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LATTICE QCD AND THE STANDARD MODEL*

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ABSTRACT

Most of the poorly known parameters of the Standard Model cannot be determined without reliable calculations in nonperturbative QCD. Lattice gauge theory provides a first-principles definition of the required functional integrals, and hence offers ways of performing these calculations. This paper reviews the progress in computing hadron spectra and electroweak matrix elements needed to determine α_S, the quark masses, and the Cabibbo-Kobayashi-Maskawa matrix.

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1 Introduction

Many contemporary reviews of elementary particle physics start by celebrating (or lamenting!) the success of the Standard Model. Indeed, with some nineteen* parameters the SU(3)×SU(1)×U(1) gauge theory explains an enormous array of experiments. Even a terse compendium[1] of the experiments is more than big enough to fill a phone book. A glance at Table 1 shows, however, that roughly half of the parameters are not so well determined. To test the Standard Model stringently, and thus to gain an inkling of what lies beyond, we must learn the values of these parameters more precisely.

Except for the mass of the Higgs boson (or any other undiscovered remnant of electroweak symmetry breaking), the poorly known parameters all involve quarks. Other than top,[2] which decays too quickly for confinement to play a role, the masses of the quarks are a bit better than wild guesses. The information on the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix is spotty, especially when one relaxes the assumption of three-generation unitarity, as shown in Table 2. They are poorly determined simply because experiments measure properties not of quarks, but of the hadrons inside which they are confined. Of course, everyone knows what to do: calculate with QCD, the part of the Standard Model that describes the strong interactions. But then, the strong coupling is known only at the 5% level; not bad, but nothing like the fine structure or Fermi constants. Moreover, the binding of quarks into hadrons is nonperturbative—the calculations cannot be done on the back of an envelope.

The most systematic technique for understanding nonperturbative QCD is lattice gauge theory. The lattice provides quantum field theory with a consistent and mathematically well-defined ultraviolet regulator. At fixed lattice spacing, the quantities of interest are straightforward (combinations of) functional integrals. These integrals can be approximated by a variety of techniques borrowed from statistical mechanics. Especially promising is a numerical technique, the Monte Carlo method with importance sampling, which has become so pre-eminent that the young and uninitiated probably haven't heard of any other.

Results from lattice-QCD Monte Carlo calculations have begun to influence Table 1. The world average for the SU(3) gauge coupling α_S includes results from lattice calculations of the quarkonium spectrum,[4, 5, 6] and at the time of this conference an even more precise result had appeared.[7] The same calculations are also providing some of the best information on the charm[8] and bottom[9] masses. This is an auspicious beginning. Over the next several years the lattice QCD calculations will mature. They will help to determine the other unknowns—light quark masses and the CKM matrix. The third column of Tables 1 and 2 lists relevant quantities or processes, and the rest of this talk explains how the program fits together.

*I assume the neutrinos are massless, but count the "vacuum angle" of QCD.

Table 1: Parameters of the standard model and lattice calculations that will help determine them. Numerical values taken from the 1994 Review of Particle Properties,[1] except m_t (Ref. 2), $\sin\delta$, and $\theta_{\rm QCD}$. The strong coupling α_S refers to the $\overline{\rm MS}$ scheme at M_Z. Adapted from Ref. 3.

parameter	value or range	related lattice calculations
gauge couplings		
$\alpha_{\rm em}$	1/137.036	
$10^5 G_F$	1.166 GeV^{-2}	
α_S	0.116 ± 0.005	Δm_{1P-1S}; scaling
electroweak masses		
m_Z	91.19 GeV	
m_H	> 58 GeV	
lepton masses		
m_e	0.51100 MeV	
m_μ	105.66 MeV	
m_τ	1777 MeV	
quark masses		
m_u	2–8 MeV	m_π^2, m_K^2
m_d	5–15 MeV	m_π^2, m_K^2
m_s	100–300 MeV	m_K^2
m_c	1.0–1.6 GeV	$m_{J/\psi}$
m_b	4.1–4.5 GeV	m_Υ
m_t	$174 \pm 10^{+13}_{-12}$ GeV	
CKM matrix		
s_{12}	0.218–0.224	$K \to \pi e \nu$
s_{23}	0.032–0.048	$B \to D^* l \nu$
s_{13}	0.002–0.005	$B \to \pi l \nu$
$\sin\delta$	$\neq 0$	B_K, B_B, B_{B_s}
QCD vacuum angle		
$\theta_{\rm QCD}$	$< 10^{-9}$	d_n

Sect. 2 gives the non-expert some perspective on the conceptual and numerical strengths and weaknesses of lattice QCD. Sect. 3 reviews 1) the status of the light hadron spectrum and the propects for extracting m_u, m_d, and m_s; and 2) results for the quarkonium spectrum, which yield α_S, m_b, and m_c. Sect. 4 outlines lattice QCD calculations of electroweak, hadronic matrix elements that are needed to pin down the unitarity triangle of the CKM matrix. There are, of course, many other interesting applications to electroweak phenomenology; for more comprehensive reviews

Table 2: Ranges for CKM matrix elements $|V_{qr}|$ assuming unitarity but *not* three generations. Numerical values taken from the 1994 Review of Particle Properties. In three generations $|V_{ud}| = s_{12}$, $|V_{cb}| = s_{23}$, and $|V_{ub}| = s_{13}$, to excellent approximation.

parameter	value or range	related lattice calculations
$\|V_{ud}\|$	0.974	
$\|V_{us}\|$	0.218–0.224	$K \to \pi e \nu$
$\|V_{ub}\|$	0.002–0.005	$B \to \pi l \nu$
$\|V_{cd}\|$	0.180–0.228	$D \to \pi l \nu$
$\|V_{cs}\|$	0.800–0.975	$D \to K l \nu$
$\|V_{cb}\|$	0.032–0.048	$B \to D^* l \nu$
$\|V_{td}\|$	0.0–0.13	$f_B^2 B_B$; B_K
$\|V_{ts}\|$	0.0–0.56	$f_{B_s}^2 B_{B_s}$
$\|V_{tb}\|$	0.0–0.9995	

the reader can consult some of the papers listed in the bibliography.[3, 10]

2 Rudiments of Lattice Gauge Theory

In quantum field theory physical measurements are related to vacuum expectation values $\langle \mathcal{O} \rangle$. Feynman's functional integral representation is

$$\langle \mathcal{O} \rangle = \lim_{L \to \infty} \lim_{a \to 0} Z^{-1} \int \prod_{x,\mu} dA_\mu(x) \prod_{x,a} d\psi_a(x) d\bar{\psi}_a(x)\, \mathcal{O}\, e^{-S(A_\mu, \psi, \bar{\psi})}. \tag{1}$$

The formula is easier to understand if read from right to left. S is the action—in our case the action of QCD, so it depends on the gluons A_μ and the quarks ψ and anti-quarks $\bar{\psi}$. $\mathcal{O}$ depends on the physics under investigation; the most useful kinds of $\mathcal{O}$'s are given below. The integration over all components and positions of the basic fields, with weight e^{iS} would reproduce the familiar Schrödinger or Heisenberg formulations of quantum mechanics. The more convergent weight e^{-S} provides some benefits and imposes some restrictions—see below. Z is the same integral without $\mathcal{O}$ in it, so that $\langle 1 \rangle = 1$.

