

Fermi National Accelerator Laboratory

FERMILAB-Pub-94/108-A

May 1994

gr-qc/9404044

Submitted to *Class. Quantum Grav.*

Quantum Cosmology and Higher-Order Lagrangian Theories

Henk van Elst^{1a}, James E. Lidsey^{2b} & Reza Tavakol^{1c}

*¹School of Mathematical Sciences
Queen Mary & Westfield College
Mile End Road
London E1 4NS, UK*

*²NASA/Fermilab Astrophysics Center
Fermi National Accelerator Laboratory
Batavia IL 60510, USA*

In this paper the quantum cosmological consequences of introducing a term cubic in the Ricci curvature scalar R into the Einstein-Hilbert action are investigated. It is argued that this term represents a more generic perturbation to the action than the quadratic correction usually considered. A qualitative argument suggests that there exists a region of parameter space in which neither the tunneling nor the no-boundary boundary conditions predict an epoch of inflation that can solve the horizon and flatness problems of the big bang model. This is in contrast to the R^2 -theory.

Electronic addresses: ^ahve@maths.qmw.ac.uk;

Electronic addresses: ^bjim@fnas09.fnal.gov;

Electronic addresses: ^creza@maths.qmw.ac.uk

PACS number(s): 98.80.Hw; 04.50.+h; 04.60.Kz; 98.80.Cq



1 Introduction

An important motivation for the development of the quantum cosmology programme has been to explain the initial conditions for the emergence of the Universe as a classical outcome. In principle one must find the form of the wave function Ψ satisfying the Wheeler–DeWitt equation [1]. This equation describes the annihilation of the wave function by the Hamiltonian operator and since it admits an infinite number of solutions, one must also choose the boundary conditions in order to specify the wave function uniquely. Such boundary conditions must be viewed as an additional physical law since, by definition, there is nothing external to the Universe. In practice one assumes, at least implicitly, that a finite subset of all possible boundary conditions is favoured by cosmological observations, in the sense that the wave functions corresponding to such boundary conditions predict outcomes which are compatible with observations. For example, if one believes in the inflationary scenario, the requirement that sufficient inflation occurred in order to solve the assorted problems of the standard big bang model can, in principle, restrict the number of plausible boundary conditions.

Among the set of all possible choices the *tunneling from nothing* boundary condition proposed by Linde [2] and Vilenkin [3] and the *no-boundary* boundary condition proposed by Hartle and Hawking [4] have been the subject of intense discussion. Given the non-uniqueness of such conditions, the question arises as to the consequences of choosing different boundary conditions for the resulting wave function of the Universe and its corresponding probability measures. An important study in this regard is due to Vilenkin [5], who considered the effects of the above boundary conditions within the context of Einstein gravity minimally coupled to a self-interacting scalar field. He restricted his analysis to the minisuperspace corresponding to the spatially closed, isotropic and homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) Universe and showed that the tunneling wave function predicts initial states that are likely to lead to sufficient inflation, whereas the Hartle–Hawking wave function does not.

It is sometimes argued that this result indicates that observations favour the tunneling as opposed to the no-boundary boundary condition. However, the precise relation between the boundary conditions and the observations is determined by the specific models employed and since such models always involve idealisations in the form of a set of simplifying assumptions, it follows that the above conclusion cannot be made *a priori*. Indeed it only makes sense in general if the correspondence between the observations and the boundary conditions is robust under physically motivated perturbations to the underlying quantum cosmological model.

Consequently, it is important to consider the ‘stability’ of the above conclusions. In particular, are the conclusions robust under higher-order perturbations to the Einstein–Hilbert action? Quadratic and higher-order terms in the Riemann curvature tensor and its traces appear in the low-energy limit of superstrings [6] and they also

arise when the usual perturbation expansion is applied to General Relativity [7, 8]. Such terms diverge as the initial singularity is approached, but can in principle be eliminated if higher-order corrections are included in the action. In four-dimensional space-times the Hirzebruch signature and Euler number imply that the most general, four-dimensional gravitational action to quadratic order is

$$S = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} \left[R - \gamma C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + \epsilon_1 R^2 \right] , \quad (1)$$

where R is the Ricci curvature scalar of the space-time with metric tensor $g_{\mu\nu}$, $g = \det g_{\mu\nu}$, $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, κ^2 is the gravitational coupling constant and ϵ_1 and γ are coupling constants of dimension (length)². The action simplifies further for spatially homogeneous and isotropic four-geometries, since the conformal flatness of these space-times implies that the Weyl tensor vanishes. The effects of including quadratic terms have been investigated in Refs. [9, 10, 11]. In particular Mijić et al [11] studied the effects of such perturbations on Vilenkin's result [5] and found that those results remain robust in the sense that the inflationary scenario still favours the tunneling boundary condition in the presence of quadratic terms in the action. On the other hand Biswas and Guha have recently arrived at the opposite conclusion [9].

The renormalisation of higher loop contributions introduces terms into the effective action that are higher than quadratic order. Consequently it is important to also study the effects of these additional terms. In this paper we shall investigate what happens to the wave function if an R^3 -contribution is present. By employing the conformal equivalence of higher-order gravity theories with Einstein gravity coupled to matter fields, we argue that this term represents a more general perturbation to the Einstein–Hilbert action than the R^2 -correction, at least within the context of four-dimensional FLRW space-times. We then consider the conditional probability that an inflationary epoch of sufficient duration can occur. We estimate how the qualitative behaviour of this quantity changes when higher-order perturbations to the action are included. Our main result is that for the R^3 -theory there exists a finite region of parameter space in which neither of the boundary conditions discussed above predict an epoch of inflationary expansion that leads to the observed Universe. We use (dimensionless) Planckian units defined by $\hbar = c = G = 1$ throughout and define $\kappa^2 = 8\pi$.

2 Higher-Order Lagrangians as Einstein Gravity plus Matter

The wave function of the Universe in higher-order Lagrangian theories can be determined in one of two ways. It is well known that theories with a Lagrangian given by a differentiable function of the Ricci curvature scalar are conformally equivalent to Einstein gravity with a matter sector containing a minimally coupled, self-interacting scalar field [12, 13]. The precise form of the self-interaction is uniquely determined

by the higher-derivative metric terms in the field equations. It follows that one can start either from the original action or the conformal action and derive the corresponding Wheeler–DeWitt equation [14]. One takes the related Lagrangian as the defining feature of the theory and then applies the canonical quantisation rules. The advantage of the conformal transformation is that it allows the known results from Einstein gravity to be carried over to the higher-order examples and we shall follow such an approach in this paper.

