest. In a loose analogy with the lattice cutoff, in which the Fourier measure of eq. (2.10) is truncated, one can carry out the truncation of the modes throughout. Thus, the regulated measure is

$$(\mathcal{D}\psi \mathcal{D}\bar{\psi})_N = \prod_{n=1-N_0}^N da_n d\bar{a}_n, \quad (3.7)$$

with eigenvalues ordered as above. The functional integral is now over fields given

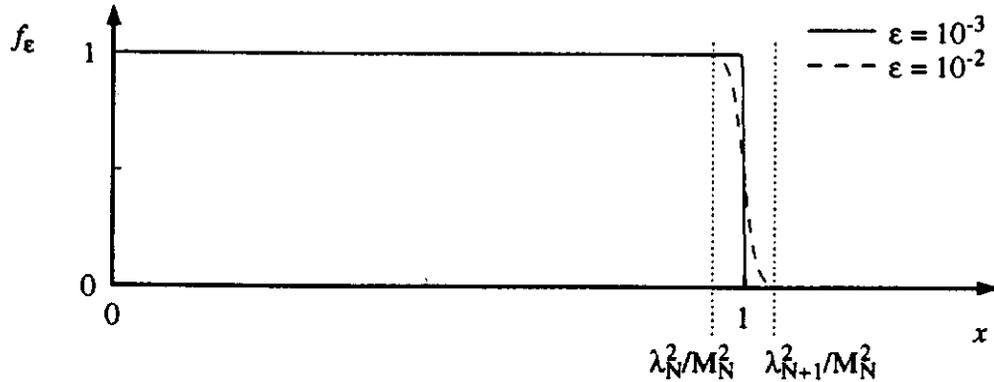


Figure 1: The shape of the regulator function $f_\epsilon(x)$, which is designed so that modes with eigenvalues $\lambda_n^2 > \lambda_N^2$ are omitted.

by

$$\begin{aligned}\psi(x) &= \sum_{n=1-n_0}^N a_n \varphi_n(x), \\ \bar{\psi}(x) &= \sum_{n=1-n_0}^N \bar{a}_n \varphi_n^\dagger(x).\end{aligned}\tag{3.8}$$

The effective Boltzmann factor becomes

$$e^{-\Gamma_N(A)} = \prod_{n=1-n_0}^N (-i\lambda_n + m) = m^{n_0} \prod_{\substack{n=1 \\ \lambda_n > 0}}^N (\lambda_n^2 + m^2).\tag{3.9}$$

Similarly, the regulated fermion propagator is

$$\int (\mathcal{D}\psi \mathcal{D}\bar{\psi})_N \psi(x) \bar{\psi}(y) e^{-S} = \sum_{n=1-n_0}^N \varphi_n(x) \frac{1}{-i\lambda_n + m} \varphi_n^\dagger(y) e^{-\Gamma_N(A)}.\tag{3.10}$$

The finite-mode cutoff has been examined before [15], with emphasis on anomalies. Those papers did not notice the disappearance of higher-dimensional terms as the Fermi function becomes infinitely sharp, a feature that is especially important in the chiral gauge theory, cf. sect. 7.

In perturbation theory, one must be careful to impose the sharp cutoff with \mathcal{D} , rather than with loop momenta. For example, the dependence of the regulator on \mathcal{D} , and hence on A_μ , induces additional terms in the gauge current, to maintain gauge invariance.

How should one choose N ? One ought to take nonzero modes in pairs: it would be silly to take λ_n and not $-\lambda_n$. One also ought to retain all zero modes, because they are the most infrared of all! But from the Atiyah-Singer index theorem [11] the number of zero modes is even (odd) if the topological charge Q is even (odd). Thus, N must depend on the gauge field and be even (odd) if Q is even (odd).

The order of magnitude of N can be specified only vaguely. An examination of cutoff effects, Appendices A and B, introduces expansions. For a generic cutoff, one would need to maintain $A_\mu/M \ll 1$. The absence of non-universal terms with the finite-mode cutoff, however, permits a lower cutoff. If, as in the next section, A_μ is obtained from a lattice gauge field, $|A| < C/a$, where C is a gauge-group dependent constant. With the finite-mode cutoff it is thus natural to take $M_N \sim a^{-1}$. Indeed, on an $N_S^3 \times N_T$ lattice one expects

$$N_{\text{lat}} = 4RN_S^3 N_T\tag{3.11}$$

fermionic degrees of freedom in the lattice fermion field (per flavor), where R is the dimension of the fermion's gauge-group representation. An obvious choice would be to augment the usual number with the zero modes, i.e. $N = n_0 + N_{\text{lat}}$.

4 Coupling to a lattice gauge field

To extend the formulation to lattice gauge theory, one must provide an interpolation determining a connection $A_\mu(x)$ from the lattice gauge field $U_\mu(x)$. Ref. [10] gives a prescription adequate for defining traces of products of $F_{\mu\nu}$, but does not prescribe A_μ . Other possibilities, based on definitions of the topological (instanton) charge of a lattice gauge field [17, 18], do prescribe A_μ explicitly [19, 20]. The latter have two crucial properties:

1. The only singularities in A_μ are instanton-like.
2. The interpolation A_μ transforms as a connection under *lattice* gauge transformations.

Ref. [10] does prove a useful theorem on the spectrum of the Dirac operator for a bounded gauge field. The (manifestly) topological interpolations obey the hypothesis of the theorem.

The great insight of ref. [17] was to recognize that the lattice gauge field can be used to define a fiber bundle. Then the topological charge is the second Chern number of the bundle. Lüscher originally provided expressions for transition functions, which encode changes of gauge

$$A_\mu^{(\alpha)} = v_{\alpha\beta}(\partial_\mu + A_\mu^{(\beta)})v_{\beta\alpha} \quad (4.1)$$

from a patch α of space-time to a neighboring one β . For consistency, $v_{\beta\alpha} = v_{\alpha\beta}^{-1}$. In the fiber-bundle formalism, the gauge is fixed separately within each patch, such that $A_\mu^{(\alpha)}$ has no singularities. The winding responsible for topological charge then resides in the transition functions [17, 21]. The more familiar patch-independent connection is

$$A_\mu = w_\alpha^{-1}(x)(\partial_\mu + A_\mu^{(\alpha)})w_\alpha(x), \quad (4.2)$$

where the section w_α is related to the transition functions by

$$v_{\alpha\beta} = w_\alpha w_\beta^{-1}. \quad (4.3)$$

The first step of refs. [19, 20] is a patch-wise continuous, bounded interpolation for the nonsingular $A_\mu^{(\alpha)}$. Eq. (4.2) produces a continuous, bounded connection A_μ , but when the transition functions have nontrivial winding, the sections, and thus A_μ , have directional singularities. By construction, therefore, the *only* singularities in the globally defined A_μ are those induced by the instanton-winding of the section.

The other crucial property of the reconstructed gauge potentials of refs. [19, 20] is the response to a lattice gauge transformation. The transformation law of the lattice gauge field is

$${}^g U_\mu(s) = g(s)U_\mu(s)g^{-1}(s + \hat{\mu}), \quad (4.4)$$

where $s, s + \hat{\mu}$ denote lattice sites. For x in the patch σ associated with s , the reconstructed section obeys [22, 20]

$${}^g w_\sigma(x) = g(s)w_\sigma(x)g^{-1}(x). \quad (4.5)$$

The interpolation $g(x)$ is independent³ of σ , but, of course, $g(s)$ is independent of x . The interpolated connection transforms as

$${}^g A_\mu = g(x)(\partial_\mu + A_\mu)g^{-1}(x), \quad (4.6)$$

with the *same* function $g(x)$.