The limits are there to satisfy the mathematicians; without them the integrals are not well defined. These "cutoffs" also have a physical significance: we do not claim to understand physics either at distances smaller than a (the ultraviolet cutoff), or at distances larger than L (the infrared cutoff). The limit $a \to 0$ requires the renormalization group; it must be carried out holding L and physical, infrared scales fixed. In particular, the integration variables $A_\mu(x)$, $\psi(x)$, and $\bar{\psi}(x)$ really represent

all degrees of freedom in a block of size a^4. The limit "$a \to 0$" can be obtained not only literally, but also by improving the action of the blocked fields.

These observations apply to any cutoff scheme for quantum field theory. A nice introduction to the renormalization-group aspects is a summer-school lecture by Lepage.[11] In lattice gauge theory a is nothing but the spacing between lattice sites. If there are N on a side, $L = Na$. For given N one can compute the integrals numerically. With the 10^7–10^{10}-dimensional integrals that arise, the only viable technique is a statistical one: Monte Carlo with importance sampling.

To compute masses the observable $\mathcal{O} = \Phi(t)\Phi^\dagger(0)$, where $\Phi(t)$ is an operator at time t with the flavor and angular-momentum quantum numbers of the state of interest. One can construct such operators using symmetry alone. The radial quantum number would require a solution of the theory, but that's what we're after. The "two-point function"

$$\langle \Phi(t)\Phi^\dagger(0)\rangle = \sum_n |\langle 0|\Phi|n\rangle|^2\, e^{-m_n t}, \tag{2}$$

where the sum is over the radial quantum number. The exponentials are a happy consequence of the weight e^{-S} in eq. (1). It is advantageous because at long times t only the lowest-lying state survives. In a numerical calculation masses are obtained by fitting two-point functions, once single-exponential behavior is verified. Since Φ is largely arbitrary, some artistry enters: if single-exponential behavior sets in sooner, the statistical quality of the mass estimate is better.

To compute a matrix element of part of the electroweak Hamiltonian, $\mathcal{H}$, the observable $\mathcal{O} = \Phi_\pi(t_\pi)\mathcal{H}(t_h)\Phi_B^\dagger(0)$ for the transition from hadron "B" to hadron "π." At long times t_h and $t_\pi - t_h$ the "three-point function"

$$\langle \Phi_\pi(t_\pi)\mathcal{H}(t_h)\Phi_B^\dagger(0)\rangle \approx \langle 0|\Phi_\pi|\pi\rangle e^{-m_\pi(t_\pi - t_h)}\langle \pi|\mathcal{H}|B\rangle e^{-m_B t_h}\langle B|\Phi_B^\dagger|0\rangle, \tag{3}$$

plus excited-state contributions. If, as in decays of hadrons to leptons, the hadronic final-state is the vacuum, a two-point function will do:

$$\langle \mathcal{H}(t)\Phi_B^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{H}|B_n\rangle e^{-m_n t}\langle B_n|\Phi_B^\dagger|0\rangle. \tag{4}$$

The desired matrix elements $\langle \pi|\mathcal{H}|B\rangle$ and $\langle 0|\mathcal{H}|B\rangle$ can be obtained from eq. (3) and (4), because the masses and Φ-matrix elements are obtained from eq. (2).

To obtain good results from eqs. (2)–(4), it is crucial to devise nearly optimal operators in the two-point analysis. Consumers of numerical results from lattice QCD should be wary of results, still too prevalent in the literature, that are contaminated by unwanted states.

In the numerical work that mostly concerns us here, the integrals are computed at a sequence of fixed a's and L's. One adopts a standard mass, say m_ρ, and defines

$$a = \frac{(am_\rho)^{\mathrm{lQCD}}}{m_\rho^{\mathrm{expt}}} \tag{5}$$

to obtain the latice spacing in physical units, and other quantities are predicted via

$$m_B = \frac{(am_B)^{\text{lQCD}}}{a}. \tag{6}$$

For continuum-limit, infinite-volume results this is the same as extrapolating dimensionless ratios, e.g.

$$\frac{m_B}{m_\rho} = \lim_{L\to\infty} \lim_{a\to 0} \frac{am_B(a,L)}{am_\rho(a,L)}. \tag{7}$$

There is theoretical guidance for both limits. According to general properties of massive quantum field theories in finite boxes,[12] the infinite-volume limit is rapid for $m_\pi L \gg 1$, exponential for masses. In non-Abelian gauge theories the renormalization-group $a \to 0$ limit is controlled by asymptotic freedom.

The main strength of lattice QCD is that it *is* QCD. It has $1 + n_f$ adjustable parameters, corresponding to the gauge coupling and the quark masses. From the renormalization group, the adjustment of the gauge coupling is equivalent to setting the lattice spacing in physical units, eq. (5). Once the parameters are determined by $1 + n_f$ experimental inputs, QCD should predict all other strong-interaction phenomena. There is no need to introduce condensates, as in ITEP sum rules, or non-renormalizable couplings, as in chiral perturbation theory or heavy-quark effective theory. If theory and experiment disagree, it is a signal of new physics.

There are some disadvantages. A practical, though not conceptual, problem is that large-scale computational work is more labor-intensive than traditional theoretical physics. Careful work is needed to estimate the uncertainties reliably. The improvements in computer power and algorithms of recent years have helped practitioners understand their uncertainties better and better. As the consumers of their results become commensurately sophisticated, this trend will continue. After all, in the context of Table 1, meaningful error bars are just as important as the central value.

2.1 The quenched approximation

The biggest disadvantage of most of the numerical results mentioned in this talk is something called the "quenched" approximation. A meson consists of a valence quark and anti-quark exchanging any number of gluons. The gluons can turn into virtual quark loops and back again. The latter process costs a factor of 100-1000 in computer time, so many Monte Carlo programs just omit the virtual quark loops. To accentuate the positive—gluons and valence quarks are treated better than in non-QCD models of hadrons—the omission is sometimes called the *valence* approximation. To admit the negative, it is less often called the *loopless* approximation. But most often lattice mavens borrow a jargon from condensed-matter physics and call it the *quenched* approximation.

If quenched QCD makes any sense, it is as a kind of model or effective theory. The parameters of quenched QCD can be tuned to reproduce physics at one scale.