Consider the general, D -dimensional, vacuum theory

$$S = \int d^D x \sqrt{-g_D} [f(R)] , \quad (2)$$

where the Lagrangian $f(R)$ is some arbitrary differentiable function of the Ricci curvature scalar satisfying $\{f(R), df(R)/dR\} > 0$ and g_D is the determinant of the D -dimensional space-time metric $g_{D\mu\nu}$. If we perform the conformal transformation [13]

$$\tilde{g}_{D\mu\nu} = \Omega^2 g_{D\mu\nu} \quad \Omega^2 = \left(2\kappa^2 \frac{df(R)}{dR} \right)^{2/(D-2)} , \quad (3)$$

and define a new scalar field

$$\kappa \bar{\phi} \equiv \left(\frac{D-1}{D-2} \right)^{1/2} \ln \left[2\kappa^2 \left(\frac{df(R)}{dR} \right) \right] , \quad (4)$$

the conformally transformed action takes the Einstein–Hilbert form

$$S = \int d^D x \sqrt{-\tilde{g}_D} \left[\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2} (\tilde{\nabla} \bar{\phi})^2 - U(\bar{\phi}) \right] , \quad (5)$$

where the self-interaction potential is given by

$$U(\bar{\phi}) = \left(2\kappa^2 \frac{df[R(\bar{\phi})]}{dR} \right)^{-D/(D-2)} \left(R(\bar{\phi}) \frac{df[R(\bar{\phi})]}{dR} - f[R(\bar{\phi})] \right) . \quad (6)$$

Definition (4) yields a correspondence between the values of the Ricci curvature scalar R and the values of the scalar field $\bar{\phi}$. We shall consider the quadratic and cubic Lagrangians

$$f_2(R) = \frac{1}{2\kappa^2} (R + \epsilon_1 R^2) \quad (7)$$

$$f_3(R) = \frac{1}{2\kappa^2} (R + \epsilon_1 R^2 + \epsilon_2 R^3) , \quad (8)$$

in four dimensions, where the parameters ϵ_1 and ϵ_2 have dimensions $(\text{length})^2$ and $(\text{length})^4$ respectively before the introduction of Planckian units. The corresponding

potentials for positive ϵ_1 and ϵ_2 are given by [11, 15]:

$$U_{f_2}(\bar{\phi}) = \frac{1}{8\kappa^2 \epsilon_1} \left[1 - \exp(-\sqrt{2/3} \kappa \bar{\phi}) \right]^2 \quad (9)$$

$$U_{f_3}(\bar{\phi}) = \frac{\epsilon_1^3}{27\kappa^2 \epsilon_2^2} \exp(-2\sqrt{2/3} \kappa \bar{\phi}) \left[-1 + \frac{9\epsilon_2}{2\epsilon_1^2} \left[1 - \exp(\sqrt{2/3} \kappa \bar{\phi}) \right] + \left(1 - \frac{3\epsilon_2}{\epsilon_1^2} \left[1 - \exp(\sqrt{2/3} \kappa \bar{\phi}) \right] \right)^{3/2} \right], \quad (10)$$

and are semi-positive definite for all values of $\bar{\phi}$.

Figures 1a & 1b

In the classical R^3 -theory the requirement that the inflationary epoch lasts sufficiently long implies that the coupling constants must satisfy $|\epsilon_2| \ll \epsilon_1^2$ [15]. Moreover, the observed isotropy of the cosmic microwave background radiation requires that $\epsilon_1 \approx 10^{11}$ [11]. In view of these constraints we specify $\epsilon_1 = 10^{11}$ in the subsequent numerical calculations. Figures 1a and 1b illustrate the behaviour of the potentials (9) and (10) for $\epsilon_1 \approx 10^{11}$ and $\epsilon_2 \approx 10^{20}$. The effect of decreasing the value of the parameter ϵ_1 is to increase the height of the plateau and the relative maximum of the potentials in the quadratic and cubic cases respectively. This reflects the fact that decreasing this parameter is equivalent to increasing the energy scales involved. In this sense there exists no continuous transformation from an R^2 -theory to the ordinary Einstein-Hilbert action as this parameter approaches zero. In the neighbourhood of the origin of $\bar{\phi}$ corresponding to smaller values of R the quadratic term in the action dominates and the potentials in this region are equivalent. This can be seen by expanding the last of the three terms in the square brackets of Eq. (10). The first-order contribution cancels the remaining terms in U_{f_3} and the second-order term reduces the form of U_{f_3} to that of U_{f_2} . Hence the two potentials are effectively identical if the third- and higher-order terms in the expansion can be neglected. It is straightforward to show that this is a consistent approximation if

$$\kappa \bar{\phi} \ll \kappa \bar{\phi}_{limit} \equiv \sqrt{\frac{3}{2}} \ln \left(\frac{2\epsilon_1^2}{\epsilon_2} \right). \quad (11)$$

For polynomial Lagrangians with $f(R) = \left(\sum_{k=1}^n \epsilon_{k-1} R^k \right) / 2\kappa^2$, the detailed form of the corresponding potential $U(\bar{\phi})$ is extremely complicated and generally not expressible in an analytically closed form. Nevertheless, one can determine the qualitative behaviour of the potential at small and large $\bar{\phi}$. Close to the origin the quadratic term in the action again dominates and the potential in this region is therefore similar to Eq. (9). The asymptotic behaviour at infinity, however, depends critically upon

the combination of the highest degree n of the polynomial and the dimensionality D of the space-time [13]. More precisely, for $D > 2n$ the potential is unbounded from above, for $D = 2n$ it flattens into a plateau and for $D < 2n$ the potential has an exponentially decaying tail [15]. In particular, if $D < 2n$ the effective scalar field potential $U(\bar{\phi})$ is qualitatively equivalent to the cubic potential (10). As a result, when $D = 4$ the qualitative behaviour of $U(\bar{\phi})$ does not change relative to the cubic case as terms with $n > 3$ are considered, although the relative position of the maximum of $U(\bar{\phi})$ will be n -dependent. This implies that the $n = 2$ contribution is rather special in four dimensions, whereas the R^3 -term is in fact a more generic perturbation. Thus, it is instructive to consider this case further.

