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On the Choice of Dispersion Relation to Calculate the QCD Correction to $\Gamma(H \to \ell^+\ell^-)$.

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ABSTRACT

We use the Operator Product Expansion (OPE) of quark vacuum polarization functions to show that the dispersion relation of Kniehl and Sirlin will yield the correct result to all orders in α, when applied to the QCD correction to the leptonic decay width of the Higgs boson.

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I. INTRODUCTION

Dispersion relations (DR's) are widely used to calculate higher order electroweak radiative corrections [1, 2]. However, the freedom to make subtractions in DR's often makes the choice of one DR over another a delicate issue [3, 4], since a certain DR may work for some observables, but not for others. The use of a particular DR is often justified by comparing the result with that of a direct calculation using dimensional regularization or some other computational technique at a given order [5].

We wish to emphasize that in so far as higher order QCD corrections are involved, the Operator Product Expansion (OPE) can be used to decide on a DR without explicitly doing a multi-loop calculation. In a previous paper [6] we illustrated this with the example of higher order corrections to $\Delta\rho$. There we found that the subtracted DR's employed in Refs. [1, 3], as well as the naïve unsubtracted DR give the correct result for $\Delta\rho$. In this paper, we wish to illustrate this use of the OPE by considering corrections to the decay width $\Gamma(H \to \ell^+\ell^-)$, and show that in this particular case one does need a subtracted DR, and that furthermore the subtraction of Ref. [3] gives the correct result, whereas that of Ref. [1] does not.

This paper is organized as follows. In Section II, we discuss the role of subtractions in DR's and the use of the OPE in determining the correctness of the subtraction. In Section III we show that the DR of Ref. [3] gives the correct answer to all orders in α_s when used to calculate the QCD correction to $\Gamma(H \to \ell^+\ell^-)$. In Section IV we show that the DR of Ref. [1] gives the incorrect answer for the same correction. Section V concludes.

II. THE NECESSITY AND CORRECTNESS OF SUBTRACTIONS

In this section, we will look at the self energies that must be computed to obtain the QCD correction to $\Gamma(H \to \ell^+\ell^-)$. We will show that a naïve application of Cauchy's theorem will not lead to a dispersion relation for this correction and that a subtraction must be introduced. We then discuss how one may check which choice of subtraction is the correct one.

QCD corrections to the decay width $\Gamma(H \to \ell^+\ell^-)$ enter through the quark contribution to the self energies of the W and the Higgs. In Ref. [7], it was shown that

$$\Gamma(H \to \ell^+\ell^-) = \Gamma_0(H \to \ell^+\ell^-)\Big[1 + \delta + \cdots\Big], \tag{1}$$

where

$$\delta = \frac{\Pi_{WW}(0)}{M_W^2} + \Pi'_{HH}(M_H^2). \tag{2}$$

The tree level value Γ_0 is assumed to be expressed in terms of G_μ which is why $\Pi_{WW}(0)$, the self–energy of the W evaluated at zero momentum transfer, appears in this formula. The derivative of the self–energy of the Higgs, $\Pi'_{HH}(M_H^2)$, comes from the wave–function renormalization constant of the Higgs field.

In contrast to the $\Delta\rho$ case that was considered in a previous paper [6], one must introduce a subtraction in order to write down a dispersion relation for δ. This can be seen as follows: applying Cauchy's theorem to $\Pi_{WW}(s)$ and $\Pi_{HH}(s)$, we can write

$$\begin{aligned} \Pi_{WW}(s) &= \frac{1}{\pi}\int^{\Lambda^2} ds'\frac{\mathrm{Im}\Pi_{WW}(s')}{s'-s-i\epsilon} + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds'\frac{\Pi_{WW}(s')}{s'-s}, \\ \Pi_{HH}(s) &= \frac{1}{\pi}\int^{\Lambda^2} ds'\frac{\mathrm{Im}\Pi_{HH}(s')}{s'-s-i\epsilon} + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds'\frac{\Pi_{HH}(s')}{s'-s}. \end{aligned} \tag{3}$$

