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Comment on: "Negative $\delta\rho$ with four families in the Standard Model"

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ABSTRACT

We consider the contribution of a heavy fourth generation to the ρ parameter. We show that recent claims of a large negative contribution to $\delta\rho$ from a hypothetical heavy fourth family are not well founded. We discuss certain difficulties in the application of dispersion relations to the calculation of $\delta\rho$, and how these give rise to uncertainties in both the magnitude and the sign of $\delta\rho$.

I. INTRODUCTION

A recent paper [1] has suggested the interesting possibility that a fourth family of heavy, nearly degenerate fermions may give a large negative contribution to the ρ parameter. In particular, it was claimed that if the fourth family is heavy enough to form fermion–antifermion bound states from Higgs exchange, those bound states will give a large negative contribution to $\delta\rho$. This shift in $\delta\rho$ is sizable, and in some cases is much larger in magnitude than the usual heavy top contribution in the minimal standard model. This result goes against the conventional wisdom [2] and, if true, has interesting implications with regard to the top mass, as well as various extensions of the standard model.

We have studied the contribution of a heavy fourth generation to $\delta\rho$, and found that the conclusions of Ref. [1] are unfounded. In particular, it is possible to show that the bound state contribution to the dispersive representation of $\delta\rho$ does not dominate over that of the continuum and possible 'surface' terms. This is because there exist two dispersion relations (DR's) for $\delta\rho$, the *transverse* and the *longitudinal*, to which the bound states contribute with the same magnitude but with opposite sign [3]. If both dispersion relations are to give the same result for $\delta\rho$, the contribution of the continuum and 'surface' terms must account for the difference. Ref. [1] only calculates the bound state contribution to the transverse DR so its results cannot be considered a good estimate of $\delta\rho$.

Even if one could argue that the bound states would dominate in the transverse DR but not in the longitudinal, the result of Ref. [1] is still problematic: the expression for the bound state contribution to the transverse DR is not symmetric under the interchange of up–type and down–type fermion masses ($m_U \leftrightarrow m_D$) as it should be. If one calculates the expression of [1] with $m_U > m_D$, then it is indeed negative. However, when the up–type and down–type masses are interchanged, it gives a positive result. This asymmetry was introduced by approximating the masses of both the $U\bar{U}$ and $D\bar{D}$ bound states with $2m_U$. A more symmetric approximation results in an expression which is always positive.

We will elaborate on these points in this comment and show that considerable care is needed to calculate a reliable estimate of $\delta\rho$.

II. DISPERSIVE CALCULATIONS OF $\delta\rho$

In this section we will discuss two different DR's that can be used to calculate $\delta\rho$. We will find that although the two relations must give the same result for $\delta\rho$, they disagree with each other when restricted to the bound state contributions.

The ρ–parameter is defined as the ratio of the strengths of neutral to charged current interactions in the low energy effective Lagrangian of weak interactions

[2]. In the Standard Model, $\rho = 1$ at tree level and the shift of ρ away from its tree level value due to radiative corrections is given by

$$\delta\rho = \frac{1}{v^2}\left[\Pi_{+-}(0) - \Pi_{33}(0)\right] = \frac{1}{v^2}\left[\Delta_{+-}(0) - \Delta_{33}(0)\right]. \tag{1}$$

In this expression $\Pi_{+-}(s)$ and $\Pi_{33}(s)$ represent the transverse parts of the vacuum polarization tensors of the $W^{\pm}$ and W^0 bosons, respectively, while $\Delta_{+-}(s)$ and $\Delta_{33}(s)$ represent the longitudinal parts [3, 4], and $v \simeq 246$ GeV is the Higgs vacuum expectation value.

The fact that $\delta\rho$ can be expressed either as a difference of the transverse parts of the vacuum polarization tensors or the difference of the longitudinal parts leads to two possible DR's.