If $g(s) = g_2(s)g_1(s)$ at each site, the three interpolated gauge transformation fields obey the composition law

$$g(x; U) = g_2(x; {}^{g_1}U)g_1(x; U). \quad (4.7)$$

Note that the interpolation depends on the underlying lattice gauge field, which is emphasized here by the second argument. Consequently, every interpolated $g(x; U)$ can be built up from infinitesimal, site-by-site steps.

For the present discussion the complicated expressions for the interpolated $g(x)$ and A_μ are not illuminating. Interested readers can consult refs. [22, 19] for Lüscher's bundle. (For Phillips and Stone's bundle, analogous results can be obtained [20].)

Our prescription for coupling fermions to lattice gauge fields is to start with the lattice gauge field U_μ , interpolate to obtain the connection A_μ , use the associated Dirac operator to define the measure, and regulate the determinant using the sharp cutoff. Because the Dirac operator is constructed using fiber bundles, it automatically satisfies the Atiyah-Singer index theorem

$$n_+ - n_- = Q, \quad (4.8)$$

where Q is the topological charge (as defined by ref. [17]) of the lattice gauge field. Moreover, from eq. (4.6) the eigenvalues of the Dirac operator, and hence the effective action, are invariant under lattice gauge transformations. From eq. (3.9) the Boltzmann factor $e^{-\Gamma_N(U)}$ is also positive and finite.

5 Relation to lattice fermion fields⁴

It is intriguing to contrast the regulated measure of eq. (3.7) with the usual one

$$(\mathcal{D}\psi\mathcal{D}\bar{\psi})_{\text{lat}} = \prod_s d\psi(s)d\bar{\psi}(s), \quad (5.1)$$

where $\psi(s)$ and $\bar{\psi}(s)$ denote the lattice fermion field and its conjugate at site s . There is an interpolation of the fermion field analogous to A_μ . In particular, under a lattice gauge transformation the interpolated fermion field transforms as

$${}^g\psi(x) = g(x)\psi(x), \quad {}^g\bar{\psi}(x) = \bar{\psi}(x)g^{-1}(x), \quad (5.2)$$

with the same interpolated $g(x)$ mentioned in sect. 4.

³If $x \in \sigma \cap \tau$, eq. (4.3) requires that the interpolation obey $g(x)|_\sigma = g(x)|_\tau$.

⁴This section is a diversion, and the rest of the paper makes no use of it.

Let us expand the interpolated field $\psi(x)$ as in eq. (2.8), use the orthonormality to solve for the a_n and take differentials. Then

$$\begin{aligned} da_n &= \sum_s \int d^4x \varphi_n^\dagger(x) \frac{\partial \psi(x)}{\partial \psi(s)} d\psi(s) =: \sum_s m_{ns} d\psi(s), \\ d\bar{a}_n &= \sum_s \int d^4x d\bar{\psi}(s) \frac{\partial \bar{\psi}(x)}{\partial \bar{\psi}(s)} \varphi_n(x) =: \sum_s d\bar{\psi}(s) \bar{m}_{sn}. \end{aligned} \quad (5.3)$$

If the number of modes N equals the number of lattice sites, m and \bar{m} are $N \times N$ matrices and one can write

$$\prod_n da_n d\bar{a}_n = (\det^{-1} m \times \det^{-1} \bar{m}) \prod_s d\psi(s) d\bar{\psi}(s). \quad (5.4)$$

The determinants here are akin to ones appearing in ref. [16]. Formally, one can combine the determinants into a “metric”

$$g_{\bar{s}s} = \bar{m}_{\bar{s}n} m_{ns} = \int d^4x \frac{\partial \bar{\psi}(x)}{\partial \bar{\psi}(\bar{s})} \frac{\partial \psi(x)}{\partial \psi(s)}, \quad (5.5)$$

and rewrite the measure as

$$\prod_n d\bar{a}_n da_n = \det g^{-1} \prod_s d\psi(s) d\bar{\psi}(s). \quad (5.6)$$

The metric depends covariantly on the gauge field, by construction of the interpolation.

One can develop a geometric picture by imagining a Grassman line element

$$\sum_{\bar{s}s} g_{\bar{s}s} d\bar{\psi}(\bar{s}) d\psi(s) = \sum_n d\bar{a}_n da_n \quad (5.7)$$

that is gauge invariant. In this language the customary lattice fields are curvilinear coordinates, and the expansion coefficients are rectilinear. According to this picture, the usual lattice measure mistakenly neglects the curvature. Note that without the gauge interaction, the metric becomes flat: the two bases are then Fourier transforms of one another.

The metric also becomes trivial in the naive continuum limit. Then the interpolation is unnecessary and $\partial \psi(x)/\partial \psi(s) = \delta(x-s)$. The factor $\det g^{-1}$ is essential, however, for obtaining the correct anomaly, index theorem, etc. In particular, the anomaly arises because one may remove the regulators only after calculating with the functional integral, not (as in the naive continuum limit) before.

Of course, eq. (5.6) is merely heuristic. To make the manipulations rigorous, the interpolated fields must be smooth enough that the Dirac eigenmode expansions leading to eq. (5.3) stop at the N th term. Here N is both the number of modes kept and the number of lattice sites. This condition presumably puts constraints on the smoothness of the lattice gauge field similar to, but perhaps more stringent

than, those imposed by uniqueness considerations in the fiber bundle constructions [17, 18]. One should emphasize, however, that such constraints are on the connection between the present measure and the usual lattice one. They do not detract from eq. (2.9) as a definition of the functional integral, or $e^{-\Gamma_N(U)}$ as a definition of the effective action.

6 Computational Considerations

Since the effective action $\Gamma_N(U)$ is real and positive, the functional integral over the gauge bosons is amenable to the Monte Carlo method with importance sampling. The Dirac eigenvalues, and hence the effective action, depend on the lattice gauge field U so they must be recomputed for every change of U . If the effective action is defined with a Gaussian $e^{-\lambda_n^2/M^2}$ [16] or standard Pauli-Villars [10] regulator, one would need to compute all eigenvalues of \hat{D} and weight them appropriately—an infinite computation. With eq. (3.9), however, only the lowest N eigenvalues are needed—a finite computation.⁵

To compute the eigenvalues one can introduce an auxiliary lattice much finer than the original one. Parallel transporters for the fine lattice are constructed from the interpolated gauge field. Now consider any discretization of the Dirac operator, and denote its eigenvalues by d_n . The discretization and auxiliary lattice spacing must be chosen such that $d_n = \lambda_n$ (up to tolerable floating-point precision) for $n \leq N$. On a fine enough auxiliary lattice, “any” discretization becomes precise enough. To avoid problems sorting the eigenvalues, however, a discretized operator without a doubled spectrum is preferable. Suitable examples would be the Wilson discretization [3], or one derived by gauging the fixed-point action of a free fermion [23]. The discretization may break chiral symmetry, provided the breaking is numerically significant only for modes above the cutoff N .

Similar remarks apply to the construction for chiral gauge theories, sect. 7. An important difference is that the Boltzmann factor $e^{-\Gamma_N(U)}$ can be complex (for fermions in “complex” representations), and in any case not positive definite. Monte Carlo integration is then much more difficult, because fluctuations in sign reduce the effectiveness of importance sampling, cf. sect. 7.4. Nevertheless, $e^{-\Gamma_N(U)}$ can be evaluated with finite computation.

7 Chiral fermions

7.1 General remarks

The preceding sections provide a definition of the functional integral for vector-like fermions. It is nonperturbative, has the correct axial anomaly, and by construction provides a natural association between a Dirac operator and a topological charge,

⁵Conceding round-off error, the “infinite” computation is ϵ^{-1} times more costly, where ϵ characterizes machine precision.

so that the index theorem is obeyed. The ideas are now extended to chiral fermions, which appear in the Standard Model and in grand unified theories.