We consider the $\Pi(s)$'s to be regularized and finite so that both sides of these equations are well defined. Note that from this point of view, the radius of the contour Λ^2 has no relation to the ultraviolet regulator which makes the $\Pi(s)$'s finite. Substituting these expressions into Eq. (2), we find

$$\delta = \delta_0(\Lambda^2) + R_0(\Lambda^2), \tag{4}$$

where

$$\begin{aligned} \delta_0(\Lambda^2) &= \frac{1}{\pi}\int^{\Lambda^2} ds\left[\frac{1}{M_W^2}\frac{\mathrm{Im}\Pi_{WW}(s)}{s} + \frac{\mathrm{Im}\Pi_{HH}(s)}{(s-M_H^2-i\epsilon)^2}\right], \\ R_0(\Lambda^2) &= \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds\left[\frac{1}{M_W^2}\frac{\Pi_{WW}(s)}{s} + \frac{\Pi_{HH}(s)}{(s-M_H^2)^2}\right]. \end{aligned} \tag{5}$$

Since $\Pi_{WW}(s) \sim s$ and $\Pi_{HH}(s) \sim s$ as $s \to \infty$, both $\delta_0(\Lambda^2)$ and $R_0(\Lambda^2)$ diverge quadratically as $\Lambda^2 \to \infty$ even though their sum is finite and independent of Λ^2. Therefore, we find that δ cannot be replaced with $\delta_0(\infty)$, and that the naïve substitutions of Eq. (3) do not lead to a dispersion relation which expresses δ as an integral involving only the imaginary parts of the $\Pi(s)$'s.

This problem can be solved by noticing that the representation of the $\Pi(s)$'s as an integral along the real s axis plus an integral around the circle at $|s| = \Lambda^2$ is not unique. It is always possible to introduce an analytic function $f(s)$ such that

$$0 = \frac{1}{\pi}\int^{\Lambda^2} ds\,\mathrm{Im}f(s) + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds\, f(s), \tag{6}$$

and write

$$\Pi_{WW}(s) = \frac{1}{\pi}\int^{\Lambda^2} ds'\left[\frac{\mathrm{Im}\Pi_{WW}(s')}{s'-s-i\epsilon} + \mathrm{Im}f(s')\right]$$

$$+\frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds' \left[\frac{\Pi_{WW}(s')}{s'-s} + f(s')\right], \tag{7}$$

without changing the value of $\Pi_{WW}(s)$. (We do not consider a similar subtraction on $\Pi_{HH}(s)$ here because the subtraction will not contribute to the derivative $\Pi'_{HH}(s)$.)

Substituting Eq. (7) into Eq. (2) gives us

$$\delta = \delta_f(\Lambda^2) + R_f(\Lambda^2), \tag{8}$$

where

$$\begin{aligned}
\delta_f(\Lambda^2) &= \frac{1}{\pi}\int^{\Lambda^2} ds \left[\frac{1}{M_W^2}\left\{\frac{\mathrm{Im}\Pi_{WW}(s)}{s} + \mathrm{Im}f(s)\right\} + \frac{\mathrm{Im}\Pi_{HH}(s)}{(s-M_H^2-i\epsilon)^2}\right], \\
R_f(\Lambda^2) &= +\frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds \left[\frac{1}{M_W^2}\left\{\frac{\Pi_{WW}(s)}{s} + f(s)\right\} + \frac{\Pi_{HH}(s)}{(s-M_H^2)^2}\right].
\end{aligned} \tag{9}$$

If the subtraction $f(s)$ is judiciously chosen so that

$$\lim_{\Lambda^2\to\infty} R_f(\Lambda^2) = 0, \tag{10}$$

then one obtains the dispersion relation

$$\delta = \delta_f(\infty), \tag{11}$$

i.e. we can represent δ by an integral over the real s axis only.

