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CBR Anisotropy and the Running of the Scalar Spectral Index

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Abstract

Accurate ($\lesssim 1\%$) predictions for the anisotropy of the Cosmic Background Radiation (CBR) are essential for using future high-resolution ($\lesssim 1^\circ$) CBR maps to test cosmological models. In many inflationary models the variation ("running") of the spectral index of the spectrum of density perturbations is a significant effect and leads to changes of around 1% to 10% in the CBR power spectrum. We propose a general method for taking running into account which uses the derivative of the spectral index ($dn/d\ln k$). Conversely, high-resolution CBR maps may be able to determine $dn/d\ln k$, giving important information about the inflationary potential.

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The cosmic background radiation contains a wealth of information about the spectrum of primeval density perturbations. This is because CBR anisotropy on a given angular scale arises largely due to density perturbations on a (comoving) length scale $L \simeq (\theta/1^\circ)100h^{-1}$ Mpc. Since the COBE detection of CBR anisotropy on angular scales of 10° to 90° [1], more than ten additional detections on angular scales from about 0.5° to 20° have been reported [2]. In addition, plans are being made for a satellite-borne experiment within the decade that will map the CBR sky with an angular resolution of better than 1° and an accuracy that is an order of magnitude better than current measurements [3]. Thus, in the near future CBR anisotropy should be able probe inhomogeneity on length scales from about $30h^{-1}$ Mpc to $30,000h^{-1}$ Mpc.

A key to using CBR measurements to reveal the underlying spectrum of density perturbations is the accurate calculation of the expected anisotropy in a given model. Much progress has been made in understanding and taking into account all the relevant microphysics [4], and several groups are now making a concerted effort to calculate expected CBR anisotropies with an accuracy of better than 1% [5].

Much of this effort is directed at inflation, as CBR anisotropy has the potential to both test the inflation hypothesis and reveal important information about the underlying scalar-field potential [6]. Inflationary models predict approximately scale-invariant spectra of density (scalar metric) perturbations [7] and gravity-wave (tensor metric) perturbations [8], and both contribute to CBR anisotropy. The following parameters have been identified as important for accurately computing the expected anisotropy [9]: the power-law indices of the scalar and tensor spectra, $n_T \approx 0$ and $n \approx 1$; the overall amplitudes of the scalar and tensor perturbations, often quantified by their contributions to the variance of the quadrupole anisotropy, Q_S and Q_T ; the Hubble parameter $h = H_0/100 \text{ km s}^{-1} \text{ Mpc}^{-1}$; the baryon density, which is constrained by primordial nucleosynthesis to the interval $\Omega_B h^2 \simeq 0.009 - 0.022$ [10]; and possible contribution of a cosmological constant to the energy density of the Universe today Ω_Λ . (In addition, some have considered the possibility of a total energy density less than the critical density predicted by almost all models of inflation, nonstandard ionization histories for the Universe, and variations in the nonbaryonic component of the matter density, e.g., adding a small amount of hot dark matter.)

In this paper we emphasize that the spectral indices n and n_T in general vary with scale and point out that for many interesting models of inflation (chaotic, natural, and new) the variation in scalar spectral index leads to significant corrections (1% to 10%) in the predicted CBR anisotropy. Conversely, this means that a high-resolution CBR map could be used to extract information about the variation of n with scale and thereby reveal additional information about the inflationary potential. We thus make the case that the variation of the scalar spectral index should be taken into account when calculating CBR anisotropy, and suggest that it is most sensibly done by using $dn/d \ln k$.

CBR anisotropy on the sky is usually expanded in spherical harmonics,

$$\delta T(\Omega)/T = \sum_{lm} a_{lm} Y_{lm}(\Omega). \quad (1)$$

Inflation makes predictions about the statistical properties of the multipole moments; since isotropy in the mean guarantees that $\langle a_{lm} \rangle = 0$ and the underlying perturbations in almost all inflationary models are gaussian, the variance $C_l \equiv \langle |a_{lm}|^2 \rangle$ serves to specify all statis-

tical properties. (Here and throughout brackets refer to the average over an ensemble of observers.) Measurements of the CBR temperature on the sky can be used to estimate the statistical properties of the underlying density perturbations. In particular, the C_l 's can be estimated. Because the sky is but a finite sample, a fundamental limit to the accuracy of the estimate (referred to as cosmic variance) is given by

$$\langle (C_l - C_l^{\text{est}})^2 \rangle = \frac{2C_l^2}{2l+1}. \quad (2)$$

Other major (and presently dominant) sources of uncertainty include receiver noise, various instrumental systematic errors, foreground sources (our own galaxy, radio sources, etc.), limited sky coverage, and finite resolution (a map with angular resolution θ is only sensitive to multipoles with $l \lesssim 200^\circ/\theta$).