The essential feature of chiral gauge theories is that positive and negative chirality fermions transform under different representations of the gauge group. In a basis of the Dirac matrices with γ_5 diagonal it is useful to split the four-component Dirac spinor into two two-component Weyl spinors. Without loss of generality one can charge-conjugate the negative chirality part and assemble everything into one positive chirality field. This Weyl spinor is henceforth denoted ψ_+ , and the representation of the gauge group under which it transforms is denoted ρ , with generators t^a . The kinetic term of the action is

$$S = \int d^4x \psi_+^\dagger D_+ \psi_+. \quad (7.1)$$

As before one wants to define the Boltzmann factor

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi_+ \mathcal{D}\psi_+^\dagger e^{-S(A, \psi_+, \psi_+^\dagger)}. \quad (7.2)$$

But D_+ maps positive chirality Weyl spinors into negative chirality Weyl spinors. The underlying difficulty in constructing a chiral gauge theory is that the right-hand side of eq. (7.2) is *not* a (functional) determinant.

A well-formulated chiral gauge theory should also exhibit fermion nonconservation [24]. If the vector-like operator \hat{D} has zero modes, the right-hand side of eq. (7.2) should vanish: Recall that the zero modes have definite chirality. Thus the zero modes possess natural projections from Dirac spinors φ_\pm onto Weyl spinors ϕ_\pm . For positive chirality the projection implies $\hat{D}_+ \phi_+ = 0$, and for negative chirality $\phi_-^\dagger \hat{D}_+ = 0$ (integration by parts implied). In eq. (7.2) integration over the ϕ_+ component of ψ_+ or over the ϕ_-^\dagger component of ψ_+^\dagger yields zero. On the other hand, if there are n_\pm zero modes of each chirality (counting individual species appropriately) the integral

$$\mathcal{Z}(\nu_+, \nu_-) = \int \mathcal{D}\psi_+ \mathcal{D}\psi_+^\dagger (\psi_+^\dagger)^{\nu_-} (\psi_+)^{\nu_+} e^{-S(A, \psi_+, \psi_+^\dagger)} \quad (7.3)$$

does not vanish if $\nu_\pm \geq n_\pm$. (The notation $(\psi_+)^{\nu_+}$ is schematic for a product of n_+ suitable components or positions of ψ_+ .) Amplitudes for fermion nonconservation are proportional to integrals like $\mathcal{Z}(\nu_+, \nu_-)$.

Before discussing how to regulate the integrals in eqs. (7.2) and (7.3), one should list the properties of eq. (7.2) that the regulator should respect. In addition to the connection between zero modes and fermion nonconservation, one wants

1. $\text{Re } \Gamma(gA) = \text{Re } \Gamma(A)$, under all circumstances.
2. $\text{Im } \Gamma(gA) = \text{Im } \Gamma(A)$, only if ρ is ‘‘anomaly-free.’’
3. $\text{Im } \Gamma(A) \neq 0$, if ρ is complex;⁶ indeed $\text{Im } \Gamma(\rho^*(A)) = -\text{Im } \Gamma(\rho(A))$.

⁶If there is a unitary matrix u such that $t^a = u t^{a*} u^\dagger$, then ρ is real; otherwise it is complex. The representation generated by t^{a*} is denoted ρ^* . If ρ is real, then $e^{-\Gamma(A)}$ is real.

Here ${}^g A_\mu = g(\partial_\mu + A_\mu)g^{-1}$ is a gauge transform of A_μ , and the notation $\Gamma(\rho(A))$ stresses the fermion representation. Condition 2 is imposed so that the nonperturbative definition reproduces perturbation theory. Condition 3 identifies a diagnostic feature of the effective action of chiral fermions.

Even though the right-hand side of eq. (7.2) is by nature not a determinant, ultraviolet regulators almost always turn it into one. For example, in the most naive (and unsuccessful) lattice formulation, D_+ becomes a large, square, numerical matrix, and $\det D_+$ is the matrix determinant. So let us provisionally assume

$$e^{-\Gamma(A)} = \prod_n^N (-i\lambda_n). \quad (7.4)$$

In anticipation of a cutoff analogous to the one for vector-like theories (sect. 3), the product runs only over the first N eigenvalues. The most transparent way to realize Conditions 1 and 2 is as follows: Write $\lambda_n = ie^{l_n + i\theta_n}$, with l_n and θ_n real, and note that

$$\operatorname{Re} \Gamma(A) = -\sum_n^N l_n, \quad \operatorname{Im} \Gamma(A) = -\sum_n^N \theta_n \pmod{2\pi}. \quad (7.5)$$

If the moduli $|\lambda_n| = e^{l_n}$ of the eigenvalues are gauge invariant and one orders the eigenvalues by $|\lambda_n|$, Condition 1 is satisfied. By Condition 2, in anomalous (sub)representations the phases of the eigenvalues would be gauge variant, but the variation from one species could cancel that of another. Unfortunately, it seems that eigenvalue problems with these simple gauge-transformation properties leave the total phase $\operatorname{Im} \Gamma$ unspecified, flouting Condition 3.⁷

7.2 A specific formulation

A standard way to cast the effective action as a determinant is to introduce a new negative chirality partner ψ_- with no dynamics. The action is now

$$S = \int d^4x (\psi_+^\dagger D_+ \psi_+ + \psi_-^\dagger \not{\partial} \psi_-) = \int d^4x \bar{\psi} \hat{D} \psi, \quad (7.6)$$

where the four-component spinor

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \bar{\psi} = (\psi_-^\dagger \quad \psi_+^\dagger), \quad (7.7)$$

and

$$\hat{D} := D_+ + \not{\partial} = \begin{pmatrix} 0 & \not{\partial} \\ D & 0 \end{pmatrix}. \quad (7.8)$$

⁷Ref. [14] adopts the spirit of this realization. There $\operatorname{Re} \Gamma$ is related to the vector-like theory and $\operatorname{Im} \Gamma =: \eta$ to the spectral asymmetry of a certain operator [25]. This method, however, requires three regulators: one for the gauge fields, one for the vector-like fermions, and one for η .

The matrix forms of eqs. (7.7) and (7.8) presumes a γ -matrix basis with γ_5 diagonal. Formally, the functional integral over $\mathcal{D}\psi_- \mathcal{D}\psi_+^\dagger$ is trivial, but the combined integral

$$e^{-\Gamma} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S(\psi, \bar{\psi}, A)} \quad (7.9)$$

can be expressed as a determinant, as in the vector-like theory.

The operator $i\hat{D}$ is not self-adjoint, but it is elliptic [26], so on a compact space-time it still has a discrete spectrum. The eigenvalue problem now has different right and left eigenfunctions. There are zero modes

$$i\hat{D}\varphi_{+,n} = 0, \quad (i\hat{D})^\dagger\varphi_{-,n} = 0, \quad (7.10)$$

where $\varphi_{\pm,n}$ are zero-mode eigenfunctions of the vector-like operator \hat{D} , with chirality ± 1 . Nonzero modes come in pairs⁸

$$\begin{aligned} i\hat{D}\eta_n &= \lambda_n \eta_n, \\ (i\hat{D})^\dagger\chi_n &= \lambda_n^* \chi_n \end{aligned} \quad (7.11)$$

that are mutually orthonormal:

$$\int d^4x \chi_n^\dagger(x) \eta_m(x) = \delta_{nm}. \quad (7.12)$$

In addition $\int d^4x \chi_n^\dagger \varphi_{+,m} = \int d^4x \varphi_{-,n}^\dagger \eta_m = \int d^4x \varphi_{-,n}^\dagger \varphi_{+,m} = 0$. Furthermore, since $\gamma_5 \hat{D} = -\hat{D} \gamma_5$, if η_n has eigenvalue λ_n , then $\gamma_5 \eta_n$ has eigenvalue $-\lambda_n$, and similarly for χ_n , $\gamma_5 \chi_n$.