We would like to emphasize here the viewpoint that the purpose of subtractions in any DR is to make the integral around the circle at $|s| = \Lambda^2$ disappear. If the integral disappears automatically in the linear combination of vacuum polarization functions that one wishes to calculate, then subtractions are not necessary. For instance, this has been shown to be the case for $\Delta\rho$ [6].

Whether a certain choice of the subtraction $f(s)$ is correct or not can be checked in two ways. The first is to calulate $\delta_f(\infty)$ and see if it reproduces the correct result to a given order in α_s. This was the strategy used in Ref. [5]. This technique is useful for motivating the choice of one dispersion relation over another, but cannot rigorously establish the correctness of such a choice to all orders in α_s.

A second method is to see if the condition (10) is satisfied. Since we only need to know the behavior of the integrand in the limit $s \to \infty$, the operator product expansion (OPE) will suffice to tell us whether this condition is satisfied to *all* orders in α_s. This is the approach we will use in the following.

III. THE KNIEHL–SIRLIN SUBTRACTION

In this section, we will look at the subtraction introduced in Ref. [3] and show that Eq. (10) is indeed satisfied.

Following Ref. [3], we define the following notation:

$$\begin{aligned}
\Pi^{V,A}_{\mu\nu}(q,m_1,m_2) &= -i\int d^4x e^{iq\cdot x}\langle 0|T^*\left[J^{V,A}_\mu(x)J^{V,A\dagger}_\nu(0)\right]|0\rangle \\
&= g_{\mu\nu}\Pi^{V,A}(s,m_1,m_2)+q_\mu q_\nu\lambda^{V,A}(s,m_1,m_2) \\
&= \left(g_{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\Pi^{V,A}(s,m_1,m_2)+\left(\frac{q^\mu q^\nu}{q^2}\right)\Delta^{V,A}(s,m_1,m_2),
\end{aligned} \tag{12}$$

where $s=q^2$, and $J^{V,A}_\mu(x)$ represents the vector and axial vector currents constructed from quark fields, respectively. Note that

$$\Pi^{V,A}(s)=\Delta^{V,A}(s)-s\lambda^{V,A}(s), \tag{13}$$

so that

$$\Pi^{V,A}(0)=\Delta^{V,A}(0), \tag{14}$$

unless $\lambda^{V,A}(s)$ has a pole at $s=0$. We further introduce the notation

$$\begin{aligned}
\Pi^{V,A}_\pm(s) &= \Pi^{V,A}(s,m_1,m_2), \\
\lambda^{V,A}_\pm(s) &= \lambda^{V,A}(s,m_1,m_2), \\
\Delta^{V,A}_\pm(s) &= \Delta^{V,A}(s,m_1,m_2), \\
\Pi^{V,A}_0(s) &= \frac{1}{2}\left[\Pi^{V,A}(s,m_1,m_1)+\Pi^{V,A}(s,m_2,m_2)\right], \\
\lambda^{V,A}_0(s) &= \frac{1}{2}\left[\lambda^{V,A}(s,m_1,m_1)+\lambda^{V,A}(s,m_2,m_2)\right], \\
\Delta^{V,A}_0(s) &= \frac{1}{2}\left[\Delta^{V,A}(s,m_1,m_1)+\Delta^{V,A}(s,m_2,m_2)\right].
\end{aligned} \tag{15}$$

The conservation of the neutral vector currents implies the Ward Identities:

$$\Pi^V_0(s)=-s\lambda^V_0(s), \qquad \Delta^V_0(s)\equiv 0. \tag{16}$$

These definitions let us write the contribution of a quark doublet, with masses m_1, and m_2, to $\Pi_{WW}(0)$ as

$$\Pi_{WW}(0)=\frac{g^2}{8}\left[\Pi^V_\pm(0)+\Pi^A_\pm(0)\right]=\frac{g^2}{8}\left[\Delta^V_\pm(0)+\Delta^A_\pm(0)\right]. \tag{17}$$