3 Behaviour of the Wave Function

Within the context of the spatially closed FLRW minisuperspace, the Wheeler-DeWitt equation derived from theory (5) has been solved for an arbitrary potential, subject to the condition that the momentum operator for the scalar field can be neglected [2, 3]. This is self-consistent if $|dV/d\phi| \ll \max\{|V|, a^{-2}\}$, where a represents the cosmological scale factor and

$$V \equiv \frac{16}{9} U \quad \phi \equiv \sqrt{\frac{4\pi}{3}} \bar{\phi}. \quad (12)$$

The wave functions satisfying the quantum tunneling boundary condition (Ψ_T) and the no-boundary proposal (Ψ_{NB}) take the forms [5]

$$\Psi_T = (1 - a^2 V)^{-1/4} \exp \left[\frac{(1 - a^2 V)^{3/2} - 1}{3V} \right] \quad (13)$$

$$\Psi_{NB} = (1 - a^2 V)^{-1/4} \exp \left[\frac{1 - (1 - a^2 V)^{3/2}}{3V} \right] \quad (14)$$

in the classically forbidden (Euclidean signature) region defined by $a^2 V < 1$, and

$$\Psi_T = e^{i\pi/4} (a^2 V - 1)^{-1/4} \exp \left[-\frac{1}{3V} \right] \exp \left[-i \frac{(a^2 V - 1)^{3/2}}{3V} \right] \quad (15)$$

$$\Psi_{NB} = 2(a^2 V - 1)^{-1/4} \exp \left[\frac{1}{3V} \right] \cos \left[\frac{(a^2 V - 1)^{3/2}}{3V} - \frac{\pi}{4} \right] \quad (16)$$

in the classically allowed (Lorentzian signature) region $a^2 V > 1$. Substituting for $V(\phi)$ from the potentials of the quadratic and cubic Lagrangians of Section 2, it can readily be seen that the wave functions corresponding to the quadratic and cubic theories have very different types of behaviour, at least for large ϕ . In the quadratic case both Ψ_T and Ψ_{NB} remain bounded. However, for the cubic case Ψ_{NB} becomes divergent in the classically allowed region whilst Ψ_T remains regular. In this sense

then the qualitative behaviour of the wave function satisfying the no-boundary proposal is fragile with respect to cubic perturbations to the action. This is significant because often the quadratic corrections to the action are taken as representative of higher-order perturbations.

To proceed it is important to ensure that for the regimes under consideration the conformal transformation (3) remains non-singular. This is the case if the condition $df(R)/dR \neq 0$ is valid for all values of R . The conformal transformation is singular at the point

$$R = -\frac{1}{2\epsilon_1}, \quad (17)$$

in the R^2 -theory and at the point

$$R = -\frac{\epsilon_1}{3\epsilon_2} \left[1 \pm \sqrt{1 - \frac{3\epsilon_2}{\epsilon_1^2}} \right] \quad (18)$$

for the R^3 -theory. Since ϵ_1 and ϵ_2 are taken to be positive, these conditions imply that in both cases the problematic values of R lie in the region $R < 0$. However, for a classical, spatially closed FLRW model, the Ricci curvature scalar is given by

$$R = 6(1 - q) \left(\frac{\dot{a}}{a} \right)^2 + \frac{6}{a^2}, \quad (19)$$

where $q \equiv -\ddot{a}a/\dot{a}^2$ defines the deceleration parameter and a dot denotes differentiation with respect to cosmic proper time. Now if, as is generally assumed, the Universe tunnels into the Lorentzian region in an inflationary phase ($q < 0$), it follows that R will be positive-definite. Thus, the conformal transformation is self-consistent in these theories.

4 Interpretation of the Wave Function

In the previous section we saw that the wave functions corresponding to the tunneling and the Hartle–Hawking boundary conditions have qualitatively different modes of behaviour for the quadratic and cubic theories. To see what predictive effects such changes might have, we employ the notion of a probability density ρ as is usually done. For the cases of the tunneling and the no boundary boundary conditions respectively, ρ takes the form [5]

$$\rho_T(a, \phi) = C_T \exp \left[-\frac{2}{3V(\phi)} \right] \quad (20)$$

$$\rho_{NB}(a, \phi) = C_{NB} \exp \left[\frac{2}{3V(\phi)} \right] \quad (21)$$

on surfaces of constant scale factor in the classically allowed region of minisuperspace, where the normalisation constants C_T and C_{NB} are given by

$$C_T^{-1} = \int_{V(\phi)>0} d\phi \exp\left[-\frac{2}{3V(\phi)}\right] \quad (22)$$

$$C_{NB}^{-1} = \int_{V(\phi)>0} d\phi \exp\left[\frac{2}{3V(\phi)}\right]. \quad (23)$$

Since $\rho(\phi)$ is usually not normalisable, the common practice is to employ the notion of a conditional probability [14]. One argues that the initial values of the scalar field must lie in the range $\phi_{min} < \phi_i < \phi_P$. The lower limit ϕ_{min} follows from the requirement that the Universe expands at least until the formation of large-scale structure and the upper bound follows from the condition that $V(\phi_P) \approx 1$, since the minisuperspace approximation is unlikely to be valid when the potential energy of the matter sector exceeds the Planck density. However, in a chaotic inflationary scenario there is a critical value of the scalar field, ϕ_{suf} , and sufficient inflation occurs if $\phi_i > \phi_{suf}$ but not for $\phi_i < \phi_{suf}$. We must therefore calculate the conditional probability that sufficient inflation occurs given that ϕ_i is bounded by ϕ_{min} and ϕ_P . This quantity takes the form [14]

$$P(\phi_i > \phi_{suf} | \phi_{min} < \phi_i < \phi_P) = \frac{\int_{\phi_{suf}}^{\phi_P} \rho(\phi) d\phi}{\int_{\phi_{min}}^{\phi_P} \rho(\phi) d\phi}, \quad (24)$$

and allows us to determine which of the two boundary conditions considered here “naturally” predicts a phase of sufficiently long inflationary expansion. Sufficient inflation is a prediction of a theory if $P \approx 1$, whereas it is not if $P \ll 1$.