The functional integral is now defined to be over fields

$$\begin{aligned} \psi(x) &= \sum_{n=1}^{n_+} z_n \varphi_{+,n}(x) + \sum_{n=1}^N a_n \eta_n(x), \\ \bar{\psi}(x) &= \sum_{n=1}^{n_-} \bar{z}_n \varphi_{-,n}(x) + \sum_{n=1}^N \bar{a}_n \chi_n^\dagger(x), \end{aligned} \quad (7.13)$$

i.e. the measure is

$$\mathcal{D}\psi_- \mathcal{D}\psi_+^\dagger \mathcal{D}\psi_+ \mathcal{D}\psi_-^\dagger = \mathcal{D}\psi \mathcal{D}\bar{\psi} = \prod_{n=1}^N da_n d\bar{a}_n \prod_{n=1}^{n_+} dz_n \prod_{n=1}^{n_-} d\bar{z}_n. \quad (7.14)$$

As in sect. 3 the functional integral is cut off by retaining only the lowest N nonzero modes. The principle for ordering the eigenvalues is revealed below, but clearly

⁸The eigenvalue problem described here is implicitly adopted by ref. [10] and many other papers [15, 27, 28]. Often the literature discusses models that couple fermions to external fields via $\not{\partial} + \not{Y} + A\gamma_5$. This operator is again not self-adjoint, though elliptic, so there are left and right eigenfunctions $\chi_n \neq \eta_n$. Most papers either ignore this subtlety, or try to circumvent it. For example, some tricks turn $i\hat{D}$ into a Hermitian operator, thus leaving $\text{Im } \Gamma$ unspecified. They cannot be adopted here.

the two modes with the same λ_n^2 should be adjacent. Analogously to sect. 3, it is convenient to take n odd (even) if $\text{Re } \lambda_n$ is negative (positive).

Let us first integrate over the a_n and \bar{a}_n . Eqs. (7.13) and (7.14) yield

$$e^{-\tilde{\Gamma}_N(A)} = \prod_{n=1}^N (-i\lambda_n) = \prod_{\substack{n=1 \\ \text{Re } \lambda_n > 0}}^N \lambda_n^2. \quad (7.15)$$

for the functional integral. In a real representation these eigenvalues come in pairs $(-i\lambda_n, i\lambda_n^*)$; $\tilde{\Gamma}_N$ satisfies Condition 3. But one should not expect $\tilde{\Gamma}_N(A)$ to be a suitable definition of the effective action—hence the tilde—because neither the moduli nor the phases of the nonzero eigenvalues are gauge invariant.

Under the gauge transformation $g = e^\omega$ the chiral Dirac operator transforms as ${}^g\hat{D} = e^{\omega P_-} \hat{D} e^{-\omega P_+}$, where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$. The zero modes are gauge invariant. To first order in ω the nonzero eigenvalues vary by

$${}^g\lambda_n = \lambda_n \left(1 - \int d^4x \chi_n^\dagger(x) t^a \gamma_5 \eta_n(x) \omega^a(x) \right), \quad (7.16)$$

which immediately yields the variation of $\tilde{\Gamma}_N$:

$$\delta_\omega \tilde{\Gamma}_N = \int d^4x \mathcal{A}_{\text{reg}}^a(x) \omega^a(x), \quad (7.17)$$

where

$$\mathcal{A}_{\text{reg}}^a(x) = \sum_{n=1}^N \chi_n^\dagger(x) t^a \gamma_5 \eta_n(x) = \lim_{\epsilon_N \rightarrow 0} \sum_{n=1}^{\infty} \chi_n^\dagger(x) t^a \gamma_5 f_{\epsilon_N}(-\hat{D}^2/M_N^2) \eta_n(x). \quad (7.18)$$

The last expression applies if the eigenvalues are ordered by increasing $\text{Re } \lambda_n^2$, which is justified because it reproduces the consistent gauge anomaly.

From eq. (7.18) and Appendix A, the imaginary part of $\delta_\omega \tilde{\Gamma}_N$ is

$$i \text{Im } \mathcal{A}_{\text{reg}}^a = \frac{1}{24\pi^2} \epsilon_{\mu\nu\sigma\tau} \partial_\mu \text{tr}_\rho [t^a (A_\nu \partial_\sigma A_\tau + \frac{1}{2} A_\nu A_\sigma A_\tau)], \quad (7.19)$$

the familiar consistent anomaly [29, 30, 31]. It vanishes if $\text{tr}_\rho(t^a \{t^b, t^c\}) = 0$ in representation ρ [31]; $\tilde{\Gamma}_N$ satisfies Condition 2.

Even if the anomaly cancels, however, the real part $\delta_\omega \tilde{\Gamma}_N$ is not gauge invariant. With α_2^a and α_{4R}^a from Appendix A

$$\delta_\omega \text{Re } \tilde{\Gamma}_N = \frac{1}{16\pi^2} \int d^4x \left(-M_N^2 \alpha_2^a(x) + \alpha_{4R}^a(x) \right) \omega^a(x). \quad (7.20)$$

After taking $\epsilon_N \rightarrow 0$, all higher-dimension terms drop out because they are proportional to e^{-1/ϵ_N} . Following Bardeen, the gauge variation of the real part can be compensated by counter-terms [29]. Let

$$\begin{aligned} S_2 &= -\frac{M_N^2}{16\pi^2} \int d^4x \text{tr}_\rho(A^2), \\ S_4 &= \frac{1}{48\pi^2} \int d^4x \text{tr}_\rho \left[\frac{1}{2} A_\mu \partial^2 A_\mu + (\partial \cdot A)^2 + \frac{1}{4} A_\mu A_\nu A_\mu A_\nu - \frac{1}{2} (A^2)^2 \right], \end{aligned} \quad (7.21)$$

and $S_{\text{ct}} = S_2 + S_4$. If S_{ct} is computed using the *interpolated* gauge field, its gauge variation cancels that of $\text{Re}\tilde{\Gamma}_N$ exactly.⁹ The choice of S_4 is not unique, because one could also add a gauge invariant term proportional to $\text{tr}_\rho F^2$. But the ambiguity corresponds to a shift in the (inverse) bare gauge coupling, so it should make no difference once all cutoffs are removed.

The counter-term S_2 is supposed to remove a quadratic divergence, but the number of modes N —not the mass M_N that appears in eq. (7.21)—defines the cutoff. The number of fermion modes with momentum below a cutoff M is

$$N = 4R \sum_k f_\epsilon(k^2/M^2). \quad (7.22)$$

Approximating the sum by an integral yields $N = 4R(LM)^4/32\pi^2 + O(e^{-bLM})$. But N is an integer, so the error in this approximation can be eliminated by taking

$$M_N^4 = \frac{32\pi^2 N}{4RL^4}. \quad (7.23)$$

This value of M_N is the one needed to cancel the quadratic divergence. If one chooses $N = N_{\text{lat}}$, then $M_N^2 = 4\sqrt{2}\pi/a^2 \approx (4.2/a)^2$.

The combination

$$\Gamma_N = \tilde{\Gamma}_N + S_{\text{ct}} \quad (7.24)$$

is thus gauge invariant under infinitesimal gauge transformations. By eq. (4.7) this is enough to show that $\Gamma_N(U)$ is invariant under all lattice gauge transformations. Hence, $\Gamma_N(U)$ satisfies all three conditions. This is the main result.

Finally, let us integrate over the zero modes. Unless there are enough factors of the fermion field in the amplitude, the integral vanishes. With the minimal number of fields in eq. (7.3)

$$\mathcal{Z}(n_+, n_-) = e^{-\Gamma_N(A)} \prod_{n=1}^{n_-} \varphi_{-,n}^{j_n}(y_n) \prod_{n=1}^{n_+} \varphi_{+,n}^{i_n}(x_n), \quad (7.25)$$

where i_n, j_n and x_n, y_n denote discrete indices and positions of the fields in eq. (7.3). Functional integration alone would lead to eq. (7.25) with $\tilde{\Gamma}_N$ instead of Γ_N . The zero modes present no substantive changes in the computation of the gauge variation of $\tilde{\Gamma}_N$, so the same counter-terms restore gauge symmetry.