But the β function of quenched and genuine QCD differ, as one sees in perturbation theory, so one cannot expect agreement at all scales. As with any model, only in special cases can one argue that these effects are correctable or negligible; these cases will be highlighted in the rest of the talk.

3 From Hadron Spectra to the QCD Parameters

3.1 Light hadrons and light-quark masses

Over the past few years a group at IBM has carried out a systematic calculation of the light-hadron spectrum using the dedicated supercomputer GF11.[13] They have numerical data for 5 different combinations of (a, L). At $L \approx 2.3$ fm there are three lattice spacings varying by a factor of ~ 2.5. At the coarsest lattice spacing ($a^{-1} \approx 1.4$ GeV) there are three volumes, up to almost 2.5 fm. A variety of quark masses are used, and the physical strange quark is reached by interpolation, whereas the light (up and down) through extrapolation. The mass dependence is assumed linear, as expected from weakly broken chiral symmetry, and the data substantiate the assumption.

The units (i.e. lattice spacing) has been fixed with m_ρ and the quark masses with m_π and m_K. The final results, after extrapolation to the continuum limit and infinite volume, are shown in Fig. 1 for two vector mesons and six baryons. (The quark-mass interpolation could reach only the combination $m_\Xi + m_\Sigma - m_N$.) Despite the quenched approximation the agreement with experiment is spectacular.

Fig. 1 also includes results from the same investigation for decay constants. The agreement of f_π/m_ρ and f_K/m_ρ is not as good as for the masses. Because of the quenched approximation, this is not entirely unexpected. Recall the argument concerning distance scales and effective theories in sect. 2.1. The binding mechanism responsible for the masses encompasses distances out to the typical hadronic radius. The decay constant, on the other hand, is proportional to the wavefunction at the origin and thus is more sensitive to shorter distance scales. One sees better agreement when forming the ratio f_K/f_π, which—recall eq. (5) and subsequent discussion—is like retuning to the shorter distance.

One would like to use the hadron masses to extract the quark masses. Because of confinement, the quark mass more like a renormalized coupling than the classical concept of mass. Calculations like the one described above yield immediately the bare mass of the lattice theory. More useful to others would be the $\overline{\text{MS}}$ scheme of dimensional regularization. A one-loop perturbative calculation can be used to convert from one scheme to another.[15, 16, 17, 8]

For the light quarks it is convenient to discuss the combinations $\hat{m} = \frac{1}{2}(m_d + m_u)$, $\Delta m_{du}^2 = m_d^2 - m_u^2$, and m_s. Ratios of the light-quark masses are currently best estimated using chiral perturbation theory.[18] To set the overall scale requires a dynamical calculation in QCD. In lattice QCD, $\hat{m}$ and m_s can be extracted from

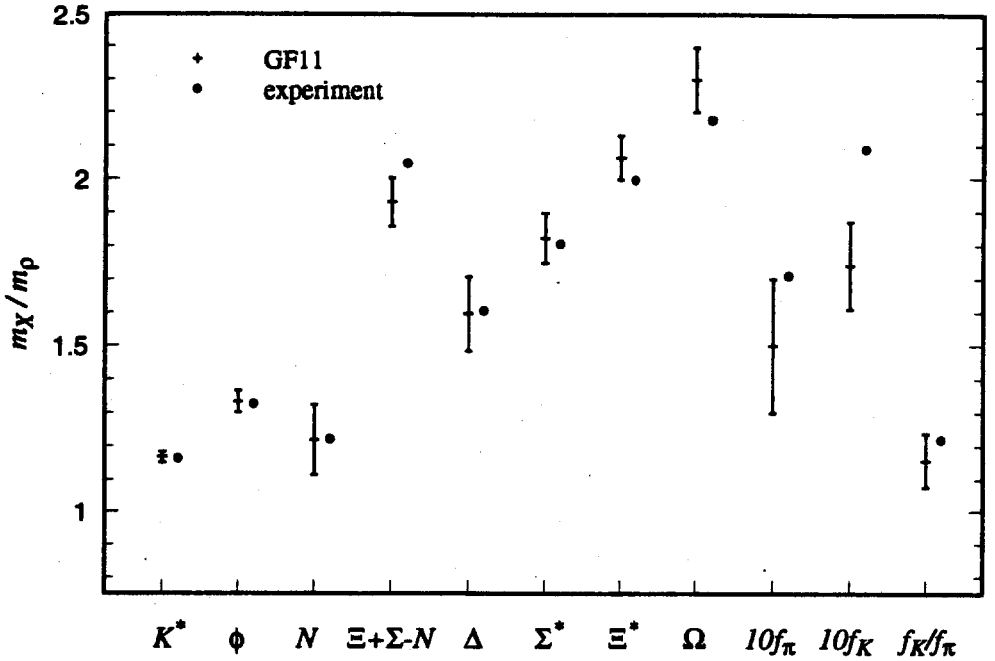

Figure 1: The spectrum and decay constants of the light hadrons. Error bars are from lattice calculations in the quenched approximation,[13, 14] and • denotes experiment.

the variation in the square of the pseudoscalar mass between m_π^2 and m_K^2. The most difficult quark-mass combination is Δm_{du}^2, which causes the isospin-violating end of the splittings in hadron multiplets. Since chiral perturbation theory provides a formula for $\Delta m_{du}^2/m_s^2$ with only second-order corrections, it is likely that the best determination of Δm_{du}^2 will come from combining the formula with a lattice QCD result for m_s.

Using the compliation of quenched and unquenched results of Ukawa,[19] Mackenzie[20] has estimated $\hat{m}_{\overline{\rm MS}}(1\ {\rm GeV}) \sim 2.3$ MeV and $m_{s,\overline{\rm MS}}(1\ {\rm GeV}) \sim 65$ MeV. The symbol $\sim$ stresses the lack of error bar. This is outside the ranges of 3.5–11.5 MeV and 100–300 MeV indicated in Table 1. A more recent analysis of the strange quark finds $m_{\overline{\rm MS},s}(2\ {\rm GeV}) = 127 \pm 18$ MeV,[21] in the lower part of the range in Table 1. None of these results should be taken seriously until a more complete error analysis exists, but it is intriguing that the conventional estimates might be too high.

3.2 *Quarkonia, α_S, and heavy-quark masses*

Quarkonia are bound states of a heavy quark and heavy anti-quark. Three families of states exist, charmonium (η_c, J/ψ, etc), bottomonium (η_b, Υ, etc), and the as yet unobserved B_c ($b\bar{c}$ and $\bar{b}c$ bound states). Compared to light hadrons, these systems are simple. The quarks are nonrelativistic, and potential models give an excellent empirical description. But a fundamental treatment of these systems requires nonperturbative QCD, i.e. lattice QCD.† Potential models can be exploited, however, to estimate lattice artifacts, and in the quenched approximation they can be used to make corrections. Many states have been observed in the lab, providing cross-checks of the methodology of uncertainty estimation.