For standard reheating the minimum amount of inflation that solves the horizon problem is determined by the condition $N \equiv \ln(a_f/a_i) \approx 65$, where subscripts i and f denote the values of the scale factor at the onset and end of inflation respectively [16]. It is then straightforward to deduce from the classical field equations that

$$N \approx 65 \approx 6 \int_{\phi_f}^{\phi_{suf}} V(\phi) \left(\frac{dV(\phi)}{d\phi}\right)^{-1} d\phi, \quad (25)$$

where the value of the scalar field at the end of inflation, ϕ_f , is computed from the relation

$$\frac{1}{12} \left[V^{-1}(\phi) \frac{dV(\phi)}{d\phi} \right]_{\phi=\phi_f}^2 = 1. \quad (26)$$

This condition corresponds to the breakdown of the slow-roll approximation [17].¹

¹Strictly speaking, conditions (25) and (26) are only valid in spatially flat FLRW models, but we are considering spatially closed cases in this work. However, during inflation the curvature term in the Friedmann equation is redshifted to zero within one Hubble expansion time and the Universe effectively becomes spatially flat at an exponentially fast rate. For our purposes, therefore, these expressions remain valid.

Once ϕ_f is known, the value of ϕ_{suf} can be determined numerically by evaluating the integral in Eq. (25).

To understand how the probability densities (20) and (21) change in the quadratic and cubic cases, we shall consider them in turn. Since (20) and (21) are usually *not* normalisable (unless the range of values that ϕ can take is bounded), we set the “normalisation constants” equal to one as is the common practice.

4.1 The Quadratic case

To begin with, we note that the shape of $V(\phi)$ does not qualitatively change with changes in the coupling constant ϵ_1 . This parameter only fixes the height of the plateau and as a result leaves the shapes of the two probability densities unchanged. Consequently the qualitative behaviours of the probability densities are robust with respect to changes in ϵ_1 . Figure 2a gives a plot of ρ_T showing that it starts at zero when $\phi = 0$ and asymptotically approaches a constant value. On the other hand, as can be seen from Figure 2b, ρ_{NB} decreases from infinity and asymptotically approaches a constant value. We should emphasise here that since the probability distribution functions (20) and (21) typically take values of the order $\exp(\pm 10^{14})$, we, for the sake of graphical representation, applied non-linear scalings of the kinds $\tilde{\rho}_T = \rho_T^{1/C}$ and $\tilde{\rho}_{NB} = \ln(\rho_{NB}^{1/C})$ respectively (where C is a constant) to the two probability distribution functions. Note, however, that the values of the argument ϕ remain unaffected by this scaling.

Contrary to the claim of Biswas and Guha [9], the two probability distribution functions reveal *no* qualitative changes as compared to the case of “chaotic” type potentials (e.g. $V(\phi) = m^2 \phi^2/2$) as discussed by Vilenkin [5] and Halliwell [14]. This means that the tunneling wave function has its maximum nucleation probability for the Universe coming into existence somewhere on the plateau of the potential $V(\phi)$, whereas the no boundary wave function peaks near the true minimum of the potential at $\phi = 0$. Translated into initial values of the Ricci curvature scalar, this means that the tunneling wave function prefers values of R_i near the Planck scale, whereas the no-boundary wave function favours a Universe of large initial size, i.e. small R_i [11].

Figures 2a & 2b

We now consider the conditional probability (24). The range of values of ϕ_i is specified by the range of initial values R_i . In Planckian units, where $R_P = 1$, we deduce that $\phi_P = 13.0$. The value of ϕ_f is calculated from (26) to be $\phi_f = 0.38$ and condition (25) is therefore satisfied for $\phi_{suf} = 2.27$. Since the conditional probability measure (24) essentially amounts to a comparison of areas between the $\rho(\phi)$ curve and the positive ϕ -axis in Figures 2a and 2b, it seems obvious that the tunneling wave function leads to sufficient inflation whereas the no-boundary wave function does not. This is in line with the conclusions of Vilenkin [5] and Mijić et al [11] and in contrast

to what is claimed by Biswas and Guha [9].

4.2 The Cubic Case

We now consider the effects of adding a cubic term to the action. In general ρ_T is peaked around the maximum of $V(\phi)$ at ϕ_{max} and falls off to zero on both sides. In contrast ρ_{NB} decreases from infinity near $\phi = 0$ to a minimum at ϕ_{max} and diverges again as $\phi \rightarrow \infty$. In this sense the presence of the cubic term drastically alters the shapes of the two probability distributions. This qualitative behaviour is illustrated in Figures 3a and 3b for $\epsilon_1 = 10^{11}$ and $\epsilon_2 = 10^{20}$.

Figures 3a & 3b

Now, regarding the location of the maximum nucleation probability, the tunneling case is unambiguous since there is only a single peak in the probability distribution function. Note, however, that in the cubic case this wave function favours *smaller* values of the initial curvature R_i (viz. ϕ_i) as compared to those in the quadratic case, where they are of Planckian order. On the other hand, the case of the Hartle–Hawking boundary condition is ambiguous because of the presence of two peaks in the probability distribution function, corresponding respectively to low and high values of R_i .

From a practical point of view, the question arises as to whether the tunneling wave function still predicts a phase of sufficiently long inflationary expansion immediately after tunneling into the Lorentzian signature region. To investigate this, we confined ourselves to the region on the left of the maximum in the potential (10), i.e. $\phi \leq \phi_{max}$. Although inflation occurs on both sides of the turning point, there is no end to the superluminal expansion if the field rolls down the right-hand side and consequently there is no reasonable mechanism of reheating [15]. On the basis of these physical considerations it is therefore more appropriate to *identify* the upper limit ϕ_P of the integrals in Eq. (24) with ϕ_{max} rather than with the Planck limit.

The specific value of the conditional probability depends on the magnitude of ϵ_2 and it is therefore necessary to determine the relevant range of values for this parameter. We noted in Section 2 that ϵ_2 is bounded from above by the condition $\epsilon_2 \ll \epsilon_1^2$. As ϵ_2 is decreased relative to a *fixed* ϵ_1 , the location of the maximum is shifted to larger values of ϕ and eventually beyond the Planck limit ϕ_P . This follows since the model reduces to the R^2 -theory for which the potential exhibits a plateau, i.e. the maximum is effectively located at infinity in this case. However, according to condition (11) the region over which the cubic and quadratic potentials are equivalent also increases as ϵ_2 decreases. The question then is whether ϕ_{limit} grows faster or slower than ϕ_{max} . By explicitly calculating the values of ϕ_{max} and ϕ_{limit} it is found that ϕ_{limit} exceeds ϕ_{max} for all parameter values $\epsilon_2 \leq 10^{20}$. This implies that the R^2 - and R^3 -theories are equivalent for $\phi < \phi_{max}$ in this range. Hence the results

in Section 4.1 for R^2 -theory may be carried over directly to the cubic case in this region of the variable ϕ , although there is the important difference that the upper bound on ϕ_i is now identified with ϕ_{max} and not ϕ_P .