7.3 Relation to ref. [10]

Although ref. [10] focuses primarily on the vector-like theory with Pauli-Villars cutoff, it does prove its important convergence theorem for theories with vector and axial-vector couplings. This suggests that the chiral coupling is also intended as an application. With the cutoff proposed there, the analysis of cutoff effects leads to somewhat different conclusions. Appendix C recasts 't Hooft's cutoff in

⁹In other methods [32, 33, 34] the cancellation is either approximate or subject to tuning.

a way that makes Appendix A directly applicable. One finds that the quadratic divergence drops out, but the universal term α_{4R}^2 still spoils gauge invariance of the real part. Moreover, the coefficients of the higher-dimension terms, α_{6R}^2 and so forth, no longer vanish, though they are suppressed by powers of M^2 . To eliminate these violations of gauge symmetry, one must take M to infinity, on each gauge field separately. This is cumbersome, and perhaps logically inconsistent, because to obtain a renormalized theory, the bare gauge coupling must be adjusted to keep physical masses—properties of the ensemble average—cutoff independent.

7.4 How to compute baryon violation

To summarize the results of this section it is worth sketching how to compute correlation functions. Because the integrals \mathcal{Z} sometimes vanish trivially, let us denote the sector of lattice gauge fields with n_{\pm} zero modes $\mathcal{U}_{(n_+, n_-)}$. Nonzero integrals with then be over one or so sectors.

Consider first fermion-conserving observables. One would like to compute a ratio of the form

$$\langle O \rangle = \frac{\int \mathcal{D}U_{(0,0)} O e^{-\Gamma_N - S_g}}{\int \mathcal{D}U_{(0,0)} e^{-\Gamma_N - S_g}}, \quad (7.26)$$

where S_g is the lattice-gauge-field action. Here the gauge-invariant observable O is constructed from the gauge field and fermion propagators. For eq. (7.26) one requires an ensemble of fields in $\mathcal{U}_{(0,0)}$, distributed with weight

$$W = e^{-\Gamma_N(U) - S_g(U)}. \quad (7.27)$$

For amplitudes of this type, all other sectors carry weight 0, because of the zero modes, so in the Monte Carlo they are simply omitted.

Consider next a fermion-violating amplitude. For simplicity, suppose that O does not contain fermion species that are being created or annihilated. Now one would like to compute a ratio of the form

$$\langle O \psi^i(x) \psi^j(y) \rangle = \frac{\int \mathcal{D}U_{(2,0)} O \varphi_+^i(x) \varphi_+^j(y) e^{-\Gamma_N(U) - S_g}}{\int \mathcal{D}U_{(0,0)} e^{-\Gamma_N(U) - S_g}}. \quad (7.28)$$

The numerator and denominator are averages over different sectors.¹⁰ and recall that in sectors with zero modes $e^{-\tilde{\Gamma}_N}$ is defined to be the product of the first N nonzero eigenvalues, ordered by $\text{Re } \lambda_n^2$. One can re-write eq. (7.28) as

$$\langle O \psi^i(x) \psi^j(y) \rangle = \frac{\int \mathcal{D}U_{(2,0)} e^{-\Gamma_N(U) - S_g}}{\int \mathcal{D}U_{(0,0)} e^{-\Gamma_N(U) - S_g}} \langle O \varphi_+^i(x) \varphi_+^j(y) \rangle_{(2,0)}, \quad (7.29)$$

¹⁰The two-zero-mode sector $\mathcal{U}_{(2,0)}$ is typically the sector with topological charge $Q = 1$, which would have one zero mode for each species.

where the average $\langle \bullet \rangle_{(2,0)}$ is over $\mathcal{U}_{(2,0)}$. In addition to the weight W , one must compute the zero-mode eigenfunctions $\varphi_+(x)$ for created and annihilated species.

The other factor in the fermion-nonconserving amplitude is the ratio of two partition functions. To compute it accurately some special numerical techniques [35, 36] are available, which keep track of the system's preference for one sector or the other. Various versions of these "histogram methods" [37] may also prove useful in obtaining the nontrivial phase of W , inherent to a complex representation.

7.5 Global anomalies

Some theories, the simplest of which is $SU(2)$ with one Weyl doublet, are afflicted by a global anomaly [38]. The representations in question are real, and therefore $e^{-\Gamma_N(U)}$, as defined by eqs. (7.15) and (7.24), is real and positive. Thus $\Gamma_N(U)$ is real.

Let us focus on $SU(2)$. Because $\pi_4(SU(2)) = \mathbb{Z}_2$, there are nontrivial gauge transformations, for which the proof of gauge invariance of $\Gamma_N(U)$ breaks down.¹¹ Let w be in the nontrivial class. The variation $\Gamma_N({}^wU) - \Gamma_N(U)$ is real, but does it vanish? Following ref. [39] one can compute the difference by embedding $SU(2)$ into $SU(3)$ and taking a trajectory from $g = 1$ to $g = w$ in $SU(3)$. $\text{Re } \Gamma_N$ does not change for any infinitesimal step, and since $\pi_4(SU(3)) = 0$, the trajectory can be constructed from infinitesimal steps. Thus $\Gamma_N({}^wU) - \Gamma_N(U) = 0$ for the embedded field, and hence likewise for the $SU(2)$ fields themselves.

On the other hand, Witten argued that the two configurations U and wU should have Boltzmann weights equal in magnitude but *opposite* in sign [38]. Indeed the gauge variation of $\text{Im } \Gamma_N$ integrated along the $SU(3)$ trajectory supports his conclusion [39]. But, given a lattice gauge field U' , the algorithm for $e^{-\Gamma_N(U')}$ cannot determine whether $U' = U$ or $U' = {}^wU$. And thanks to the pains taken ensure gauge invariance, the computed weight is the same in either case. This is not a serious drawback, however. If the global anomaly applies, one can replace numerator and denominator of eq. (7.26) by Witten's original result, $0/0$, eliminating all computation.

8 Conclusions

The seeming incompatibility of lattice fermions and chiral symmetry has inspired the recurring idea [9, 14, 10] of treating the fermions in the continuum, even if the underlying gauge field is on the lattice. Smit calls the idea desperate [13]. How desperate are the main results presented here? The colorful terminology refers to the tacit assumption that a continuum requires an infinite number of degrees of freedom (per unit volume). Then the arithmetic needed to evaluate functional integrals is infinite: we are desperate because a computer cannot do the job. But

¹¹Although lattice gauge transformations can be built up slowly, site-by-site, and eq. (4.7) shows that the interpolations inherit this property, the two classes can be separated by lattice gauge transformations for which the interpolation is ill-defined.

with a finite-mode cutoff there are N eigenvalues; the computation is finite. Alas, even in the vector-like theory the effort needed to obtain higher eigenvalues will be high (cf. sect. 6). The construction for chiral fermions is yet more computationally intensive, because, first, the counter-terms must now be computed accurately, and, second, the phase, which stems not from the regulator but from the chiral coupling itself, requires extra care.

There is a potential shortcoming to the finite-mode regulator. One would like to verify that the spectrum remains chiral, in perturbation theory and beyond. Appendix B demonstrates perturbative universality for fermion loops. (Ref. [15] and Appendix A do the same for anomalous diagrams only.) But there is (as yet) no comparable proof for diagrams with external fermion lines, which are needed to examine the fermion spectrum. The resolution is not straightforward, because it depends on details of the gauge-boson propagator, i.e. on the lattice gauge action. And even if the perturbative test is a success, one should be cautious until the nonperturbative spectrum has been checked.