Once the checks are satisfactory, one can use the spectra to determine α_S, m_c, and m_b. One can also have some confidence in further applications, such as the phenomenology of D and B mesons discussed in sect. 4.

For charm, and especially for bottom, the quark mass is close to the ultraviolet cutoff, $1/a$ or π/a, of present-day numerical calculations. Originally lattice gauge theory was formulated assuming $m_q a \ll 1$, so quarks $m_q a \sim 1$ require some reassessment. There are four ways to react. The patient, stolid way is to wait ten years, until computers are powerful enough to reach a cutoff of 20 GeV—not very inspiring. The naive way is to extrapolate from smaller masses, assuming the $m_q a \ll 1$ interpretation of the lattice theory is adequate; history shows that naive extrapolations can lead to naive and, thus, unacceptable error estimates. The insightful way is to formulate an effective theory for heavy quarks with a lattice cutoff;[22, 23] this is the computationally most efficient approach, and when the effectiveness of the heavy-quark expansion is *a priori* clear, it is the method of choice. The compulsive way to examine a wide class of lattice theories without assuming either $m_q a \ll 1$ or $m_q \gg (\Lambda_{\rm QCD}, a^{-1})$; by imposing physical normalization conditions on masses and matrix elements, it is possible to interpret the correlation functions at *any* value of $m_q a$.[24] The underlying reason is that the lattice theory is completely compatible with the heavy-quark limit, so the mass-dependent interpretation connects smoothly onto both the insightful method for $m_q a \gg 1$ and the standard method for $m_q a \ll 1$.

Fig. 2 shows the charmonium spectrum, on a scale appropriate to the spin-averaged spectrum. Light quark loops are quenched in these calculations.[25, 26] The agreement with experimental measurements is impressive, but Fig. 2 barely displays the attainable precision. Fig. 3 shows the fine and hyperfine structure of the P states, now for bottomonium.[25] (The $^1\mathrm{P}_1$ state h_b has not been observed in the lab; the h_c has been seen.) The authors of Ref. 25 also have results with the virtual quark loops from two light quarks, i.e. up and down are no longer quenched, but strange still is. The agreement is comparable.[27]

To obtain these results only two parameters have been adjusted. The standard

†The utility of quarkonia for testing the methodology of lattice gauge theory and the theory of QCD has been stressed over and over by Peter Lepage.

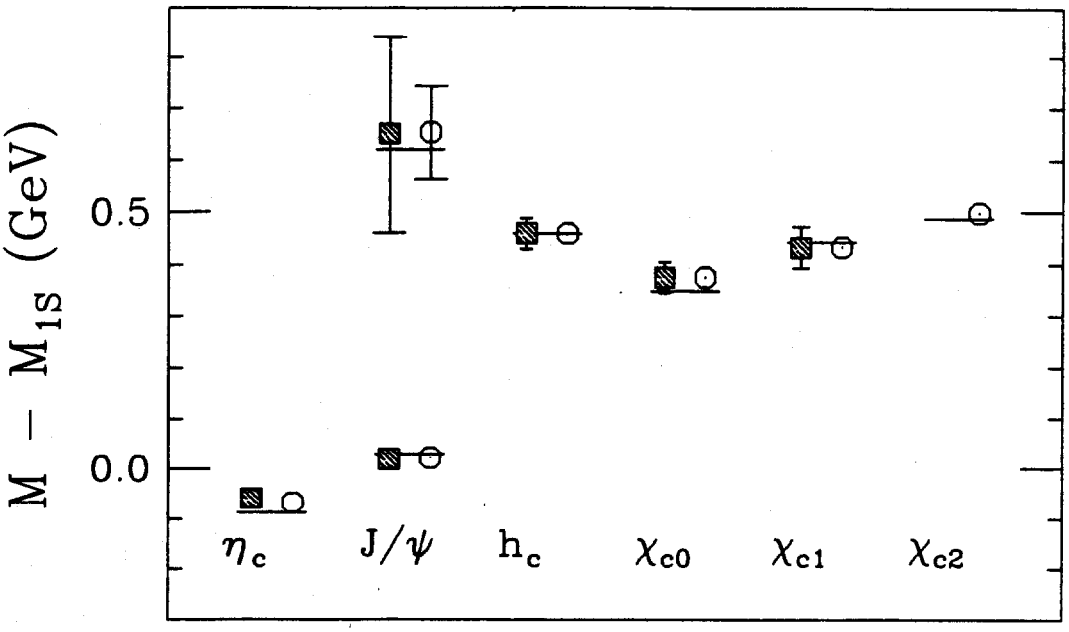

Figure 2: A comparison of the charmonium spectrum as calculated in lattice QCD, using two different methods. Ref. 25: o, Ref. 26: □. From Ref. 6.

mass in eq. (5) is Δm_{1P-1S}, the spin-averaged splitting of the 1P and 1S states, which is insensitive to the quark mass. By the renormalization group, this is equivalent to eliminating the bare gauge coupling, or to determining Λ_{QCD}. The bare quark mass is adjusted to obtain the spin average of the 1S states that is measured in the lab. Otherwise figs. 2 and 3 represent predictions of quenched QCD.

The success of these calculations permits one to extract the basic parameters, α_S and m_q. There are four steps:

1. Compute the charm- and bottomonium spectra with $n_{f,\mathrm{MC}} = 0, 2$ or 3 flavors of virtual quark loops. ($n_{f,\mathrm{MC}} = 0$ corresponds to the quenched approximation; $n_{f,\mathrm{MC}} = 2$ quenches just the strange quark; $n_{f,\mathrm{MC}} = 3$ would be the real world.)

2. With perturbation theory, convert the bare lattice coupling $\alpha_0^{(n_{f,\mathrm{MC}})}$ to the quark-potential (V) or $\overline{\mathrm{MS}}$ scheme; convert the bare lattice mass $(m_0 a)^{(n_{f,\mathrm{MC}})}$ to the pole or $\overline{\mathrm{MS}}$ scheme. The natural scale for this conversion is near (but not quite[29]) π/a.

3. Unless $n_{f,\mathrm{MC}} = 3$, correct for the quenched approximation.

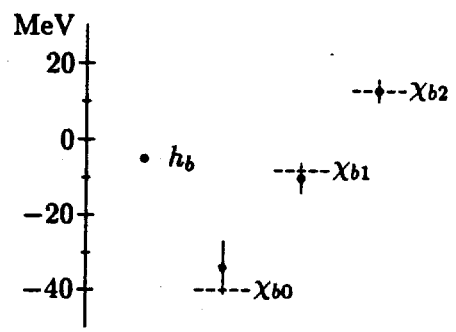

Figure 3: Lattice QCD results for the spin-dependent splittings of the lowest-lying P states in bottomonium. The dashed lines are the experimental values, where available. Energies are measured relative to the spin average of the χ states. From Ref. 25.