Applying Cauchy's theorem to the representation of $\delta\rho$ in terms of the transverse parts, we can write

$$\delta\rho = \delta\rho^{\mathrm{T}}(\Lambda^2) + \delta R^{\mathrm{T}}(\Lambda^2), \tag{2}$$

where

$$\begin{aligned}
\delta\rho^{\mathrm{T}}(\Lambda^2) &\equiv \frac{1}{v^2}\left[\frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\left\{\mathrm{Im}\Pi_{+-}(s) - \mathrm{Im}\Pi_{33}(s)\right\}\right], \\
\delta R^{\mathrm{T}}(\Lambda^2) &\equiv \frac{1}{v^2}\left[\frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\left\{\Pi_{+-}(s) - \Pi_{33}(s)\right\}\right].
\end{aligned} \tag{3}$$

Alternatively, we can apply Cauchy's theorem to the representation of $\delta\rho$ in terms of the longitudinal parts and write

$$\delta\rho = \delta\rho^{\mathrm{L}}(\Lambda^2) + \delta R^{\mathrm{L}}(\Lambda^2), \tag{4}$$

where

$$\begin{aligned}
\delta\rho^{\mathrm{L}}(\Lambda^2) &\equiv \frac{1}{v^2}\left[\frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\left\{\mathrm{Im}\Delta_{+-}(s) - \mathrm{Im}\Delta_{33}(s)\right\}\right], \\
\delta R^{\mathrm{L}}(\Lambda^2) &\equiv \frac{1}{v^2}\left[\frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\left\{\Delta_{+-}(s) - \Delta_{33}(s)\right\}\right].
\end{aligned} \tag{5}$$

In Ref. [3], it has been shown that for the case of QCD corrections, one has

$$\lim_{\Lambda^2\to\infty}\delta R^{\mathrm{T}}(\Lambda^2) = \lim_{\Lambda^2\to\infty}\delta R^{\mathrm{L}}(\Lambda^2) = 0, \tag{6}$$

and hence

$$\delta\rho = \delta\rho^{\mathrm{T}}(\infty) = \delta\rho^{\mathrm{L}}(\infty). \tag{7}$$

This result was derived by making use of the asymptotic freedom of QCD, together with the operator product expansion of current–current correlators. However, if we consider corrections due to Higgs exchange, these tools are unavailable: the Yukawa couplings of heavy fermions in the mass range considered in Ref. [1] tend to become *stronger* asymptotically, invalidating the application of the operator product expansion. Consequently it is unclear whether $\delta R^{\mathrm{T}}(\Lambda^2)$ and $\delta R^{\mathrm{L}}(\Lambda^2)$ have any simple limit as $\Lambda^2 \to \infty$. Consequently, we must retain these 'surface' terms in our expressions for $\delta\rho$.

Fermions in the mass range considered in Ref. [1] will be strongly coupled by Higgs exchange, resulting in the formation of fermion–antifermion bound states. If these states are weakly bound (*i.e.* if the binding energy is small compared to the fermion masses) then we may use the leading non-relativistic approximation to compute the contribution of these bound states to the vacuum polarization functions.

In this approximation, the contributions from the bound states of fourth generation quarks (U, D) to the transverse and longitudinal parts are given by

$$\begin{aligned}
\mathrm{Im}\Pi_{+-}(s) &\approx -2N_c\pi\sqrt{s}\sum_n \left|\Psi_n^{\pm}(0)\right|^2 \delta(s - s_n^{\pm}),\\
\mathrm{Im}\Pi_{33}(s) &\approx -N_c\pi\sqrt{s}\sum_n \left\{\left|\Psi_n^{U}(0)\right|^2 \delta(s - s_n^{U}) + \left|\Psi_n^{D}(0)\right|^2 \delta(s - s_n^{D})\right\},
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\mathrm{Im}\Delta_{+-}(s) &\approx 2N_c\pi\sqrt{s}\sum_n \left|\Psi_n^{\pm}(0)\right|^2 \delta(s - s_n^{\pm}),\\
\mathrm{Im}\Delta_{33}(s) &\approx N_c\pi\sqrt{s}\sum_n \left\{\left|\Psi_n^{U}(0)\right|^2 \delta(s - s_n^{U}) + \left|\Psi_n^{D}(0)\right|^2 \delta(s - s_n^{D})\right\}.
\end{aligned} \tag{9}$$