The counter-terms required in the chiral gauge theory may be unsettling at first sight. But their necessity arises from the unassailable observation [1] that a *regulated* functional integral either respects a symmetry or it does not. To obtain the anomaly (and hence the physically powerful requirement of anomaly cancellation) the functional integral $e^{-\hat{\Gamma}_N(U)}$ cannot be gauge symmetric. Consequently, the final result for the chiral Boltzmann factor

$$e^{-\Gamma_N(U)} = e^{-\hat{\Gamma}_N(U)} e^{-S_{ct}(U)} \quad (8.1)$$

is the product of a functional integral and a symmetry-restoring factor.

A related peculiarity is the fate of Witten's global anomaly [38]. Again, an Ansatz for the effective action is either gauge invariant or not. The effective action $\Gamma_N(U)$ is gauge invariant, even when the gauge transformation is in the nontrivial class. Without asserting that this paper's formulation succeeds at defining the globally anomalous theories, one might suggest that it could shed light on the dynamical puzzles that originally motivated ref. [38]. An optimistic possibility is that the dynamics of $\Gamma_N(U)$ fail in globally anomalous theories, but not otherwise.

A serious complication is that the formulation is in Euclidean field theory. At the nonperturbative level the Wick rotation does no good, and instead one constructs the Minkowski theory via the imaginary-time evolution operator on the Hilbert space of states. This procedure defines a Hamiltonian that can then be used to propagate the states in real time. The eigenfunctions used to define the functional integral are fundamentally four-dimensional, so the constructive approach [40] does not seem helpful. Perhaps the axiomatic approach [41] will prove more promising.

Finally, let us compare the present chiral construction with the overlap formalism [6]. Both define an effective action with a gauge-invariant real part. Both generate an imaginary part in a complex fermion representation, but not in a real representation. With the interpolation and \hat{D} the gauge variation of the imaginary part is the consistent anomaly and with the finite-mode regulator (or in the limit

$M \rightarrow \infty$ [10]) nothing else. With the overlap the gauge variation of the imaginary part contains the anomaly and higher-order gauge breaking terms, analogous to α_{6I}^a (notation of Appendix A), as well. Ref. [6] argues they should be tolerably small, and tests in two dimensions [42] indicate that this conclusion may be correct. Both methods provide fermion nonconserving amplitudes: here the gauge-field topology [17] drives fermion nonconservation, and in ref. [6] the fermion nonconserving amplitudes define the gauge-field topology. (For smooth lattice gauge fields the two topologies coincide.) Whereas Appendix B includes an explicit verification of fermion-loop coupling-constant renormalization, it does not seem that an explicit calculation starting from the lattice overlap is available yet [43]. On the other hand, ref. [6] includes several numerical cross checks that have not been done here. An important, dynamical test is whether the fermion spectrum remains chiral after integrating over gauge fields.¹² Neither construction has been subjected to this test yet, because it requires a full-fledged Monte Carlo calculation.

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While this paper was being written, a preprint appeared by S.D.H. Hsu (Yale University report YCTP-P5-95, hep-th/9503058), expanding on the idea of Gökeler and Schierholz [14] to exploit the η -invariant formulated in ref. [25].

A Cutoff effects in the anomaly

This appendix fills in the steps from eq. (3.1) to eq. (3.5). The modifications needed to obtain eq. (7.19) are provided in sect. A.1. The analysis is standard, but it is provided to demonstrate the form of higher-dimension terms suppressed by powers of $1/M_N^{2n}$. The sharp limit $\varepsilon_N \rightarrow 0$ turns out to be special, because it eliminates these terms *before* taking $M_N \rightarrow \infty$.

First some preliminaries on Fourier transforms. For definiteness the space-time is a box with sides L , and volume $V = L^d$. The Fourier transform

$$\tilde{\varphi}(k) = \int d^4x e^{-ik \cdot x} \varphi(x). \quad (\text{A.1})$$

Allowed values of k depend on the boundary conditions. This Appendix presents

¹²This is the physical crux of recent skepticism [44] of the overlap formalism.

details for a wide class of almost periodic boundary conditions,¹³ such that $k_\mu = 2\pi(\nu_\mu + \eta_\mu)/L$, $\nu_\mu \in \mathbb{Z}$. Strictly periodic directions have $\eta_\mu = 0$; anti-periodic directions have $\eta_\mu = 1/2$; C -periodic directions have a certain combination of the foregoing and $\eta_\mu = 1/4$. The inverse transform (assuming convergence) is

$$\varphi(x) = V^{-1} \sum_k \tilde{\varphi}(k) e^{ik \cdot x}. \quad (\text{A.2})$$

The application needed here is the Fourier transform of eq. (2.7)

$$\sum_n \tilde{\varphi}_n^i(p) \tilde{\varphi}_n^{j*}(q) = V \delta_{pq} \delta^{ij}. \quad (\text{A.3})$$

Consider any function f obeying $f(0) = 1$ and $f(\infty) = f'(\infty) = f''(\infty) = \dots = 0$ [16], and let

$$\mathcal{A}_f^a(x) = (\gamma_5 T^a)_{ji} \sum_n \left[\varphi_n^i(x) f(-D^2/M^2) \varphi_n^j(x) \right]_{ij}, \quad (\text{A.4})$$

with the summation convention over the spin-flavor-color multi-indices i, j . In sect. (3) the Fermi function f_{ϵ_N} appears, with mass M_N and the limit $\epsilon_N \rightarrow 0$. For comparison with refs. [10, 16], however, it is convenient to keep f arbitrary.

Fourier transforming the eigenfunctions and using eq. (A.3) one obtains

$$\mathcal{A}_f^a(x) = V^{-1} (\gamma_5 T^a)_{ji} \sum_k \left[e^{-ik \cdot x} f(-D^2/M^2) e^{ik \cdot x} \right]_{ij}. \quad (\text{A.5})$$

From eq. (2.15) one recalls that here $D e^{ik \cdot x} = e^{ik \cdot x} (ik + D)$. Hence,

$$\mathcal{A}_f^a(x) = V^{-1} (\gamma_5 T^a)_{ji} \sum_k \left[f \left((k^2 - D^2 + 2ik \cdot D) / M^2 \right) \right]_{ij}, \quad (\text{A.6})$$

Expanding in D/M

$$\mathcal{A}_f^a(x) = V^{-1} (\gamma_5 T^a)_{ji} \sum_k \sum_n \frac{(-1)^n}{n!} \left(D^2/M^2 + 2ik \cdot D/M \right)_{ij}^n f^{(n)}(k^2), \quad (\text{A.7})$$

where now $k_\mu = 2\pi(\nu_\mu + \eta_\mu)/(LM)$, and $f^{(n)} = d^n f / dx^n$.

Because the functions under consideration are smooth and vanish rapidly at infinity, the sums can be approximated by integrals

$$\frac{1}{(LM)^d} \sum_k (1, k_\mu k_\nu, \dots) f(k^2) = \int \frac{d^d k}{(2\pi)^d} (1, k_\mu k_\nu, \dots) f(k^2) + O(e^{-bLM}) \quad (\text{A.8})$$

with little error. Odd powers vanish. Note that the finite-size effects also vanish when the ultraviolet regulator is removed.

¹³Other boundary conditions such as Dirichlet, von Neumann, or fixed lead to the same final conclusions.