High-resolution maps of the CBR probably offer the best means of studying the scalar and tensor metric perturbations predicted by inflation [11,12]. Such maps may also provide valuable information about the Hubble constant H_0 , the cosmological constant Λ , the baryon density Ω_B , and the total density of the Universe Ω ; however, other measurements will complement the determination of these parameters. If the four parameters describing the scalar and tensor perturbations are measured to some level of accuracy, properties of the underlying inflationary potential $V(\phi)$ can be determined [13]:

$$V_N = 1.65 Q_T m_{\text{Pl}}^4, \quad (3)$$

$$V'_N = \pm \sqrt{-8\pi n_T} V_N / m_{\text{Pl}} = \pm \sqrt{\frac{8\pi r}{7}} V_N / m_{\text{Pl}}, \quad (4)$$

$$V''_N = 4\pi[(n-1) - 3n_T] V_N / m_{\text{Pl}}^2 = 4\pi \left[(n-1) + \frac{3}{7}r \right] V_N / m_{\text{Pl}}^2, \quad (5)$$

where $r \equiv Q_T/Q_S$, a prime indicates derivative with respect to ϕ , and the sign of V' is indeterminate. In addition, a consistency relation $n_T = -r/7$ must be satisfied, and the factors of $\frac{1}{7}$ arise from using it [14]. Subscript N indicates that the potential is to be evaluated at the value of ϕ where the scale corresponding to the present Hubble scale ($k_N = H_0$) crossed outside the horizon during inflation. This generally occurs around $N \simeq 50$ e-foldings before the end of inflation, though the precise expression depends upon the model of inflation, the reheat temperature, and any entropy production after inflation. (Only the expression for V_N depends upon the definition of N ; the other two always apply.) The expression for N can be written as

$$N \simeq 54 + \frac{1}{6} \ln(-n_T) + \frac{1}{3} \ln(T_{\text{RH}}/10^6 \text{ GeV}) - \frac{1}{3} \ln \gamma - \ln h, \quad (6)$$

where T_{RH} is the reheat temperature, γ is the ratio of the entropy per comoving volume today to that after reheating which quantifies any post-inflation entropy production, and the perturbation spectrum has been normalized to COBE. (In calculating N it has been assumed that inflation is followed immediately by a matter-dominated epoch associated with coherent oscillations of the inflaton field and then by reheating.)

The above expressions were derived in a systematic approximation scheme that relates the derivatives of an arbitrary smooth inflationary potential to CBR observables [13,15].

The expansion parameter is the deviation from scale invariance, and formally involves all the derivatives of the potential, $m_{\text{Pl}}^n V_N^{(n)}/V_N$ (constant V corresponds to the scale-invariant limit; see Ref. [16] for a discussion of this scheme). For most potentials the deviation from scale invariance of the scalar and tensor spectra, quantified by $(n-1)$ and n_T , serve as the expansion parameters. The above expressions are given to lowest order in $(n-1)$ and n_T ; the next-order corrections are given in Ref. [16].

The crucial point for the present discussion is that the spectra of scalar and tensor perturbations are only exactly power laws for an exponential potential. In general, they vary with scale, though $dn/d\ln k$ and $dn_T/d\ln k$ are second order in the deviation from scale invariance, i.e., involve terms that are $\mathcal{O}[(n-1)^2, n_T^2, (n-1)n_T]$. Since the present data indicate that scalar perturbations do not differ from scale invariance by a large amount, $n-1 = 0.10 \pm 0.32$ [17], the variation of the spectral indices is expected to be small. Further, indications are that the tensor perturbations are subdominant and in any case only contribute significantly to multipoles $l = 2$ to 50. However, we shall show that the variation of scalar spectral index is important, given the desired precision for the theoretical predictions of the multipoles.

The power spectrum for the scalar perturbations is given by

$$P(k) \equiv Ak^{n(k)} \propto \left(\frac{k}{k_N}\right)^{n_N + \ln(k/k_N)(dn/d\ln k) + \dots} \quad (7)$$

The contribution to the l th multipole comes from wavenumbers k centered around l/τ_0 , where $\tau_0 \simeq 2/H_0$ is the distance to the last scattering surface. Recalling that the characteristic scale k_N was chosen to correspond to the current horizon size, this implies an approximate scaling relation for the C_l 's which relates them to a spectrum with constant spectral index:

$$C_l[n(k)] \simeq \left(\frac{l}{2}\right)^{\ln(l/2)dn/d\ln k} C_l[n(k) = n_N]. \quad (8)$$

If $|dn/d\ln k| \gtrsim 3 \times 10^{-4}$, the effect of ignoring the ‘‘running’’ of the spectral index over the range $l = 2 - 1000$ is greater than one percent, which is significant compared to the accuracy goal for CBR anisotropy [5]. We now show that values this large are expected in interesting inflationary models.