The subtraction scheme of Ref. [3] is then given by,

$$\begin{aligned}\Pi^{V,A}_{\pm,0}(s) &= \frac{1}{\pi}\int^{\Lambda^2} ds'\left[\frac{\mathrm{Im}\Pi^{V,A}_{\pm,0}(s')}{s'-s-i\epsilon}+\mathrm{Im}\lambda^{V,A}_{\pm,0}(s')\right] \\ &\quad +\frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds'\left[\frac{\Pi^{V,A}_{\pm,0}(s')}{s'-s}+\lambda^{V,A}_{\pm,0}(s')\right]. \qquad (18)\end{aligned}$$

Using Eq. (13), this can also be written as

$$\begin{aligned}\Pi^{V,A}_{\pm,0}(s) &= \frac{1}{\pi}\int^{\Lambda^2} ds'\left[\frac{\mathrm{Im}\Delta^{V,A}_{\pm,0}(s')}{s'-s-i\epsilon}-s\frac{\mathrm{Im}\lambda^{V,A}_{\pm,0}(s')}{s'-s-i\epsilon}\right] \\ &\quad +\frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds'\left[\frac{\Delta^{V,A}_{\pm,0}(s')}{s'-s}-s\frac{\lambda^{V,A}_{\pm,0}(s')}{s'-s}\right], \qquad (19)\end{aligned}$$

which shows that the effect of the subtraction amounts to applying Cauchy's theorem to the $\Delta(s)$'s and $\lambda(s)$'s instead of the $\Pi(s)$'s.

Application of this subtraction to $\Pi_{WW}(0)$ gives us

$$\begin{aligned}\frac{\Pi_{WW}(0)}{M_W^2} &= \frac{G_\mu}{\sqrt{2}}\left[\frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\left\{\mathrm{Im}\Delta^V_\pm(s)+\mathrm{Im}\Delta^A_\pm(s)\right\}\right. \\ &\quad \left.+\frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\left\{\Delta^V_\pm(s)+\Delta^A_\pm(s)\right\}\right]. \qquad (20)\end{aligned}$$

Therefore, we can write

$$\delta = \delta_{KS}(\Lambda^2)+R_{KS}(\Lambda^2), \qquad (21)$$

where

$$\begin{aligned}\delta_{KS}(\Lambda^2) &= \frac{1}{\pi}\int^{\Lambda^2} ds\left[\frac{G_\mu}{\sqrt{2}}\left\{\frac{\mathrm{Im}\Delta^V_\pm(s)}{s}+\frac{\mathrm{Im}\Delta^A_\pm(s)}{s}\right\}+\frac{\mathrm{Im}\Pi_{HH}(s)}{(s-M_H^2-i\epsilon)^2}\right], \\ R_{KS}(\Lambda^2) &= \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds\left[\frac{G_\mu}{\sqrt{2}}\left\{\frac{\Delta^V_\pm(s)}{s}+\frac{\Delta^A_\pm(s)}{s}\right\}+\frac{\Pi_{HH}(s)}{(s-M_H^2)^2}\right]. \qquad (22)\end{aligned}$$

In order to show that

$$\lim_{\Lambda^2\to\infty} R_{KS}(\Lambda^2)=0, \qquad (23)$$

we need the following two relations. The first is that when $s \gg v^2$,

$$\Pi_{HH}(s)=\Pi_{\chi\chi}(s)\left[1+O\left(\frac{v^2}{s}\right)\right], \qquad (24)$$

where $\Pi_{\chi\chi}(s)$ is the self-energy of the neutral Goldstone boson (which is absorbed into the Z), and v is the Higgs VEV. This can easily be seem to be true since the two functions must coincide in the limit $v^2 \to 0$. The second is the Ward Identity

$$\Pi_{\chi\chi}(s) = -\frac{G_\mu}{\sqrt{2}} s \Delta_0^A(s), \tag{25}$$

which comes from the conservation of isospin currents. See Ref. [8].

