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# Microwave Stability at Transition

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## Abstract

The question of microwave stability at transition is revisited using a Vlasov approach retaining higher order terms in the particle dynamics near the transition energy. A dispersion relation is derived which can be solved numerically for the complex frequency in terms of the longitudinal impedance and other beam parameters. Stability near transition is examined and compared with simulation results.

## I. Introduction

The question of microwave stability at transition has long been an issue for machines which must pass through transition energy. Due to the fact that the relative motion of particles at transition goes to zero, Landau damping is presumed to vanish. However, growth rates may also be sufficiently long to prevent significant mode growth. Recent theoretical studies have suggested that transition is absolutely stable against microwave modes owing to a particular cancellation of resonant contributions [1], although this analysis was based on a truncated model of the particle dynamics.

In this work we would like to reconsider microwave stability at transition including a necessarily higher-order expansion of the particle motion around the transition point. This is done in order to resolve the pole-cancellation issue referred to above. In particular, we find that while a portion of the distribution may indeed be stable near transition, those particles which exist slightly off transition in a distribution of finite momentum spread will always lead to instability. By retaining higher-order terms in the particle motion, we find an extension of the usual linear stability model for longitudinal modes which shows the appearance of a new unstable branch. The resulting dispersion relation is solved numerically for the stability boundary in the impedance plane.

As a confirmation of the analytical results, we have performed particle simulations in a coasting beam, consistent with the notion of short-wavelength modes associated with microwave instability. Using this approach, we find that regions of instability always occur above transition that can lead to longitudinal emittance blowup.

## II. Theory

The following dispersion relation can be derived from the Vlasov equation [1] which expresses the relation between

the impedance and the coherent frequency of the collective mode.

$$1 = \left(\frac{e\omega_0}{2\pi}\right)^2 N Z_n \int_C d\epsilon \frac{\frac{\partial}{\partial \epsilon} \psi(\epsilon)}{i[n\omega(\epsilon) - \Omega_n]} \quad (1)$$

where  $N$  is the number of particles,  $n$  is the harmonic number,  $Z_n$  is the impedance associated with the  $n$ th harmonic,  $\psi$  is a normalized distribution function which is a solution to the Vlasov equation,  $\epsilon$  is the energy deviation from the synchronous particle which is referred to by the subscript 0, and  $\Omega_n$  is the coherent frequency. The integral contour is chosen so that  $\Omega_n$  is continuous while crossing the real axis. The frequency  $\omega(\epsilon)$  in terms of the dispersion coefficients is given by

$$\omega(\epsilon) = \omega_0 - \omega_0 \eta_0 \delta - \omega_0 \frac{\eta_1}{2} \delta^2 \quad (2)$$

where

$$\eta_0 = \alpha_0 - \frac{1}{\gamma_0^2}, \quad (3)$$

$$\frac{\eta_1}{2} = \alpha_0 - \alpha_1 - \eta_0 \alpha_0 \frac{3}{2} \frac{\beta_0^2}{\gamma_0^2} \quad (4)$$

and

$$\delta = \frac{\Delta p}{p} = \frac{1}{\beta^2} \frac{\epsilon}{E_0}. \quad (5)$$

The quantities  $\alpha_0$  and  $\alpha_1$  are the momentum compaction factors.

We have solved Eqn. 1 for  $Z_n/n$  assuming a Gaussian distribution

$$\psi(\epsilon) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2\sigma^2}}. \quad (6)$$

The integral can be reduced to evaluating the plasma dispersion integral which can be expressed in terms of the complex error function. The details are outlined in the appendix. The results are

$$\frac{Z_n}{n} = \left[ \frac{-ie^2 \omega_0^2 N \beta^4 E_0^2}{\sqrt{2}(2\pi)^2 \sigma^3 \eta_1} \times \left( A \cdot Z\left(\frac{\epsilon_1}{\sqrt{2}\sigma}\right) + B \cdot Z\left(\frac{\epsilon_2}{\sqrt{2}\sigma}\right) \right) \right]^{-1} \quad (7)$$

The quantities  $A, B, \epsilon_1$  and  $\epsilon_2$  are defined in Eqns. 15-17.