For any given ϵ_2 the end of inflation occurs at $\phi_f = 0.38$ as in the R^2 -case, since the R^3 -contribution is negligible at very small ϕ . Unfortunately a direct numerical integration of Eq. (24) cannot be performed, because the integrands are typically of the orders of $\exp(\pm 10^{14})$. However, since the probability density ρ is a single valued, positive-definite function of ϕ , it follows that a handle on the qualitative behaviour of the conditional probability can be obtained by investigating how the area under the $\rho(\phi)$ curve changes as ϵ_2 changes. The problem then reduces to determining how the limits of the integrals in the numerator and denominator vary as the parameters of the theory are altered.

The dependences of the parameters of interest on ϵ_2 are summarised in Table 1. We find that ϕ_{suf} for the potential (10) settles at the same value as in the quadratic case when ϵ_2 is of order 10^{18} or smaller. We also find that ϕ_{max} rapidly approaches ϕ_{suf} in the region $10^{18} \leq \epsilon_2 \leq 10^{20}$. This implies that the integral in the numerator of the conditional probability (24) becomes *much smaller* than the term in the denominator for $\epsilon_2 \geq 10^{18}$. Consequently the tunneling approach does not predict a phase of sufficiently long inflation in this region, contrary to the results for the R^2 -model. We further note that for the same range of initial values of ϕ , the Hartle–Hawking wave function shows no qualitative change from the quadratic case. Consequently, it appears that neither boundary condition predicts inflation for this choice of the parameters ϵ_1 and ϵ_2 . This behaviour occurs because the presence of the cubic perturbation severely restricts the range of initial field values ϕ_i for which a phase of sufficiently long inflationary expansion is likely.

Including the full range of values of ϕ_i up to the Planck limit ϕ_P would not significantly improve this result in the tunneling scheme. In the Hartle–Hawking case, however, the integral in the numerator of (24) would have a large contribution from the second peak in ρ_{NB} . However, this range of ϕ_i was excluded, as discussed above, in order to avoid the problem of exiting the inflationary expansion.

Even though the conditional probability P of Eq. (24) cannot be estimated numerically in this case, nevertheless, we present a set of values of "scaled conditional probabilities" in Appendix A which are obtained by applying a non-linear scaling to the probability distribution functions as discussed in Section 4.1. These values, which may be treated as qualitative indicators of P , also support the conclusions given in this section.

5 Discussion and Conclusions

In this paper we have investigated how the probability of realising sufficient inflation from quantum cosmology is altered when higher-order corrections to the Einstein–Hilbert action are introduced. Our results confirm that the addition of quadratic

terms to the action does not reverse the conclusions of Vilenkin [5] regarding the effects of boundary conditions on the likelihood of sufficient inflation, in contrast to some recent claims [9]. On the other hand, cubic perturbations can produce qualitative changes to the nature of the probability distribution function $\rho(\phi)$. From a physical point of view one is confined to consider initial values of the scalar field that allow an exit from the inflationary expansion. As a result the important physical (as opposed to purely mathematical) consequences of cubic perturbations are that they restrict the measure of allowed initial field values ϕ_i that lead to sufficient inflation. This is in agreement with the classical arguments [15]. By considering the conditional probability (24) (see also Appendix A) we have argued that if the coupling constant ϵ_2 , which determines the strength of the R^3 -contribution to the Lagrangian, exceeds a critical value, neither the tunneling nor the no-boundary boundary conditions predict an epoch of sufficient inflation, in the sense that the conditional probability is significantly less than unity in both cases.

Our results appear to exhibit some generality in four-dimensions. As discussed in Section 2, the qualitative shape of the self-interaction potential $V(\phi)$ remains unaltered if general polynomial perturbations with a highest order term $\epsilon_{n-1} R^n$ are considered. In general this result is true when $D < 2n$. This immediately implies that neither of the two probability distributions ρ_T and ρ_{NB} for the $n = 3$ case will be qualitatively affected under $n > 3$ perturbations. The qualitative conclusions drawn for the case of cubic perturbations in Section 4.2 therefore remain robust under higher-order perturbations to the action, although of course the details of what happens will depend on how the precise location of the maximum in the potential $V(\phi)$ is related to the highest-order term.

However, the consequences of the quadratic and the cubic perturbations (as well as those of general polynomial types) depend crucially on the values of the free parameters of the system, namely ϵ_k ($k = 1, \dots, n - 1$), D , n , as well as on the initial field values ϕ_i . In particular, the dimensionality D of the space-time is crucial in deciding the maximum degree n of perturbations allowed ($D < 2n$ say) above which the perturbations would be qualitatively inconsequential, i.e. the system would be robust.

Finally we remark that inflation is possible, at least at the classical level, if the field is initially placed to the right of the maximum in Eq. (10) and given sufficient kinetic energy to travel over the hill towards $\phi = 0$. Unfortunately, our analysis cannot consider this possibility since the scalar field momentum operator in the Wheeler–DeWitt equation then becomes important and the solutions (20) and (21) are no longer valid. Furthermore, if one is prepared to include the effects of the R^3 -contribution in the action, the cubic term $R\Box R$ should also be considered. In this case the effective theory resembles Einstein gravity minimally coupled to two scalar fields after a suitable conformal transformation on the metric [15] and in principle a similar analysis to the one presented here can be followed for this more general case. We shall return to some of these questions in future.

Acknowledgements

We thank Jonathan J Halliwell and A D Linde for helpful remarks. HVE is supported by a Grant from the Drapers' Society at QMW. JEL is supported by the Science and Engineering Research Council (SERC), UK, and is supported at Fermilab by the DOE and NASA under Grant No. NAGW-2381. RT is supported by the SERC, UK, under Grant No. H09454.