Using Eqs. (24) and (25), we find

$$\begin{aligned}
\lim_{\Lambda^2\to\infty} R_{KS}(\Lambda^2) &= \lim_{\Lambda^2\to\infty} \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds \left[\frac{G_\mu}{\sqrt{2}}\left\{\frac{\Delta_\pm^V(s)}{s} + \frac{\Delta_\pm^A(s)}{s}\right\} + \frac{\Pi_{HH}(s)}{(s-M_H^2)^2}\right] \\
&= \lim_{\Lambda^2\to\infty} \frac{G_\mu}{\sqrt{2}}\frac{1}{2\pi i}\oint_{|s|=\Lambda^2} \frac{ds}{s}\left[\Delta_\pm^V(s) + \Delta_\pm^A(s) - \Delta_0^A(s)\right]
\end{aligned} \tag{26}$$

The asympotitic forms of the $\Delta(s)$'s as $|s| \to \infty$ can be gleaned from their OPE's found in the appendix of Ref. [9]. They are:

$$\begin{aligned}
\Delta_\pm^V(-Q^2) &= \hat{C}_{\Delta 1}(Q)\left[\hat{m}_1(Q) - \hat{m}_2(Q)\right]^2 + \hat{C}_{\Delta 2}(\mu)\left[\hat{m}_1(\mu) - \hat{m}_2(\mu)\right]^2 + O\left(\frac{1}{Q^2}\right), \\
\Delta_\pm^A(-Q^2) &= \hat{C}_{\Delta 1}(Q)\left[\hat{m}_1(Q) + \hat{m}_2(Q)\right]^2 + \hat{C}_{\Delta 2}(\mu)\left[\hat{m}_1(\mu) + \hat{m}_2(\mu)\right]^2 + O\left(\frac{1}{Q^2}\right), \\
\Delta_0^A(-Q^2) &= \hat{C}_{\Delta 1}(Q)\left[2\hat{m}_1(Q)^2 + 2\hat{m}_2(Q)^2\right] \\
&\quad + \hat{C}_{\Delta 2}(\mu)\left[2\hat{m}_1(\mu)^2 + 2\hat{m}_2(\mu)^2\right] + O\left(\frac{1}{Q^2}\right).
\end{aligned} \tag{27}$$

Though the OPE's are derived in the deep Euclidean region $-s = Q^2 \gg 0$, the power dependence of the $\Delta(s)$'s on s will be the same all around the circle at $|s| = \Lambda^2$. Therefore, we can see immediately that

$$\begin{aligned}
\lim_{\Lambda^2\to\infty} R_{KS}(\Lambda^2) &= \lim_{\Lambda^2\to\infty} \frac{G_\mu}{\sqrt{2}}\frac{1}{2\pi i}\oint_{|s|=\Lambda^2} \frac{ds}{s}\left[\Delta_\pm^V(s) + \Delta_\pm^A(s) - \Delta_0^A(s)\right] \\
&= 0.
\end{aligned} \tag{28}$$

Therefore,

$$\delta = \delta_{KS}(\infty). \tag{29}$$

We believe this derivation clarifies the reason why the DR of Ref. [3] was found to give the correct answer at $O(\alpha\alpha_s)$ in Ref. [5]. In fact, since the OPE is correct to all orders in α_s, Eq. (29) is also correct to all orders in α_s.

IV. THE SUBTRACTION OF CHANG, GAEMERS AND VAN NEERVEN

Next, we will look at the subtraction introduced in Ref. [1] and show that in contrast to the subtraction of Ref. [3], Eq. (10) is not satisfied.