Here, Ψ_n^U, Ψ_n^D, and $\Psi_n^{\pm}$ are the non-relativistic wavefunctions of the $U\bar{U}$, $D\bar{D}$, and $U\bar{D}/D\bar{U}$ bound states, respectively, and are obtained by solving the non-relativistic Schrödinger equation

$$H\Psi_n(\mathbf{x}) = \left[-\frac{1}{2\mu}\Delta + V(\mathbf{x})\right]\Psi_n(\mathbf{x}) = E_n\Psi_n(\mathbf{x}), \tag{10}$$

with reduced mass $\mu = m_1 m_2/(m_1 + m_2)$, and non-relativistic Yukawa potential

$$V(\mathbf{x}) = -\alpha_Y \frac{e^{-m_H r}}{r}, \tag{11}$$

where $r = |\mathbf{x}|$ and $\alpha_Y = m_1 m_2/(4\pi v^2)$. (In the mass range considered in Ref. [1], the force due to Higgs exchange will dominate over that due to QCD.) Also, $N_c = 3$ is the number of colors and $s_n = (m_1 + m_2 + E_n)^2 \approx (m_1 + m_2)^2$.

The connection between the Schrödinger wavefunctions and the imaginary parts of the vacuum polarization functions can be found, for instance, in Ref. [5].

From Eqs. (8) and (9), we see that the bound state contribution to $\mathrm{Im}\Pi_{33}(s)$ is equal in magnitude but opposite in sign to the bound state contribution to $\mathrm{Im}\Delta_{33}(s)$. The same is true for $\mathrm{Im}\Pi_{+-}(s)$ and $\mathrm{Im}\Delta_{+-}(s)$. This is because the non-relativistic wavefunctions are the same for both vector and pseudoscalar bound states. Therefore, the bound state contributions to $\delta\rho^{\mathrm{T}}(\Lambda^2)$ and $\delta\rho^{\mathrm{L}}(\Lambda^2)$ are found to be opposite in sign:

$$\delta\rho^{\mathrm{T}}_{\mathrm{bound}} = -\delta\rho^{\mathrm{L}}_{\mathrm{bound}} \approx \frac{N_c}{v^2} \sum_n \left[\frac{|\Psi^U_n(0)|^2}{2m_U} + \frac{|\Psi^D_n(0)|^2}{2m_D} - \frac{2|\Psi^\pm_n(0)|^2}{m_U + m_D} \right]. \tag{12}$$

In the parameter range considered in Ref. [1], there is only one bound state in each of the channels so the summation over the bound states can be dropped.

This result makes two facts immediately apparent. The first is that the sign of the contribution of bound states to $\delta\rho$ depends on the DR used. If $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$ is positive, then $\delta\rho^{\mathrm{L}}_{\mathrm{bound}}$ is negative, and vice versa. This shows that it is meaningless to talk about the sign of the bound state contribution without specifying which DR we are using.

The second point is that the continuum contribution and possible surface terms cannot be neglected as was done in Ref. [1]. Since both DR's must give the same value for $\delta\rho$, the continuum and surface terms must be large enough to compensate for the difference in the bound state contributions and make the two DR's agree. Therefore, a naïve calculation of the bound state contribution to $\delta\rho$ using the transverse DR is not even a qualitative guide in estimating the contribution of heavy fermions to $\delta\rho$.

A corollary to the second point is that in order to use DR's to calculate $\delta\rho$, one must know the imaginary parts of the vacuum polarization functions along the real s axis all the way up to $s = \Lambda^2$, and both the real and imaginary parts around the large circle at $|s| = \Lambda^2$. Since there is no systematic method of calculating these values except in perturbation theory, where it is actually easier to calculate the Higgs exchange corrections to $\delta\rho$ directly, the utility of DR's to calculate $\delta\rho$ in this case is actually quite limited.

III. THE CALCULATION OF $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$

Even with the evidence given above, one may still be tempted to assume (without justification) that the bound states will dominate in the transverse DR but not in the longitudinal DR, and that the results of Ref. [1] may still be valid.