References

- [1] DeWitt B S 1967 *Phys. Rev.* **160** 1113
Wheeler J A 1968 *Battelle Rencontres* eds DeWitt C and Wheeler J A (New York: Benjamin)
- [2] Linde A D 1984 *Zh. Eksp. Teor. Fiz.* **87** 369 (*Sov. Phys. JETP* **60** 211)
Linde A D 1984 *Lett. Nuovo Cim.* **39** 401
Linde A D 1984 *Rep. Prog. Phys.* **47** 925
- [3] Vilenkin A 1984 *Phys. Rev. D* **30** 509
- [4] Hartle J B and Hawking S W 1983 *Phys. Rev. D* **28** 2960
Hawking S W 1984 *Nucl. Phys.* **B239** 257
- [5] Vilenkin A 1988 *Phys. Rev. D* **37** 888
- [6] Candelas P, Horowitz G T, Strominger A and Witten E 1985 *Nucl. Phys.* **B258** 46
- [7] Barth N H and Christensen S M 1983 *Phys. Rev. D* **28** 1876
- [8] Antoniadis I and Tomboulis E T 1986 *Phys. Rev. D* **33** 2756
- [9] Biswas S and Guha J 1993 *Class. Quantum Grav.* **10** L123
- [10] Boulware D G 1984 *Quantum Theory of Gravity* ed Christensen S M (Bristol: Hilger) p 267
Hawking S W and Luttrell J C 1984 *Nucl. Phys.* **B247** 250
Vilenkin A 1985 *Phys. Rev. D* **32** 2511
Horowitz G T 1985 *Phys. Rev. D* **31** 1169
Hawking S W and Wu Z C 1985 *Phys. Lett.* **151B** 15
Odintsov S R and Shapiro I L 1992 *Class. Quantum Grav.* **9** 873
Kasper U 1993 *Class. Quantum Grav.* **10** 869
- [11] Mijić M B, Morris M S and Suen W-M 1989 *Phys. Rev. D* **39** 1496
- [12] Whitt B 1984 *Phys. Lett.* **145B** 176

- [13] Barrow J D and Cotsakis S 1988 *Phys. Lett.* **214B** 515
Maeda K 1989 *Phys. Rev. D* **39** 3159
- [14] Halliwell J J 1991 *Quantum Cosmology and Baby Universes* eds Coleman S, Hartle J B, Piran T and Weinberg S (World Scientific: Singapore)
- [15] Berkin A L and Maeda K 1990 *Phys. Lett.* **245B** 348
Barrow J D and Cotsakis S 1991 *Phys. Lett.* **258B** 299
- [16] Guth A H 1981 *Phys. Rev. D* **23** 347
- [17] Steinhardt P J and Turner M S 1984 *Phys. Rev. D* **29** 2162
- [18] Halliwell J J 1988 *Phys. Rev. D* **38** 2468

Appendix A

As was pointed out in Section 4.1, the integrands involved in the definition of conditional probability typically have magnitudes of order $\exp(\pm 10^{14})$, which makes the numerical calculation of the integrals not possible in practice. Now due to the nature of these numbers no linear scaling of the probability function ρ can bypass this difficulty. The question then arises as to whether appropriate non-linear scalings exist which keep the conditional probability P invariant. To see that there do *not*, recall that the only scalings that leave the Wheeler–DeWitt equation, $H\Psi = 0$, of the D -dimensional minisuperspace models of Quantum Cosmology invariant are given by $\tilde{H} = \Omega^{-2}H$, $\tilde{\Psi} = \Omega^\gamma \Psi \rightarrow \tilde{H}\tilde{\Psi} = \Omega^{\gamma-2}H\Psi = 0$, ($\Omega(q)$ is an arbitrary function of the minisuperspace co-ordinates q) provided γ and ξ (a free parameter in the Wheeler–DeWitt equation) are given by $\gamma = (2-D)/2$ and $\xi = -(D-2)/8(D-1)$ respectively [18]. Effectively this amounts to a redefinition of the potential $U(q)$ and the DeWitt metric of minisuperspace $f^{\alpha\beta}(q)$, which occur in the Hamilton operator H . More importantly, under such scale transformations the conserved probability current j^α defined from Ψ remains unchanged. This freedom, however, is not of much use in bypassing the numerical difficulty mentioned above in order to obtain quantitative values for P . Nevertheless, if we confine ourselves to qualitative information, we may choose non-linear (but monotonic) scalings of ρ , which, while violating the invariance properties of the model, would nevertheless supply us with a qualitative indicator of P . This is not dissimilar to the way non-linear scalings of functions are employed for the purpose of graphical representation.

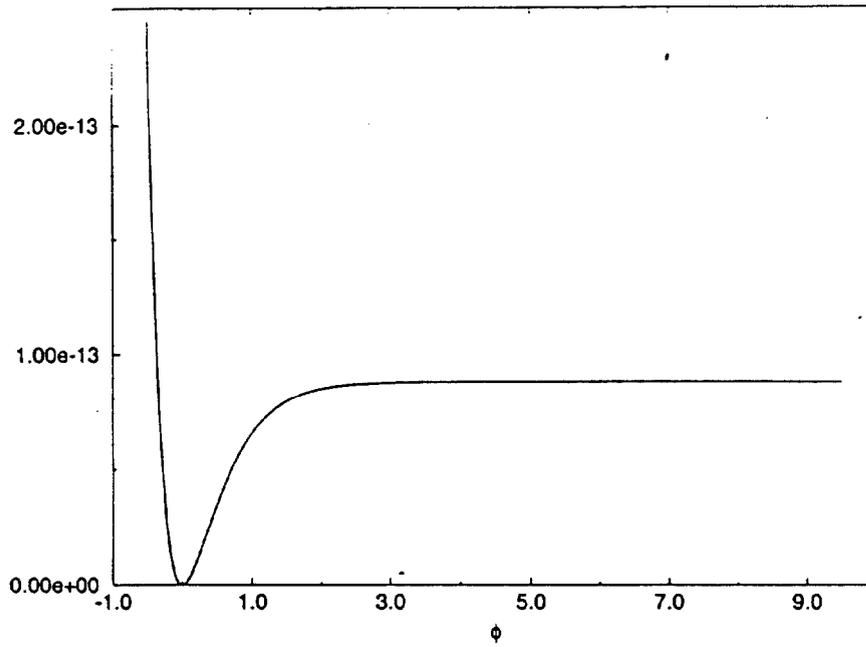
To calculate a qualitative indicator of P we define the *non-linearly scaled conditional probability* \tilde{P} as

$$\tilde{P}(\phi_i > \phi_{\text{sup}} | \phi_{\text{min}} < \phi_i < \phi_P) \equiv \frac{\int_{\phi_{\text{sup}}}^{\phi_P} \rho^{1/C}(\phi) d\phi}{\int_{\phi_{\text{min}}}^{\phi_P} \rho^{1/C}(\phi) d\phi}, \quad (27)$$