The subtraction introduced in Ref. [1] is given by

$$\begin{aligned}
\Pi_{0,\pm}^{V,A}(s) &= \frac{1}{\pi}\int^{\Lambda^2} ds' \left[\frac{\mathrm{Im}\Pi_{0,\pm}^{V,A}(s')}{s'-s-i\epsilon} + \mathrm{Im}\lambda_0^V(s')\right] \\
&\quad + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds' \left[\frac{\Pi_{0,\pm}^{V,A}(s')}{s'-s} + \lambda_0^V(s')\right]. \qquad (30)
\end{aligned}$$

Note that the difference from the scheme of Ref. [3] is that the same subtraction λ_0^V is used for all four cases, Π_0^V, Π_0^A, $\Pi_\pm^V$, and $\Pi_\pm^A$.

This time, $\Pi_{WW}(0)$ is written as

$$\begin{aligned}
\frac{\Pi_{WW}(0)}{M_W^2} &= \frac{G_\mu}{\sqrt{2}}\left[\frac{1}{\pi}\int^{\Lambda^2} ds \left\{\frac{\mathrm{Im}\Pi_\pm^V(s)}{s} + \frac{\mathrm{Im}\Pi_\pm^A(s)}{s} + 2\mathrm{Im}\lambda_0^V(s)\right\}\right. \\
&\quad \left. + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds \left\{\frac{\Pi_\pm^V(s)}{s} + \frac{\Pi_\pm^A(s)}{s} + 2\lambda_0^V(s)\right\}\right] \\
&= \frac{G_\mu}{\sqrt{2}}\left[\frac{1}{\pi}\int^{\Lambda^2} ds \left\{\frac{\mathrm{Im}\Delta_\pm^V(s)}{s} + \frac{\mathrm{Im}\Delta_\pm^A(s)}{s} + 2\mathrm{Im}\lambda_0^V(s) - \mathrm{Im}\lambda_\pm^V(s) - \mathrm{Im}\lambda_\pm^A(s)\right\}\right. \\
&\quad \left. + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds \left\{\frac{\Delta_\pm^V(s)}{s} + \frac{\Delta_\pm^A(s)}{s} + 2\lambda_0^V(s) - \lambda_\pm^V(s) - \lambda_\pm^A(s)\right\}\right]. \qquad (31)
\end{aligned}$$

This gives us

$$\delta = \delta_{CGN}(\Lambda^2) + R_{CGN}(\Lambda^2), \qquad (32)$$

where

$$\begin{aligned}
\delta_{CGN}(\Lambda^2) &= \delta_{KS}(\Lambda^2) + \frac{G_\mu}{\sqrt{2}}\frac{1}{\pi}\int^{\Lambda^2} ds \left[2\mathrm{Im}\lambda_0^V(s) - \mathrm{Im}\lambda_\pm^V(s) - \mathrm{Im}\lambda_\pm^A(s)\right], \\
R_{CGN}(\Lambda^2) &= R_{KS}(\Lambda^2) + \frac{G_\mu}{\sqrt{2}}\frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds \left[2\lambda_0^V(s) - \lambda_\pm^V(s) - \lambda_\pm^A(s)\right]. \qquad (33)
\end{aligned}$$

Using the OPE's of the $\lambda(s)$'s, again from Ref. [9]:

$$\begin{aligned}
\lambda_\pm^V(-Q^2) &= \hat{C}_{\lambda 1}(Q) + \hat{C}_{\lambda 2}(Q)\frac{[\hat{m}_1(Q) + \hat{m}_2(Q)]^2}{Q^2} \\
&\quad + \hat{C}_{\lambda 3}(Q)\frac{[\hat{m}_1(Q) - \hat{m}_2(Q)]^2}{Q^2} + O\left(\frac{1}{Q^4}\right),
\end{aligned}$$