In general, the derivatives of the scalar and tensor spectral indices are related to the inflationary potential and its derivatives. The lowest-order expression for *all* the derivatives of n and n_T can be obtained by simply differentiating the lowest-order expressions,

$$n_T = -\frac{1}{8\pi} \left(\frac{m_{\text{Pl}} V'}{V}\right)^2 \Big|_{\phi=\phi_N} \quad (9)$$

$$n-1 = n_T + \frac{m_{\text{Pl}}}{4\pi} \frac{d}{d\phi} \left(\frac{m_{\text{Pl}} V'}{V}\right) \Big|_{\phi=\phi_N}, \quad (10)$$

using the fact that to lowest order

$$\frac{d}{d\ln k} = -\frac{1}{8\pi} \left(\frac{m_{\text{Pl}}^2 V_N'}{V_N} \frac{d}{d\phi}\right) \Big|_{\phi=\phi_N}$$

If n and n_T are expressed as a function of N , one can use the fact that $d/d \ln k = -d/dN$ to obtain the desired derivatives even more easily.

It is thus a simple matter to obtain the first derivatives of n and n_T :

$$\frac{dn}{d \ln k} = -\frac{1}{32\pi^2} \left(\frac{m_{\text{Pl}}^3 V'''}{V} \right) \left(\frac{m_{\text{Pl}} V'}{V} \right) + \frac{1}{8\pi^2} \left(\frac{m_{\text{Pl}}^2 V''}{V} \right) \left(\frac{m_{\text{Pl}} V'}{V} \right)^2 - \frac{3}{32\pi^2} \left(m_{\text{Pl}} \frac{V'}{V} \right)^4 \quad (11)$$

$$\frac{dn_T}{d \ln k} = \frac{1}{32\pi^2} \left(\frac{m_{\text{Pl}}^2 V''}{V} \right) \left(\frac{m_{\text{Pl}} V'}{V} \right)^2 - \frac{1}{32\pi^2} \left(\frac{m_{\text{Pl}} V'}{V} \right)^4. \quad (12)$$

Equivalent expressions can be obtained by using the previous equations relating r and $n-1$ to the potential and its first two derivatives:

$$\frac{dn}{d \ln k} = \mp \frac{1}{16\pi^2} \sqrt{\frac{2\pi}{7}} \left(\frac{m_{\text{Pl}}^3 V_N'''}{V_N} \right) \sqrt{r} + \frac{4}{7}(n_N - 1)r + \frac{6}{49}r^2, \quad (13)$$

$$\frac{dn_T}{d \ln k} = -n_T[(n-1) - n_T] = \frac{r}{7} \left[(n-1) - \frac{1}{7}r \right], \quad (14)$$

where the upper sign applies if $V_N' > 0$ and the lower if $V_N' < 0$, and the factors of $\frac{1}{7}$ arise from using the consistency relation $n_T = -r/7$. From these expressions we see that the size of both $dn/d \ln k$ and $dn_T/d \ln k$ is controlled by the ratio of tensor to scalar perturbations, and further, that the size of $dn_T/d \ln k$ depends upon the difference between $n-1$ and n_T , which in many models is small.

We now quantify expectations in several popular models of inflation. As noted earlier, for an exponential potential $dn/d \ln k = dn_T/d \ln k \equiv 0$. For inflation models that are based upon Coleman-Weinberg like potentials, $V(\phi) = B\sigma^4/2 + B\phi^4[\ln(\phi^2/\sigma^2) - 1/2]$,

$$dn/d \ln k \simeq -1.2 \times 10^{-3} (50/N)^2, \quad (15)$$

$$d^m n/d \ln k^m \simeq -3m!/N^{m+1}. \quad (16)$$

Chaotic-inflation models are usually based upon potentials of the form $V(\phi) = a\phi^b$ (a is a constant and $b = 2, 4, \dots$ is an even integer) and

$$dn/d \ln k = -4 \times 10^{-4} (b/2 + 1)(50/N)^2, \quad (17)$$

$$d^m n/d \ln k^m = -m!(b/2 + 1)/N^{m+1}. \quad (18)$$

For the interesting cases of $b = 2$ and 4 , $dn/d \ln k = -0.8 \times 10^{-3}$ ($b = 2$, $N = 50$) and -1.2×10^{-3} ($b = 4$, $N = 50$). Finally, for the ‘‘natural’’ inflation model, where $V(\phi) = \Lambda^4[1 + \cos(\phi/f)]$, the following approximate expression applies for $f \lesssim m_{\text{Pl}}$ (which is the regime where the deviation from scale invariance is significant and $1 - n \simeq m_{\text{Pl}}^2/8\pi f^2$):