## III. Calculations

### A. Stability Diagram

A program was written to plot the real part of  $Z_n/n$  vs. the imaginary part for Eqn. 7 for different values of the coherent frequency  $\Omega_n$  and different places near transition. Figure 1 is a plot of the stability diagram below transition. The dots are for a real coherent frequency and the pluses

<sup>†</sup> Operated by the Universities Research Association, Inc, under contract with the U.S. Department of Energy

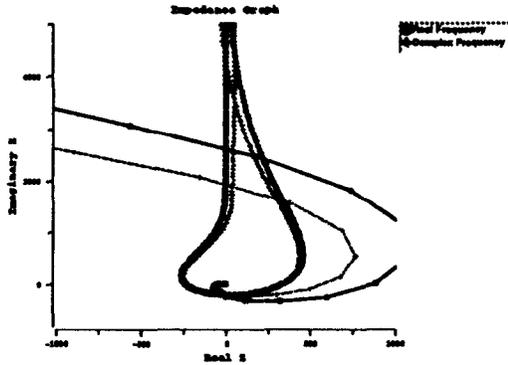


Figure 1. Stability diagram below transition. There are no regions of instability.

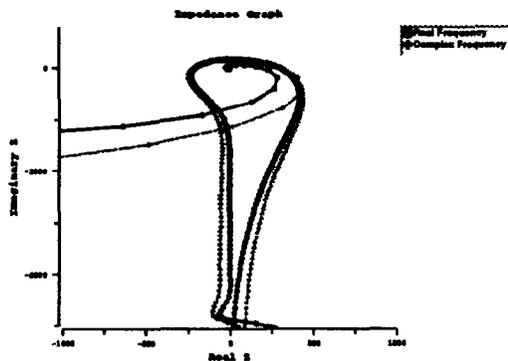


Figure 2. Stability diagram above transition.

are for a complex coherent frequency. The beam is stable.

Figure 2 is a plot of the stability diagram above transition. The dots are for a real coherent frequency and the pluses are for a complex coherent frequency. There are regions of instability.

### B. Particle Simulation

Simulations of coherent phenomena in coasting beams were first reported in 1975 [2]. The essential physics is contained in the character of the incremental kicks given to the particle's position and energy per turn, relative to the central momentum particle. These may be expressed in the form

$$\delta\theta = -\eta(\epsilon)\epsilon \quad (8)$$

$$\delta\epsilon = \frac{e^2}{(2\pi)^2 E_0} \sum_n e^{in\theta} \int_{-\infty}^{\infty} Z_{\parallel n}(\omega) e^{-i\omega t} d\omega \quad (9)$$

where  $Z_{\parallel}$  is the longitudinal impedance and is the Fourier transform of the wake function given by

$$Z_{\parallel}(\omega) = \int_{-\infty}^{\infty} e^{\frac{i\omega}{v} s} W_0'(-s) ds \quad (10)$$

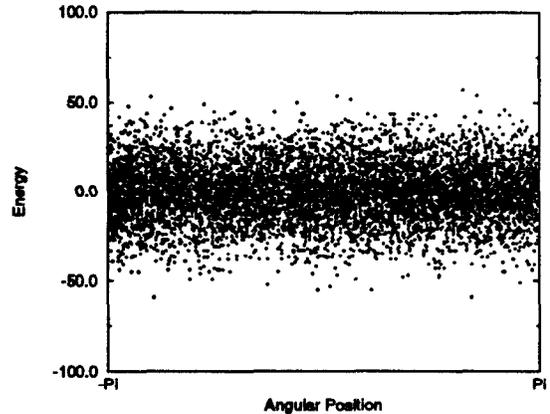


Figure 3. Particle simulation below transition. The distribution is stable.

It is readily shown that Eqs. (8) and (9), in the case of small perturbations, lead to the linear dispersion relation for longitudinal modes. We note that  $\eta$  is a function of  $\epsilon$  and may go to zero, which is the formal definition of transition. We keep both first and second-order corrections to  $\eta$  in our simulation to correspond to the analytical model described previously.