$$
\begin{aligned}
\lambda_{\pm}^{A}(-Q^2) &= \hat{C}_{\lambda 1}(Q) + \hat{C}_{\lambda 2}(Q)\frac{[\hat{m}_1(Q) - \hat{m}_2(Q)]^2}{Q^2} \\
&\quad + \hat{C}_{\lambda 3}(Q)\frac{[\hat{m}_1(Q) + \hat{m}_2(Q)]^2}{Q^2} + O\left(\frac{1}{Q^4}\right), \\
\lambda_{0}^{V}(-Q^2) &= \hat{C}_{\lambda 1}(Q) + \hat{C}_{\lambda 2}(Q)\frac{[2\hat{m}_1(Q)^2 + 2\hat{m}_2(Q)^2]}{Q^2} + O\left(\frac{1}{Q^4}\right), \\
\lambda_{0}^{A}(-Q^2) &= \hat{C}_{\lambda 1}(Q) + \hat{C}_{\lambda 3}(Q)\frac{[2\hat{m}_1(Q)^2 + 2\hat{m}_2(Q)^2]}{Q^2} + O\left(\frac{1}{Q^4}\right),
\end{aligned}
\tag{34}
$$

we find

$$
\begin{aligned}
&\lim_{\Lambda^2\to\infty} R_{CGN}(\Lambda^2) \\
&= \lim_{\Lambda^2\to\infty} \frac{G_\mu}{\sqrt{2}}\frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\left[\left\{\hat{C}_{\lambda 2}(s) - \hat{C}_{\lambda 3}(s)\right\}\frac{[2\hat{m}_1(s)^2 + 2\hat{m}_2(s)^2]}{-s} + O\left(\frac{1}{s^2}\right)\right] \\
&\neq 0,
\end{aligned}
\tag{35}
$$

which shows that the DR of Ref. [1] will give the wrong answer for $\Gamma(H \to \ell^+\ell^-)$, although it gives the correct answer for $\Delta\rho$ [6].

Ref. [9] gives the first few terms of the perturbative expansion of the Wilson Coefficients in the running coupling $\alpha_s(Q)$ and they are

$$
\begin{aligned}
\hat{C}_{\lambda 2}(Q) &= -\frac{3}{8\pi^2}\left[1 + \frac{8}{3}\frac{\alpha_s(Q)}{\pi} + \cdots\right], \\
\hat{C}_{\lambda 3}(Q) &= -\frac{3}{8\pi^2}\left[1 + 2\frac{\alpha_s(Q)}{\pi} + \cdots\right].
\end{aligned}
\tag{36}
$$

Therefore, at $O(\alpha\alpha_s)$ we find

$$
\begin{aligned}
\lim_{\Lambda^2\to\infty} R_{CGN}(\Lambda^2) &= \frac{G_\mu}{\sqrt{2}}\frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds\left[\frac{\alpha_s}{4\pi^3}\frac{(2m_1^2 + 2m_2^2)}{s}\right] \\
&= \frac{\alpha_s}{\pi}\left(\frac{m_1^2 + m_2^2}{4\pi^2 v^2}\right) \\
&= \frac{\alpha_s}{\pi}\left(\frac{\alpha_1}{\pi} + \frac{\alpha_2}{\pi}\right),
\end{aligned}
\tag{37}
$$

where

$$
\alpha_f \equiv \frac{m_f^2}{4\pi v^2}. \tag{38}
$$

This result coincides precisely with the discrepancy between δ and $\delta_{CGN}(\infty)$ found in Ref. [5].

V. CONCLUSIONS

We have shown that in so far as QCD corrections are concerned, the correctness of a particular dispersion relation is most easily checked by using the Operator Product Expansion to calculate the contribution of the circle at $|s| = \infty$. We have used the OPE to show that the DR of Ref. [3] correctly predicts the QCD correction to the the leptonic width of the Higgs $\Gamma(H \to \ell^+\ell^-)$ to all orders in α_s, whereas the DR of Ref. [1] introduces an error of $(\alpha_s/\pi)\left[(\alpha_1/\pi) + (\alpha_2/\pi)\right]$ at $O(\alpha\alpha_s)$.

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