$$dn/d \ln k = -\frac{\pi^2}{4}(n-1)^2 \exp[N(n-1)]. \quad (19)$$

Varying $(1 - n)$ from 0.04 to 0.3 and N from 40 to 50, $dn/d \ln k$ varies from about -10^{-7} to almost -10^{-3} . Even though $1 - n$ can be large in these models, r is very small when it is, and $dn/d \ln k$ never approaches $(1 - n)^2$.

It is not a complete surprise that $dn/d\ln k$ is similar in all these models. One might expect that $(n-1) \propto 1/N^m$, so that $dn/d\ln k = -dn/dN = m(n-1)/N$. This is true for new and chaotic inflation where $m = 1$. (The situation with “natural” inflation is more complicated as $n-1$ depends upon N exponentially.)

In models where $(n-1) \propto 1/N$, $dn/d\ln k$ is $\mathcal{O}(n^{-2})$ and necessarily not too large. Are there models where running is more important? For ad hoc potentials the answer is yes. Consider the potential $V(\phi) = V_0 \exp(-a\phi^b)$. Here

$$(n-1) = -\frac{8\pi}{(2-b)^\alpha} \left[\frac{ab}{8\pi} \right]^{2-\alpha} \frac{1}{N^\alpha} + \frac{\alpha}{N}, \quad (20)$$

where $\alpha = 2(1-b)/(2-b)$. For $\alpha < 1$, both $(n-1) \sim 1/N^\alpha$ and $dn/d\ln k \simeq \alpha(n-1)/N$ can be large. (For $b = 1$, this potential reduces to a simple exponential and $dn/d\ln k = 0$.)

Intermediate inflation provides an even more interesting example [18]. These models are based upon a potential of the form $V(\phi) = a\phi^{-b}$. Inflation occurs for $\phi^2 \geq b^2/16\pi m_{\text{Pl}}^2$ and never ends. (In order to construct a successful model of inflation the form of the potential must eventually change or intermediate inflation must be part of a first-order inflation model.) The interesting feature of intermediate inflation is that $(n-1)$ depends upon the number of e-folds since the *beginning* of inflation, $n-1 = (2b-4)/(4N_b+b)$. Not worrying about how inflation ends and noting that $dN = -dN_b$, it follows

$$\frac{dn}{d\ln k} = \frac{(n-1)^2}{b/2-1}. \quad (21)$$

Unlike the previous examples where $(n-1)$ depended upon $1/N$ and thus was of order $\text{few} \times 10^{-2}$, here $(n-1)$ can be arbitrarily large. Thus, $dn/d\ln k$ too can be large: taking $b = 3$ and $n-1 = 0.9$, $dn/d\ln k = 0.02$.

As mentioned earlier, the running of the tensor spectral index is expected to be less important because the tensor perturbations are likely to be subdominant and only contribute significantly for $l \lesssim 50$; in addition, $dn_T/d\ln k$ is smaller (being proportional to the difference between $n-1$ and n_T which is often small). For the potentials discussed above $dn_T/d\ln k = 0$ (exponential), $-2 \times 10^{-7}(\sigma/m_{\text{Pl}})^4(50/N)^4$ (new), $-2.0 \times 10^{-4}b(50/N)^2$ (chaotic), and $-\pi^2/4(n-1)^2 \exp[N(n-1)]$ (“natural”).

Figure 1 displays the CBR angular power spectrum for $b = 6$ chaotic inflation (where $dn/d\ln k = -0.0020$), calculated without and with the running of the scalar spectral index. The correction due to the running of scalar index is significant (about 10%) and potentially measurable. The results shown have been calculated using the power spectrum in Eq. (7). We have also calculated the C_l 's using the approximation in Eq. (8), and the maximum error in any C_l is less than 0.6%. Thus, for applications requiring $\mathcal{O}(1\%)$ accuracy, it should be sufficient to calculate a model with a fixed n and then scale the results according to Eq. (8) to obtain C_l 's for $dn/d\ln k \neq 0$. We also note that the correct k -space power spectrum is simple to include in any Boltzmann code.