Reorganizing the terms according to the power of M

$$\begin{aligned} \mathcal{A}_f^a(x) = & \frac{1}{(-4\pi)^{d/2}} \left[M^d f^{(-d/2)}(0) \alpha_0^a(x) + M^{d-2} f^{(1-d/2)}(0) \alpha_2^a(x) \right. \\ & \left. + M^{d-4} f^{(2-d/2)}(0) \alpha_4^a(x) + M^{d-6} f^{(3-d/2)}(0) \alpha_6^a(x) + \dots \right], \end{aligned} \quad (\text{A.9})$$

where $\alpha_0^a(x) = \text{tr} \gamma_5 T^a = 0$,

$$\alpha_2^a = -\text{tr}[\gamma_5 T^a (D^2 - D^2)], \quad (\text{A.10})$$

$$\alpha_4^a = \frac{1}{2} \text{tr} \left\{ \gamma_5 T^a \left((D^2 - D^2)^2 + \frac{1}{3} [D_\mu, [D_\mu, D^2 - D^2]] + \frac{1}{6} [D_\mu, D_\nu] [D_\mu, D_\nu] \right) \right\}, \quad (\text{A.11})$$

and α_6^a contains terms with 6 D 's, combined to produce a function (rather than a differential operator). The result depends on the cutoff function via the coefficients $f^{(n)}(0)$, defined by $f^{(0)}(x) \equiv f(x)$,

$$f^{(n-1)}(x) = -\int_x^\infty dx' f^{(n)}(x'), \quad f^{(n+1)}(x) = \frac{df^{(n)}}{dx}. \quad (\text{A.12})$$

The M independent term has the universal coefficient $f(0) = 1$; the other coefficients differ for different cutoffs.

With the Fermi function $f_\varepsilon^{(-n)}(0) = (-1)^n/n!$, $n > 0$, plus terms of order $\varepsilon^n e^{-1/\varepsilon}$. Hence, the α_{2j}^a , $2j < d$, are power-law divergences, unless the traces vanish. On the other hand, for $n > 0$ then $f_\varepsilon^{(n)}(0) \sim \varepsilon^{-n} e^{-1/\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus, the sharp cutoff has no "scaling violations."

For the vector-like theory $D^2 - D^2 = \frac{1}{4} [\gamma_\mu, \gamma_\nu] F_{\mu\nu}$. If $d = 2$, the only surviving term α_2^a yields the well-known result. If $d = 4$, the Dirac trace makes α_2^a vanish as well as everything in α_4^a except the term $(D^2 - D^2)^2$, yielding the familiar axial anomaly, eq. (3.5).

A.1 Modifications for the chiral gauge theory

In the chiral gauge theory one wants

$$\mathcal{A}_f^a(x) = (\gamma_5 t^a)_{ji} \sum_n \left[\chi_n^\dagger(x) f(-\hat{D}^2/M^2) \eta_n(x) \right]_{ij}. \quad (\text{A.13})$$

The left and right eigenfunctions of $i\hat{D}$ are not complete, but

$$\sum_n \eta_n^i(x) \chi_n^{j*}(y) = \delta(x-y) \delta^{ij} - \mathbf{P}_0, \quad (\text{A.14})$$

where \mathbf{P}_0 projects onto zero modes. This nuisance is easiest to handle with anti-periodic boundary conditions. If the function f drops to zero below the lowest momentum mode, then $\mathbf{P}_0 f = 0$. Thus, the projector can be dropped after Fourier transforming. The hole in f cannot affect the momentum sums, so the correct approximation in eq. (A.8) uses a function without the hole on the right-hand side.

The algebraic manipulations still hold, but one must replace T^a by t^a , D by \hat{D} and the symbol D_μ by $\bar{D}_\mu := \frac{1}{2}\{\gamma_\mu, \hat{D}\}$. From eq. (7.8)

$$\bar{D}_\mu = \partial_\mu + \frac{1}{2}A_\mu - \frac{i}{2}\sigma_{\mu\nu}\gamma_5 A_\nu. \quad (\text{A.15})$$

Then

$$\hat{D}^2 - \bar{D}^2 = \frac{1}{2}\gamma_5\partial\cdot A - \frac{i}{2}\sigma_{\mu\nu}\partial_\mu A_\nu + \frac{1}{4}(d-2)(A^2 - i\sigma_{\mu\nu}A_\mu A_\nu). \quad (\text{A.16})$$

Consequently,

$$\alpha_2^a = \begin{cases} \varepsilon_{\mu\nu} \text{tr}(t^a \partial_\mu A_\nu) - \text{tr}(t^a \partial\cdot A) & (d=2) \\ -2 \text{tr}(t^a \partial\cdot A) & (d=4), \end{cases} \quad (\text{A.17})$$

where the trace is now only over gauge indices. In $d=2$ this is the universal anomaly, plus a term that can be compensated by a local counter-term; in $d=4$ it only the latter survives, and it is a power-law divergence. The function in α_4^a is too lengthy to present explicitly, but after the trace $\alpha_4^a = \alpha_{4R}^a + \alpha_{4I}^a$, where

$$\alpha_{4I}^a = \frac{2}{3}\varepsilon_{\mu\nu\rho\sigma} \text{tr} [t^a \partial_\mu (A_\nu \partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma)] \quad (d=4) \quad (\text{A.18})$$

is the well-known consistent anomaly [29, 30, 31], and

$$\begin{aligned} \alpha_{4R}^a = \frac{1}{3} \text{tr} [t^a & \left(\partial^2 \partial\cdot A + [\partial^2 A_\mu, A_\mu] - 2[\partial_\mu \partial\cdot A, A_\mu] + \{\partial\cdot A, A^2\} \right. \\ & \left. - A_\mu (\partial\cdot A) A_\mu - [\partial_\mu A_\nu, [A_\mu, A_\nu]] + A_\mu (\partial_\mu A_\nu + \partial_\nu A_\mu) A_\nu \right)] \end{aligned} \quad (\text{A.19})$$

is the almost as well-known quantity that, like $\alpha_2^a|_{d=4}$, can be compensated by local counter-terms [29, 27, 15].

B Cutoff effects in the effective action

The analysis of the previous section can be applied directly to the effective action. It shows that the finite-mode cutoff is in the same (perturbative) universality class as Pauli-Villars regulators. As in Appendix A, the analysis is performed for an arbitrary smooth function, but once again the sharp limit is special, because it has no power corrections.

For a good infrared behavior, this section considers only anti-periodic boundary conditions.

One can write the regulated effective action as

$$\Gamma_N = -\frac{1}{2} \sum_n \log \left(\frac{\lambda_n^2 + m^2}{M_N^2} \right) f_{\varepsilon_N} \left(\frac{\lambda_n^2}{M_N^2} \right) = \int d^4x \mathcal{L}_N, \quad (\text{B.1})$$

where the effective Lagrangian

$$\mathcal{L}_N = -\frac{1}{2} \sum_n \varphi_n^\dagger(x) L_{f_{\varepsilon_N}}(-\hat{D}^2/M_N^2) \varphi_n(x), \quad (\text{B.2})$$

and, for any function f obeying $f(0) = 1$ and $f(\infty) = f'(\infty) = f''(\infty) = \dots = 0$,

$$L_f(x) = \log(x + \mu^2)f(x), \quad (\text{B.3})$$

and $\mu = m/M$. The notation applies to the vector-like theory, but the substitutions needed for the chiral gauge theory are obvious. The effective Lagrangian for arbitrary f will be denoted \mathcal{L}_f .

Fourier transforming the eigenfunctions, using eq. (A.3), and using $\mathcal{D}e^{ik \cdot x} = e^{ik \cdot x}(i\mathbf{k} + \mathcal{D})$ one obtains

$$\mathcal{L}_f(x) = -\frac{1}{2}\delta_{ji}V^{-1} \sum_k \left[L_f \left((k^2 - \mathcal{D}^2 + 2ik \cdot \mathcal{D})/M^2 \right) \right]_{ij}, \quad (\text{B.4})$$

where $k_\mu = \pi(2\nu_\mu + 1)/L$. Expanding in D/M

$$\mathcal{L}_f(x) = -\frac{1}{2}\delta_{ji}V^{-1} \sum_k \sum_n \frac{(-1)^n}{n!} \left(\mathcal{D}^2/M^2 + 2ik \cdot D/M \right)_{ij}^n L_f^{(n)}(k^2), \quad (\text{B.5})$$

where $k_\mu = \pi(2\nu_\mu + 1)/(LM)$. The derivatives take the form

$$L_f^{(n)}(x) = \sum_{l=0}^{n-1} \frac{n!}{l!} \frac{(-1)^{n-1-l}}{(n-l)} \frac{f^{(l)}(x)}{(x + \mu^2)^{n-l}} + \log(x + \mu^2)f^{(n)}(x). \quad (\text{B.6})$$

The $l = 0$ term can be called universal, because after summing over k the universal $f(0) = 1$ remains.