The time domain representation of the wake field is most convenient for computational purposes and this is given in the form [3]

$$V(\theta) = \frac{\omega_r R}{\omega_o Q} \int_{\theta}^{2\pi+\theta} d\theta' I(\theta') e^{-\frac{\omega_r}{2Q\omega_o} \theta'} \times \left( \cos \frac{\tilde{\omega}_r}{\omega_o} \theta' - \frac{1}{2Q} \left[ 1 - \frac{1}{4Q^2} \right]^{-\frac{1}{2}} \sin \frac{\tilde{\omega}_r}{\omega_o} \theta' \right) \quad (11)$$

where

$$\tilde{\omega}_r = \omega_r \left( 1 - \frac{1}{4Q^2} \right)^{\frac{1}{2}}, \quad (12)$$

$I(\theta)$  is the current distribution and  $\omega_o$  is the revolution frequency. The integration over angle is carried out at a fixed time each turn and may be extended into previous turns for long-range wakes (sufficiently high  $Q$ ).

For the simulations in this work, we typically use  $10^4 - 10^5$  particles and invoke periodic boundary conditions associated with the lowest revolution harmonic of interest. Figure 3 is a simulation of a beam before transition. The beam is stable confirming the results of Figure 1. Above transition, the simulation (Figure 4) shows that there is instability confirming the results of Figure 2.

## IV. Conclusions

We have revisited the question of microwave stability at transition and have shown by including higher-order terms of the expansion of particle motion around the transition point that particles which are slightly off transition in a distribution of finite momentum spread will always lead to instability above transition.

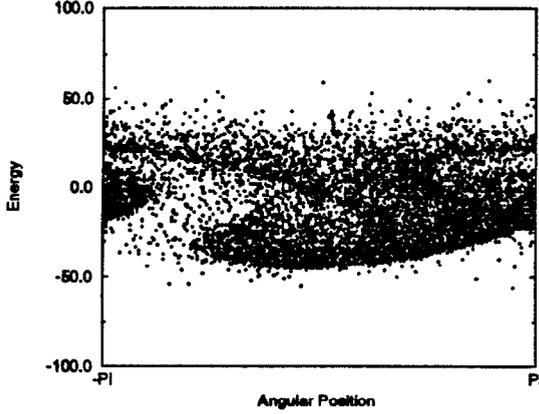


Figure. 4. Particle simulation above transition. An instability has developed.

## V. Appendix

The integral in Eqn. 1 can be written with some factorization as

$$\int_C \epsilon e^{-\frac{\epsilon^2}{2\sigma^2}} d\epsilon \left[ \epsilon^2 + \frac{\beta^2 E_0 \eta_0}{\eta_1} \epsilon - \frac{\beta^4 E_0}{\eta_1} \left( 1 - \frac{\Omega_n}{n\omega_0} \right) \right]^{-1} \quad (13)$$

The integral can be broken up into pieces by the method of partial fractions and reduces to

$$A \cdot Z\left(\frac{\epsilon_1}{\sqrt{2}\sigma}\right) + B \cdot Z\left(\frac{\epsilon_2}{\sqrt{2}\sigma}\right) \quad (14)$$

where

$$A = \frac{\epsilon_1}{\epsilon_1 - \epsilon_2}, B = \frac{\epsilon_2}{\epsilon_2 - \epsilon_1}, \quad (15)$$

$$\epsilon_1 = -\frac{\beta^2 E_0 \eta_0}{2\eta_1} + \sqrt{\left(\frac{\beta^2 E_0 \eta_0}{2\eta_1}\right)^2 + 4\frac{\beta^4 E_0}{\eta_1} \left(1 - \frac{\Omega_n}{n\omega_0}\right)} \quad (16)$$

and

$$\epsilon_2 = -\frac{\beta^2 E_0 \eta_0}{2\eta_1} - \sqrt{\left(\frac{\beta^2 E_0 \eta_0}{2\eta_1}\right)^2 + 4\frac{\beta^4 E_0}{\eta_1} \left(1 - \frac{\Omega_n}{n\omega_0}\right)} \quad (17)$$

The function

$$Z(\xi) = \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x - \xi} dx \quad (18)$$

is the plasma dispersion function which can be evaluated numerically in terms of the complex error function.

## References

- [1] A. Bogacz, "Microwave Stability at Transition - Stability Diagram Approach", *1991 IEEE Particle Accelerator Conference*, p 1815, 1991.
- [2] E. Keil and E. Messerschmid, *Nucl. Inst. and Methods*, Vol 128, p203.
- [3] J. A. MacLachlan, FN-446, Fermilab (1987).