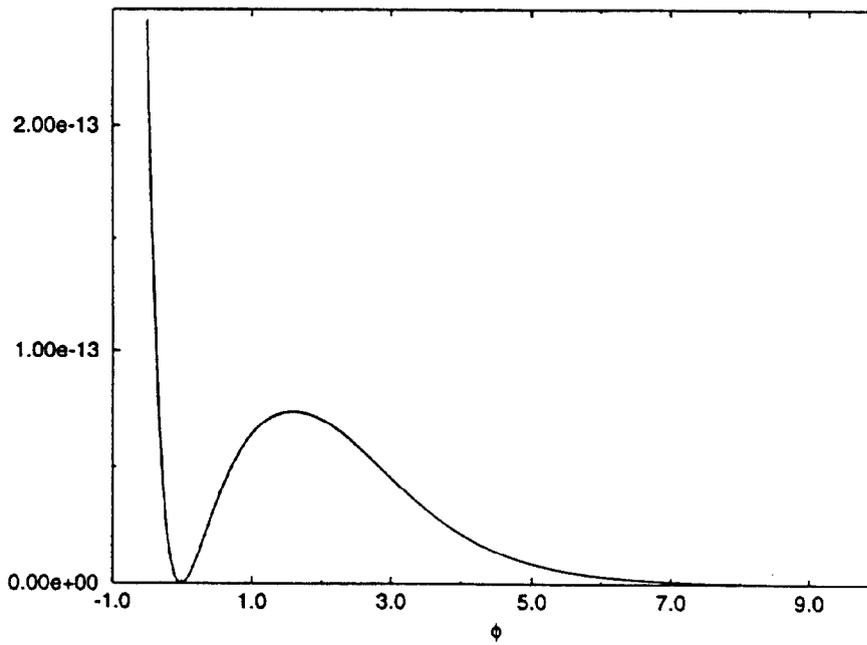
where C is the index of non-linear scaling. Clearly such a scaling will not change the qualitative behaviour of ρ_T and the values of its argument ϕ , and therefore the values of the boundaries of the integrals occurring in Eq. (24) (as listed in Table 1) remain the same. Furthermore, such scalings leave P invariant in the limiting cases where $P = 0$ and $P = 1$.

Here we chose $C = 10^{14}$. Table 2 gives the values of \tilde{P} as a function of ϵ_2 for the tunneling wave function in the case of the R^3 -model, calculated for $\epsilon_1 = 10^{11}$ and the boundary values of ϕ given in Table 1. We approximated ϕ_{min} by $\phi_f = 0.38$. For the corresponding value of \tilde{P} for the tunneling model in the R^2 -case of Section 4.1 we found $\tilde{P} = 0.85$.

As can be seen from Table 2, the behaviour of \tilde{P} supports the conclusions drawn in Section 4.2 on the basis of qualitative analysis.

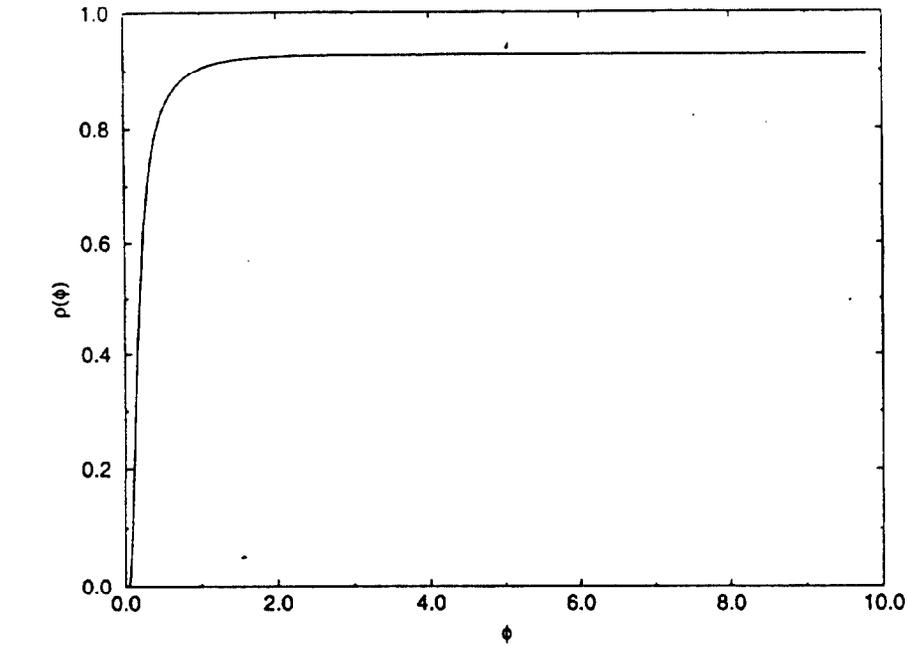


a)

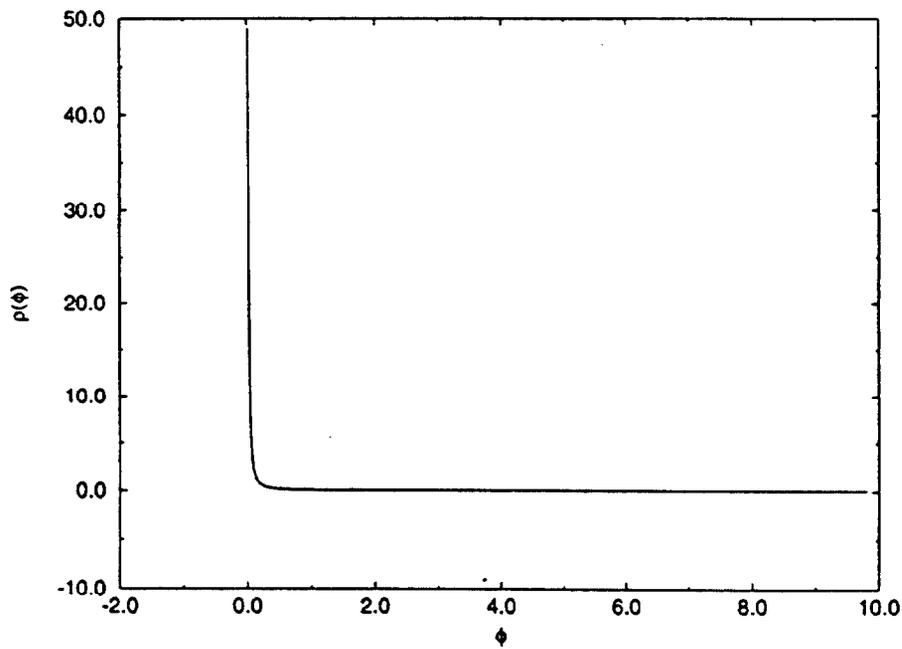


b)

Figure 1: (a) The effective self-interaction potential (9) corresponding to the R^2 -theory with $\epsilon_1 = 10^{11}$. The scalar field and magnitude of the potential have been rescaled via Eq. (12) to enable easy comparison with the results of Section 4; (b) The rescaled effective interaction potential (10) corresponding to the R^3 -theory with $\epsilon_1 = 10^{11}$ and $\epsilon_2 = 10^{20}$.