4. Eliminate a from $\alpha_{\overline{\rm MS}}(\pi/a)$ and $am_{\overline{\rm MS}}(\pi/a)$ using

$$a = \frac{a\Delta m_{\rm 1P-1S}}{460\ {\rm MeV}}, \tag{8}$$

where the numerator is the 1P–1S splitting in lattice units.

Steps 1 and 4 are explained above. Step 2 requires one-loop perturbation theory, suitably optimized.[29] Step 3 is crucial, because without it the results have no business in Table 1.

Consider first α_S, and recall the idea of treating the quenched approximation as an effective theory. One sees that the couplings are implicitly matched at some scale q_Q characteristic of quarkonia. So the matching hypothesis, supported by figs. 2 and 3, asserts

$$\alpha_S^{(n_{f,\rm MC})}(q_Q) = \alpha_S^{(3)}(q_Q). \tag{9}$$

Potential models tell us that $200 < q_c < 800$ MeV and $200 < q_b < 1400$ MeV. Step 3 yields $\alpha_S^{(n_{f,\rm MC})}(\pi/a)$, so one can use the two-loop perturbative renormalization group to run from π/a to q_Q. The perturbative running is an overestimate if q_Q is

taken at the lower end of these ranges.[‡] This argument was used for the original lattice determinations of the strong coupling,[4, 5] and its reliability was confirmed in $n_{f,\mathrm{MC}} = 2$ calculations.[28]

Currently the most accurate result is from Ref. 7,

$$\alpha_V^{(3)}(8.2\ \mathrm{GeV}) = 0.196 \pm 0.003, \tag{10}$$

based on $n_{f,\mathrm{MC}} = 0$ and $n_{f,\mathrm{MC}} = 2$ results, with an extrapolation in n_f. The V scheme is preferred for the matching argument, not only for physical reasons, but also because of its empirical scaling behavior.[29] The scaling behavior implies that one can run with the two-loop renormalization group to high scales and convert to other schemes. For comparison to other determinations, eq. (10) corresponds to

$$\alpha_{\overline{\mathrm{MS}}}(M_Z) = 0.115 \pm 0.002. \tag{11}$$

The quoted uncertainty is smaller than that reported from any other method. The largest contributor is the quenched correction; the second largest is the perturbative conversion $0 \to V \to \overline{\mathrm{MS}}$.

To determine the quark mass the one applies the same renormalization-group argument. But, quark masses don't run below threshold![9] Hence, for heavy quarks[§] $m_Q^{(n_{f,\mathrm{MC}})}(m_Q) = m_Q^{(n_{f,\mathrm{MC}})}(q_Q) = m_Q^{(3)}(q_Q) = m_Q^{(3)}(m_Q)$. The only corrections are perturbative, from lattice conventions to $\overline{\mathrm{MS}}$ or physical conventions. Using the convention of the perturbative "pole mass"

$$\begin{aligned} m_c &= 1.5 \pm 0.2\ \mathrm{MeV}\ (\text{Ref. 8, preliminary}), \\ m_b &= 5.0 \pm 0.2\ \mathrm{MeV}\ (\text{Ref. 9}). \end{aligned} \tag{12}$$

At the nonperturbative level confinement wipes out the pole, so the name "pole mass" should not be taken too literally. The perturbative pole mass is like a running mass, except that it has a fixed scale built into the definition. It is useful, because it is thought[7] to correspond to the mass of phenomenological models that do not probe energies less than Λ_{QCD}, such as potential models. In other contexts—such as the study of Yukawa couplings in unification scenarios—the $\overline{\mathrm{MS}}$ convention may be more appropriate. Eq. (12b) corresponds to $m_{b,\overline{\mathrm{MS}}}(m_b) = 4.0 \pm 0.1$ GeV.

4 From Matrix Elements to the CKM Matrix

Electroweak decays of flavored hadrons follow the schematic formula

$$\begin{pmatrix} \text{experimental} \\ \text{measurement} \end{pmatrix} = \begin{bmatrix} \text{known} \\ \text{factors} \end{bmatrix} \begin{pmatrix} \text{QCD} \\ \text{factor} \end{pmatrix} \begin{pmatrix} \text{CKM} \\ \text{factor} \end{pmatrix} \tag{13}$$

[‡]For light hadrons, $q_{\mathrm{light}} \sim \Lambda_{\mathrm{QCD}}$, so there would be no perturbative control whatsoever.

[§]For light quarks the threshold is deep in brown muck, and all bets are off.

North American, Japanese, and European taxpayers provide us with lots of money for the relevant experiments, because they want to know the CKM factors. But unless we calculate the inherently nonperturbative QCD factor, they will be sorely disappointed.

It is convenient to start with the assumption of three-generation unitarity. Then

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, \tag{14}$$

an equation that prescribes a triangle in the complex plane. Dividing by $V_{cd}V_{cb}^*$ and writing $V_{ud}V_{ub}^*/V_{cd}V_{cb}^* = \bar{\rho} + i\bar{\eta}$, one sees that unitarity predicts

$$\frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 1 - \bar{\rho} - i\bar{\eta}. \tag{15}$$

The notation[30] $(\bar{\rho}, \bar{\eta})$ is to distinguish these parameters from the standard Wolfenstein parameters $(\rho, \eta) = (\bar{\rho}, \bar{\eta})/|V_{ud}|$. The standard CKM phase in Table 1 is $\delta = \tan^{-1}(\eta/\rho) = \tan^{-1}(\bar{\eta}/\bar{\rho})$.

One would like to determine $(\bar{\rho}, \bar{\eta})$ through as many physical processes as possible. For example, the strength of CP violation in b-flavored hadrons is related to the angles of the triangle, hence the high interest in B factories. If all experiments agree, the test verifies the CKM explanation of CP violation; if not, the discrepency would have to be explained by physics beyond the Standard Model. Meanwhile, lattice QCD is useful for measuring the sides of the unitarity triangle. Depending on the shape of the triangle, the precision may be good enough to predict the angles before the B factories have been built.

Let us consider the CKM matrix elements in eq. (14). Assuming three-generation unitarity, $1 - |V_{tb}|$ is too small to worry about, and $|V_{cd}| - |V_{us}|$ is also very small. In principle, however, $|V_{us}|$ and $|V_{cd}|$ can be determined from lattice QCD and measurements of the semi-leptonic decays $K \to \pi e\nu$ and $D \to \pi l\nu$, respectively. The technique is the same as for $|V_{ub}|$ from $B \to \pi l\nu$, discussed below. It is unlikely that lattice QCD can, or will need to, improve on $|V_{ud}| = 0.9744 \pm 0.0010$ during the period relevant to this discussion.