However, as noted in the introduction, the expression for $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$ given in [1] is not symmetric under the interchange of up and down type quark masses. This asymmetry arises due to an approximation that was made for the neutral vector meson masses. Since the fourth generation fermions under consideration are nearly degenerate, it was assumed in Ref. [1] that the masses of the $U\bar{U}$ and $D\bar{D}$ bound states are approximately equal, and that they could both be approximated by $2m_U$. With this approximation, we find that the expression for the bound state contribution to the transverse DR is given by

$$\delta\rho^{\mathrm{T}}_{\mathrm{bound}} \approx \frac{N_c}{v^2}\sum_n \left[\frac{|\Psi^U_n(0)|^2 + |\Psi^D_n(0)|^2}{2m_U} - \frac{2|\Psi^{\pm}_n(0)|^2}{m_U + m_D}\right], \tag{13}$$

instead of Eq. (12). This is the form of $\delta\rho$ employed in Ref. [1]. It is manifestly asymmetric under interchange of the fermion masses, in contradiction to the very definition of the ρ parameter.

In Table 1 we give the values of $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$ calculated using Eqs. (12) and (13), as well as the value of $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$ computed using Eq. (13) with m_U and m_D interchanged. All of these results use wavefunctions computed in the variational approximation. Finally, we give the value of $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$ computed using Eq. (12), together with numerically calculated wavefunctions (i.e., wavefunctions that are computed without use of the variational approximation). We see that in every case except for that given in [1], the value of $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$ is positive.

Therefore, even if it were somehow possible to justify neglecting the continuum and surface terms in the transverse DR, a correct estimate of $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$ is positive.

This result exemplifies the danger of making ad–hoc approximations in calculating the ρ parameter. Since, $\delta\rho$ is a *small* difference between corrections to the charged and neutral $SU(2)$ gauge boson propagators, careless approximations can upset the delicate cancellation between the two corrections and result in changing the sign of $\delta\rho$.

IV. CONCLUSIONS

We have considered the contribution of a heavy fourth generation to $\delta\rho$. We have shown that the bound state contribution to $\delta\rho$ is not well defined when considered independently of the continuum. Even if ambiguities in the choice of DR are ignored and the transverse DR of Ref. [1] is employed, we find that a symmetric form of $\delta\rho$ gives a *positive* result for $\delta\rho$. Consequently we conclude that claims of a large negative contribution to $\delta\rho$ from a heavy fourth generation are not well founded.

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Table 1: Comparison of the asymmetric and symmetric calculation of the bound state contribution to $\delta\rho$ with $x = m_D/m_U$ fixed at 0.984. The number of colors N_c is set to 4 in order to include the contribution of leptons with $m_N = m_U$ and $m_E = m_D$. The first column is the value given in Ref. [1], the second is the value obtained using the expression of Ref. [1], but with the fermion masses interchanged. The last two columns are the symmetric form of $\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$ using variational and numerical wavefunctions.

		$\delta\rho$, from Ref. [1]		$\delta\rho^{\mathrm{T}}_{\mathrm{bound}}$	
m_U	m_H		$(m_U \leftrightarrow m_D)$	variational	numerical
900	300	-0.0044	0.0151	0.0044	0.0044
	360	-0.0023	0.0125	0.0042	0.0043
	435	-0.0001	0.0081	0.0032	0.0041
950	300	-0.0092	0.0261	0.0069	0.0069
	405	-0.0044	0.0203	0.0065	0.0067
	510	-0.0002	0.0127	0.0050	0.0064
1000	300	-0.0165	0.0425	0.0105	0.0106
	480	-0.0062	0.0301	0.0098	0.0100
	600	-0.0003	0.0183	0.0072	0.0096
1050	300	-0.0274	0.0665	0.0157	0.0157
	480	-0.0149	0.0516	0.0150	0.0152
	690	-0.0004	0.0280	0.0111	0.0142
1100	420	-0.0336	0.0892	0.0224	0.0226
	645	-0.0127	0.0637	0.0210	0.0215
	765	-0.0018	0.0466	0.0184	0.0208
1150	615	-0.0327	0.1091	0.0311	0.0315
	675	-0.0256	0.1003	0.0306	0.0311
	765	-0.0147	0.0864	0.0296	0.0305