In summary, expectations for $|dn/d\ln k|$ in popular inflationary models range from -2×10^{-3} to around -4×10^{-4} . Of course, the value of $dn/d\ln k$ in “the model of inflation” could be larger or smaller. At the high end of this range, neglecting the running of scalar spectral index leads to errors of 10%, more than an order of magnitude larger than the accuracy

desired [19]. The running of the scalar spectral index can be into account easily, accurately, and with generality by using $dn/d\ln k$. Based upon the models we have looked at one could adopt $dn/d\ln k \simeq (n - 1)/N$ as a default estimate.

If the running of the scalar spectral index is large enough to detect, the third derivative of the scalar potential can be measured [16]:

$$V_N'''/m_{\text{Pl}} = \pm 39\sqrt{r} [-7(dn/d\ln k)/r + 0.9r + 4(n - 1)] Q_T. \quad (22)$$

The feasibility of determining $dn/d\ln k$ from a high-resolution map of the CBR sky is currently under study [20].

Two final points. First, what about the next-order corrections? They involve $\mathcal{O}[(n - 1)^3, n_T^3, \dots]$ terms: corrections to $dn/d\ln k$, n , Q_T , Q_S and the $d^2n/d\ln k^2$ term in the expansion for n . Provided that the deviation from scale invariance is not too large, they should be small (less than about 1%) because they are suppressed by an additional factor of $\mathcal{O}[(n - 1), n_T]$; e.g., $dn^2/d\ln k^2 = -5 \times 10^{-5}(50/N)^3$ for new inflation, which leads to a correction at $l = 1000$ of about 0.5%. If the $d^2n/d\ln k^2$ should be larger, its size might be turned to good purpose; because of the qualitative difference between it and the $d/d\ln k$ term it might possibly be measured, revealing information about the fourth derivative of $V(\phi)$.

Last, but perhaps not least, the running of the scalar spectral index is also of some relevance when extrapolating a COBE-normalized spectrum to astrophysical scales; e.g., the correction to σ_8 is about -3% for $dn/d\ln k = -10^{-3}$ [21].

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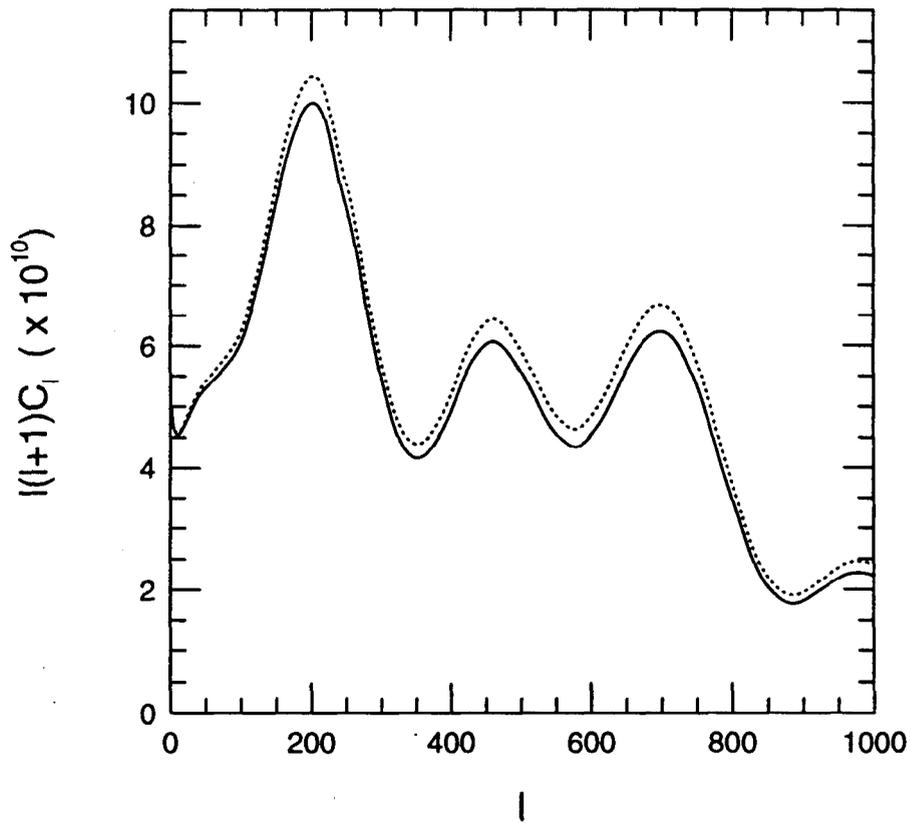


FIG. 1. Predicted angular power spectra for $b = 6$ chaotic inflation with (solid) and without (broken) the running of the scalar spectral index ($n = 0.92$, $h = 0.7$, $\Omega_B = 0.025$, and $dn/d\ln k = -0.002$).