The most poorly known elements of eq. (14) are $|V_{cb}|$, $|V_{ub}|$, and $|V_{td}|$. In principle, the first two can be determined from leptonic decays $B_q \to \tau\nu$, $q = u, b$, but the experimental prospects are bleak. The semi-leptonic decay is more promising. Near $q^2_{\max} = (m_B - m_{D^*})^2$ the differential decay rate for $B \to D^* l\nu$ is

$$\frac{d\Gamma}{dq^2} = \left[\frac{G_F^2 q^2}{64\pi^3 m_B}\Big((q^2_{\max} - q^2)(4m_{D^*}m_B + q^2_{\max} - q^2)\Big)^{1/2}\right]|A_1(q^2)|^2|V_{cb}|^2, \tag{16}$$

where q^2 is the invariant mass-squared of the lepton system. One must carry out a nonperturbative QCD calculation to obtain the form factor $A_1(q^2)$. By heavy-quark symmetry, however, $A_1(q^2_{\max})$ obeys a normalization condition, up to $1/m^2_{D^*}$.

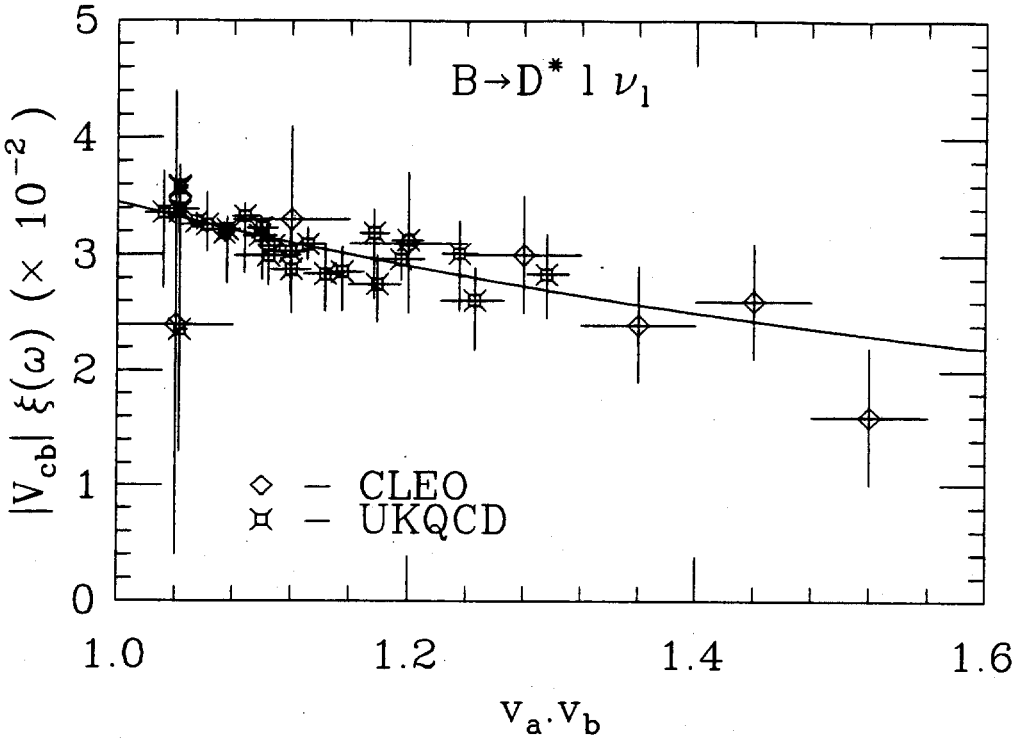

Figure 4: The Isgur-Wise function $\xi(\omega)$ (essentially the form factor A_1 of the text) from lattice QCD and CLEO. The kinematic variable $\omega = v_a \cdot v_b = 1 - (q^2_{\rm max} - q^2)/2m_B m_{D^*}$. From Ref. 36.

corrections[31] (estimated to be small) and known radiative corrections. Other form factors, which are phase-space suppressed near $q^2_{\rm max}$, are also related by heavy-quark symmetry to $A_1(q^2)$. Hence, eq. (16) provides an essentially model-independent[32] way to determine $|V_{cb}|$.

The difficulty with the model-independent analysis is that the decay rate vanishes at $q^2_{\rm max}$. To aid experimentalists' extrapolation to that point, several groups[33, 34, 35] have used quenched lattice QCD to compute the slope of A_1. A typical analysis is to fit the slope to lattice-QCD numerical data, and then fit the normalization to CLEO's experimental data, as shown in Fig. 4. For example, Simone of the UKQCD Collaboration finds[36]

$$|V_{cb}| = 0.034^{+3+2}_{-2-2}\sqrt{\frac{\tau_B}{1.49\ \text{ps}}}. \tag{17}$$

The first error is experimental; the second is from the lattice-QCD slope. Unfortunately, it is not clear how to correct for the quenched approximation, and the

associated uncertainty has not been estimated. Moreover, consistency checks of varying lattice spacing, volume, etc, are still in progress. Nevertheless, the overall consistency with experiment, shown in Fig. 4, is encouraging.

$|V_{ub}|$ can be obtained from the semi-leptonic decays $B \to \rho l \nu$ and $B \to \pi l \nu$. Expanding in q^2 near $q^2_{\rm max} = (m_B - m_\pi)^2$, the differential decay rate for $B \to \pi l \nu$ reads

$$\frac{d\Gamma}{dq^2} = \left[\frac{G_F^2 (q^2_{\rm max} - q^2)^{3/2}}{24\pi^3}\left(\frac{m_\pi}{m_B}\right)^{3/2}\right] |f_+(q^2_{\rm max})|^2 |V_{ub}|^2 \left(1 + {\rm O}(q^2_{\rm max} - q^2)\right), \quad (18)$$

where $f_+(q^2)$ is the form factor that must be calculated in lattice QCD. Now, however, heavy-quark symmetry does not restrict $f_+(q^2_{\rm max})$, so a calculation is needed to make any progress. These calculations are underway at Fermilab, and, presumably, many other places.