In the vector-like theory, the fermion mass (presumed nonzero) regulates the infrared and one can use eq. (A.8). In the chiral gauge theory (or any massless case), infrared singularities make the integrals poor estimates of the sums. For the present purposes, however, the function f can also be used as an infrared regulator. One simply chooses f to drop to zero for momentum less than the smallest allowed by the anti-periodic boundary condition, say for $k^2 < \delta^2 = (\pi/2LM)^2$.

The delicate infrared behavior has consequences when reorganizing the effective Lagrangian according to the dimension of the interactions. One finds

$$\begin{aligned} \mathcal{L}_f = \frac{-1}{2(-4\pi)^{d/2}} & \left[M^d L_f^{(-d/2)}(\delta^2) l_0(x) + M^{d-2} L_f^{(1-d/2)}(\delta^2) l_2(x) \right. \\ & \left. + M^{d-4} L_f^{(2-d/2)}(\delta^2) l_4(x) + M^{d-6} L_f^{(3-d/2)}(\delta^2) l_6(x) + \dots \right] + \Delta\mathcal{L}, \end{aligned} \quad (\text{B.7})$$

where $l_0(x) = \text{tr} 1 = 4R$ yields a constant of no dynamical significance,

$$l_2 = -\text{tr}(\mathcal{D}^2 - D^2), \quad (\text{B.8})$$

$$l_4 = \frac{1}{2} \text{tr} \left\{ (\mathcal{D}^2 - D^2)^2 + \frac{1}{3} [D_\mu, [D_\mu, \mathcal{D}^2 - D^2]] + \frac{1}{6} [D_\mu, D_\nu] [D_\mu, D_\nu] \right\}, \quad (\text{B.9})$$

and l_6 contains same combination of 6 D 's as α_6^a , which is a function rather than a differential operator. The interactions in $\Delta\mathcal{L}$ are proportional to $[\delta^2/(\delta^2 + \mu^2)]^l$; hence they survive only when $\mu^2 \ll \delta^2$.

The effective Lagrangian depends on the cutoff via the coefficients $L_f^{(n)}(\delta^2)$. If $\delta^2 \ll \mu^2$ (i.e. $mL \gg 1$) these become $L_f^{(n)}(0)$. For example, with the Fermi function

$$L_{f_e}^{(-1)}(0) = 1, \quad L_{f_e}^{(-2)}(0) = -1/4, \quad (\text{B.10})$$

plus terms of order $\varepsilon^{|n|}e^{-1/\varepsilon}$. The higher-dimension functions have coefficients $L_f^{(n)}(\delta^2)/M^{2n}$, $n > 0$. From eq. (B.6) one sees that they consist of a universal, f - and M -independent term, plus terms non-universal proportional to $f^{(m)}(0)/M^{2m}$. The remaining dimensions are balanced by infrared scales m or $1/L$. Again, for the Fermi function the non-universal terms are suppressed by $e^{-1/\varepsilon}$ and drop out in the sharp limit. Since interactions of all dimension appear in eq. (B.7), it must be interpreted as a perturbative series.

In the vector-like theory, regulated as in sect. 3, l_2 vanishes. Indeed, the gauge invariance of the *regulated* theory forbids any dimension 2 terms. In l_4 the Dirac trace eliminates the nested commutator leaving, for $d = 4$,

$$\mathcal{L}_{f,4} = \frac{\log(\mu^2, \delta^2)}{48\pi^2} \text{tr}_\rho F^2, \quad (\text{B.11})$$

which renormalizes the gauge coupling.

In the chiral gauge theory of sect. 7 this analysis applies to $\tilde{\Gamma}_N$. The dimension-two term l_2 does not vanish, because now the regulator breaks the gauge symmetry. In four dimensions one has

$$\mathcal{L}_{f,2} = \frac{M^2}{16\pi^2} L_f^{(-1)}(\delta^2) \text{tr}_\rho A^2. \quad (\text{B.12})$$

To restore gauge symmetry, one must add the counter-term S_2 in eq. (7.21). On the other hand, l_4 induces coupling constant renormalization. (The gauge non-invariant pieces cancel.) With some patience one can accumulate the nonvanishing γ -matrix traces in l_4 to obtain

$$\mathcal{L}_{f,4} = \frac{\log \delta^2}{96\pi^2} \text{tr}_\rho F^2, \quad (\text{B.13})$$

up to a total derivative. Notice that, as expected, the renormalization term is half that of the vector-like theory. The dimension-four interactions in $\Delta\mathcal{L}$, present because $\mu^2 = 0$, include gauge-breaking terms; they are cancelled by S_4 .

C Applying Appendices A and B to ref. [10]

Ref. [10] defines the fermion functional integral with a time-honored [45] Pauli-Villars regulator:

$$e^{-\Gamma_{\text{PV}}(A)} = \prod_i (\det(D + M_i))^\varepsilon_i \quad (\text{C.1})$$

in the vector-like case, and similarly $e^{-\hat{F}}$ with \hat{D} in the chiral case. The determinants are products over the infinitely numerous eigenvalues. The masses¹⁴ M_i and signatures e_i satisfy

$$\sum_{i=0}^{\infty} e_i = \sum_{i=0}^{\infty} e_i (M_i/M)^n = 0, \quad n = 1, 2, 3, \dots \quad (\text{C.2})$$

$$\sum_{i=1}^{\infty} e_i (M_i/M)^n \log(M_i/M) = 0, \quad n = 0, 1, 2, 3, \dots \quad (\text{C.3})$$

For $n = 0$, eq. (C.3) defines the overall scale M . Eqs. (C.2) and (C.3) require infinite series only if one requires the identities for all n .

The function needed to describe the effective Lagrangian, cf. eqs. (B.1)–(B.7), is

$$L_{\text{PV}}(x) = \sum_{i=0}^{\infty} e_i \log(x + M_i^2/M^2). \quad (\text{C.4})$$

Note that L_{PV} and all its derivatives vanish at infinity, by virtue of eqs. (C.2) and (C.3), so the manipulations of Appendix B still hold. Eqs. (2.17) and (7.17) also hold as before, but with \mathcal{A}_{reg} replaced by $\mathcal{A}_{f_{\text{PV}}}$. Here

$$f_{\text{PV}}(x) = x L'_{\text{PV}}(x) = - \sum_{i=1}^{\infty} \frac{e_i M_i^2}{M^2 x + M_i^2}. \quad (\text{C.5})$$

Again f_{PV} and all its derivatives vanish at infinity, so the manipulations of Appendix A hold.

To apply eq. (A.9) one notes the universal normalization $f_{\text{PV}}(0) = 1$. Power-law divergences drop out, but “scaling violations” remain ($n > 0$):

$$f_{\text{PV}}^{(-n)}(0) = 0, \quad f_{\text{PV}}^{(n)}(0) = \frac{(-1)^{n+1}}{n!} \sum_{i=1}^{\infty} e_i (M/M_i)^{2n} \neq 0, \quad (\text{C.6})$$

and analogously for $L_{\text{PV}}^{(n)}(0)$. One might remark that the absence of power-law divergences relies on integrating x to infinity. They re-appear if one truncates Γ_{PV} when, say, $L_{\text{PV}}(x)$ becomes small.

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¹⁴ $M_0 = m$ for vector-like gauge theories; $M_0 = 0$ for chiral gauge theories.

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