The third row of the CKM matrix can be probed via the box diagrams responsible for neutral meson mixing. The $CP = +$ admixture of the K_L is parameterized by $|\varepsilon_K| = 2.26 \times 10^{-3}$. The Standard Model predicts

$$\begin{aligned} |\varepsilon_K| = \left[\frac{\sqrt{2} G_F^2 m_W^2}{16\pi^2 m_K \Delta m_K}\right] \tfrac{8}{3} m_K^2 f_K^2 \hat{B}_K |V_{ud} V_{us}|^2 |V_{cb}|^2 \times \\ \bar{\eta}\Big(|V_{cb}|^2 (1-\bar{\rho}) y_t \eta_2 f_2(y_t) + y_c(\eta_3 f_3(y_t) - \eta_1)\Big), \end{aligned} \quad (19)$$

where $y_q = m_q^2/m_W^2$. This formula assumes three-generation unitarity and neglects the deviation of $|V_{cs}|$ and $|V_{tb}|$ from unity. The η_i and f_i multiplying the CKM factors arise from box diagrams and their QCD corrections.[37] The nonperturbative QCD factor is $\frac{8}{3} m_K^2 f_K^2 B_K$, which is the K–$\bar{K}$ transition matrix element of a $\Delta S = 2$ operator. The best result for B_K is[38, 39]

$$\begin{aligned} B_K({\rm NDR},\ 2\ {\rm GeV}) = 0.616 \pm 0.020 \pm 0.014 \pm 0.009 \pm 0.004 \pm 0.002 \\ \pm\ ({\rm few}\ \%) \pm 3\%. \end{aligned} \quad (20)$$

The many error bars are exihibited to show how mature the uncertainty analysis has become. The first is statistical and the others are systematic. The "few %" are for the quenched approximation. This estimate comes from repeating some of the numerical computations for full QCD,[40] though not enough to obtain the other error bars, and from an analysis of chiral logarithms.[41] The latter study is reassuring only for degenerate quarks, so the calculations are done with both quarks at $\frac{1}{2} m_s$. The 3% uncertainty is an estimate of ${\rm O}(m_s - m_d)$ contributions. Combining the errors and converting to the renormalization-group invariant that appears in eq. (19), one finds[38]

$$\begin{aligned} \hat{B}_K &= (\alpha_{\overline{\rm MS}}(\mu))^{-6/25} B_K({\rm NDR},\ \mu) \\ &= 0.825 \pm 0.027({\rm stat.}) \pm 0.023({\rm syst.}) \pm ({\rm few}\ \%) \pm 3\%. \end{aligned} \quad (21)$$

This result places a high standard on calculations of $\hat{B}_K$, whether by lattice QCD or any other method. Would-be competitors must not only reach 10% uncertainties, they must do so with an error analysis as thorough and forthright as Ref. 38.

Mixing in the B^0–$\bar{B}^0$ system is also sensitive to V_{td}. In the Standard Model the mass splitting is given by

$$x_d = \frac{\Delta m_{B_d}}{\Gamma_{B_d}} = \left[\frac{G_F^2 m_t^2 \tau_{B_d}}{16\pi^2 m_{B_d}} \eta_B f_2(y_t)\right] \tfrac{8}{3} m_{B_d}^2 f_{B_d}^2 \hat{B}_{B_d} |V_{td}^* V_{tb}|^2, \tag{22}$$

The same formula holds for the B_s–$\bar{B}_s$ system, but with the d quark replaced by s (i.e. $B_d \mapsto B_s$, $V_{td} \mapsto V_{ts}$.) The nonperturbative QCD factor is $\frac{8}{3} m_{B_q}^2 f_{B_q}^2 \hat{B}_{B_q}$, which is the B_q–$\bar{B}_q$ transition matrix element of a $\Delta B = 2$ operator. The calculation of the decay constant f_B has received a great deal of attention over the last several years,[42] but the matrix element needed here, $\frac{8}{3} m_B^2 f_B^2 \hat{B}_B$, has been mostly neglected. (There are some older, exploratory papers.[43])

It is interesting to see how the lattice results influence the unitarity triangle. Fig. 5 shows constraints from eqs. (19), $|V_{ub}/V_{cb}|$, and x_d/x_s, taking for the masses

$$\begin{aligned} m_{c,\overline{\mathrm{MS}}} &= 1.3 \pm 0.2, \\ m_{t,\overline{\mathrm{MS}}} &= 175 \pm 15, \end{aligned} \tag{23}$$

for the hadronic matrix elements

$$\begin{aligned} \hat{B}_K &= 0.825 \pm 0.050, \\ |f_{B_d}/f_{B_s}| &= 0.90 \pm 0.05, \\ |B_{B_d}/B_{B_s}| &= 1.0 \pm 0.2, \end{aligned} \tag{24}$$

for "experimental"[¶] CKM results

$$\begin{aligned} |V_{cb}| &= 0.040 \pm 0.005, \\ |V_{ub}/V_{cb}| &= 0.08 \pm 0.02, \end{aligned} \tag{25}$$

and for neutral B mixing measurements

$$\begin{aligned} x_d &= 0.72 \pm 0.08, \\ x_s &= 15 \pm 5. \end{aligned} \tag{26}$$

Other inputs are as in Ref. 30. I've made two wild guesses: $|B_{B_d}/B_{B_s}|$ and x_s. But note that I take the uncertainty estimate in $\hat{B}_K$ seriously; doubling it would not make much difference, in view of the uncertainties in m_t and $|V_{cb}|$. Alas, these and the other uncertainties are too large to make Fig. 5 interesting.

[¶]Nonperturbative QCD is needed to extract these results!

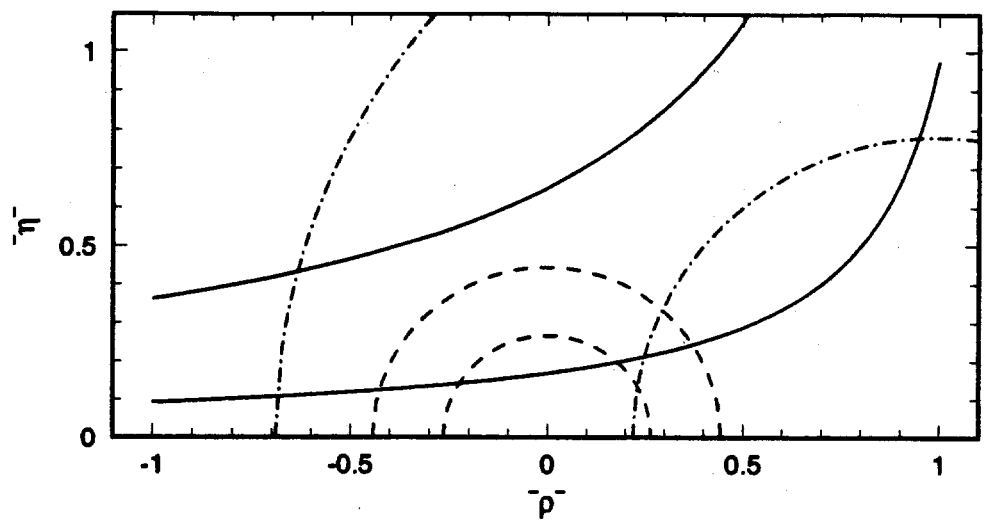

Figure 5: Constraints on $(\bar{\rho}, \bar{\eta})$ from $|\varepsilon_K|$ (solid hyperbolae), $|V_{ub}/V_{cb}|$ (dashed circles with origin (0,0)), and x_d/x_s (dash-dotted circles with origin (1,0)), and contemporary uncertainties.

What if lattice QCD calculation and the experiments improve? Consider for the masses

$$\begin{aligned} m_{c,\overline{\mathrm{MS}}} &= 1.3 \pm 0.1, \\ m_{t,\overline{\mathrm{MS}}} &= 175 \pm 5, \end{aligned} \tag{27}$$

for the hadronic matrix elements

$$\begin{aligned} \hat{B}_K &= 0.825 \pm 0.027, \\ |f_{B_d}/f_{B_s}| &= 0.90 \pm 0.02, \\ |B_{B_d}/B_{B_s}| &= 1.0 \pm 0.1, \end{aligned} \tag{28}$$

in particular eliminating almost all the statistical error in $\hat{B}_K$; for "experimental" CKM results

$$\begin{aligned} |V_{cb}| &= 0.035 \pm 0.002, \\ |V_{ub}/V_{cb}| &= 0.080 \pm 0.004 \quad \text{"low,"} \\ &= 0.091 \pm 0.004 \quad \text{"high,"} \end{aligned} \tag{29}$$

and for neutral B mixing measurements

$$\begin{aligned} x_d &= 0.72 \pm 0.04, \\ x_s &= 18 \pm 2. \end{aligned} \tag{30}$$

Fig. 6 shows how this 5–10% level of precision improves the limits on $(\bar{\rho}, \bar{\eta})$. The

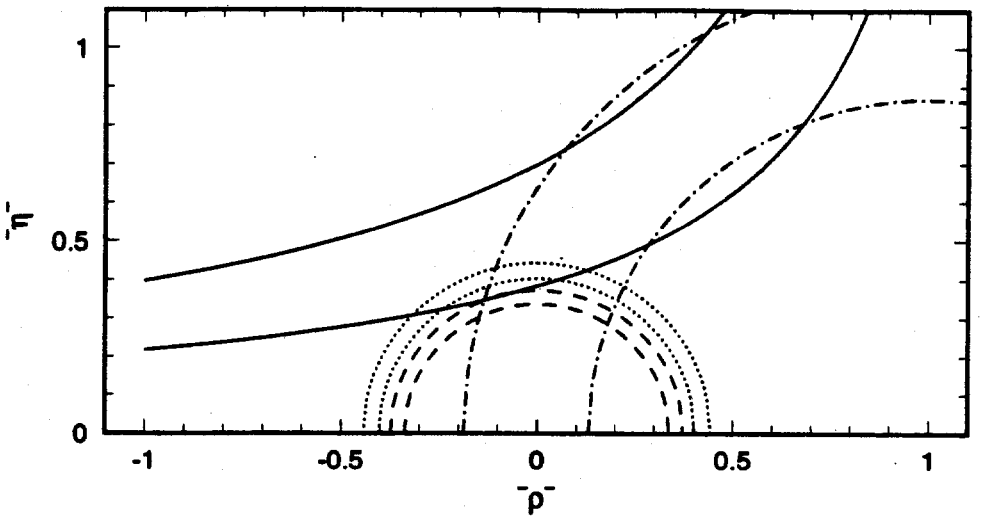

Figure 6: Constraints on $(\bar{\rho}, \bar{\eta})$ from $|\varepsilon_K|$ (solid hyperbolae), low $|V_{ub}/V_{cb}|$ (dashed circles with origin (0,0)) or high $|V_{ub}/V_{cb}|$ (dotted circles with origin (0,0)), and x_d/x_s (dash-dotted circles with origin (1,0)), and improved (5–10%) uncertainties.

wildest guess remains x_s, so ignore the dashed-dotted curves momentarily. The region allowed by the hyperbolic band from ε_K and the circular band from $|V_{ub}/V_{cb}|$ shrinks if $|V_{ub}|$ is too small. The tension between these two constraints is partly a consequence of the low value of $|V_{cb}|$ suggested by eq. (17). Increasing $|V_{cb}|$ brings the hyperbolic band down more rapidly than it shrinks the circular band.

If the real-world values of $|V_{cb}|$ and $|V_{ub}/V_{cb}|$ allow a sizable region, as for the dotted circles in Fig. 6, neutral B mixing becomes crucial. The constraint becomes more restrictive as x_s increases. Unfortunately, the experimental measurement becomes more difficut as x_s increases. If it proves impossible to obtain useful information on x_s, one can return to eq (22), and focus on x_d alone. The lattice-QCD calculations of $\frac{8}{3}m_B^2 f_B^2 \hat{B}_B$ will carry larger uncertainties, however, than the $B_d : B_s$ ratio.

5 Conclusions

This talk has examined several ways in which lattice QCD can aid the determination of standard-model couplings. The quenched lattice calculations may be divided into several classes, according to the maturity of the error analysis and the presumed reliability of the quenched approximation. One class consists of the light-hadon and quarkonia spectra and the K–$\bar{K}$ mixing parameter B_K. For them the straightforward uncertainties (statistics, a, L, excited states, perturbation theory)

seem fairly estimated. The quenched approximation is another matter. In quarkonia, one can correct for it with potential models, yielding determinations of α_S and the charm and bottom masses. The quenched error in B_K is also thought to be under control, and—taking the error bars at face value—B_K is no longer the limiting factor in the $|\epsilon_K|$ constraint on the unitarity triangle. A second class consists of f_B, the semi-leptonic form factors of K and D mesons (not discussed in this talk, but see Ref. 42), and the Isgur-Wise function. These quantities are essential for direct determinations of the first two rows of the CKM matrix. The quenched-approximation calculations are in good shape, but the the corrections to it cannot be simply estimated. A third class consists of the semi-leptonic decay $B \to \pi l \nu$ and neutral B mixing, for which only exploratory work has appeared.

Nevertheless, all QCD quantities discussed here will follow a conceptually clear path to ever-more-precise results. The next ten years or so will almost certainly witness computing and other technical improvements that will allow for wide-ranging calculations without the quenched approximation. By then the most efficient techniques for extracted the most relevant information will have been perfected.

Acknowledgements

The Adriatic glistened in the moonlight as it lapped against the quay. In the bar of the Hotel Neptun a winsome lounge singer cooed "...strangers in the night, exchanging glances,...." I looked up at the waiter and said, "Molim pivo," when a man strolled into the bar, clapped me on the back, and cried, "Ay, you kook, what's new?" In a different place and a different time he had rescued my sanity, if not my life. I peered into his eyes and replied, "It's the same old story, always the same. And even when it changes, it always ends the same way: Fermilab is operated by Universities Research Association, Inc., under contract DE-AC02-76CH03000 with the U.S. Department of Energy."

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