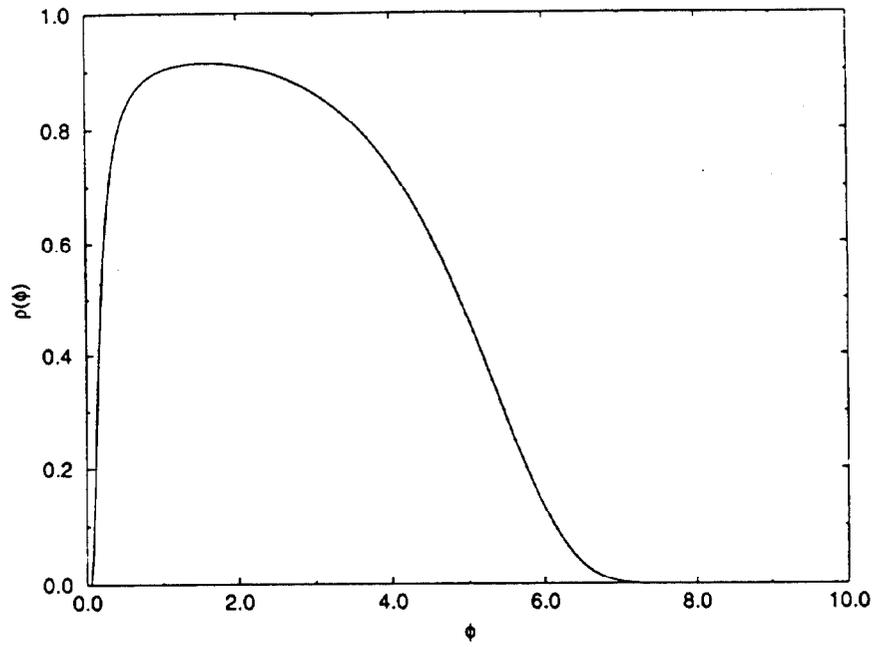


a)

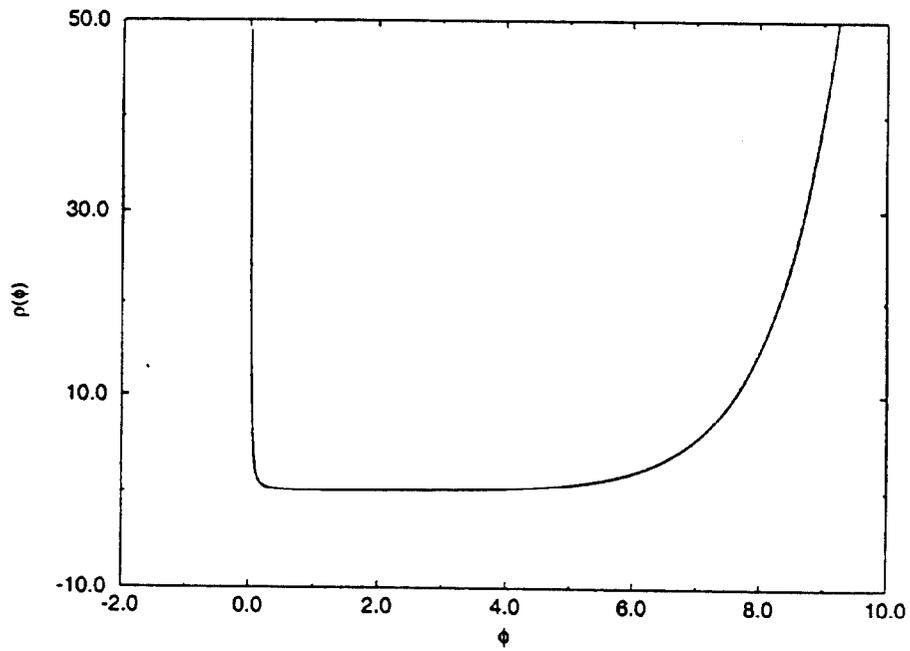


b)

Figure 2: (a) The probability distribution $\rho_T(\phi)$ associated with the tunneling boundary condition for the R^2 -theory with a rescaling $\rho_T(\phi) = [\exp(-2/3V)]^{10^{-14}}$; (b) The probability distribution $\rho_{NB}(\phi)$ associated with the no boundary boundary condition for the R^2 -theory with a rescaling $\rho_{NB}(\phi) = \ln[\exp(2/3V)]^{10^{-14}}$. We choose these particular rescaled values of $\rho(\phi)$ in order to obtain easily interpretable plots from our numerical programme.



a)



b)

Figure 3: (a) The tunneling probability distribution $\rho_T(\phi)$ for the R^3 -theory with the same rescaling as for Figure 2a; (b) The no boundary probability distribution $\rho_{NB}(\phi)$ for the R^3 -theory with the same rescaling as for Figure 2b.

ϵ_2	10^{20}	10^{18}	10^{16}	10^{14}	10^{12}	10^{10}	10^8	10^6
ϕ_P	23.6	21.3	19.0	16.7	14.4	13.1	13.0	13.0
ϕ_{limit}	2.65	4.95	7.25	9.56	11.9	14.2	16.5	18.8
ϕ_{max}	1.59	2.68	3.78	4.94	7.06	9.34	11.7	13.0
ϕ_{suf}	1.59	2.24	2.27	2.27	2.27	2.27	2.27	2.27

Table 1: Summarising, for different values of ϵ_2 , the values of the scalar field corresponding to $R_P = 1$ (ϕ_P), the limit of ϕ below which the R^2 - and R^3 -potentials are equivalent (ϕ_{limit}), the location of the maximum in the potential (ϕ_{max}) and the values of the field that just lead to sufficient inflation (ϕ_{suf}). We specify $\epsilon_1 = 10^{11}$ throughout due to microwave background considerations. As ϵ_2 increases to order of 10^{18} , the magnitudes of the quantities ϕ_{max} and ϕ_{suf} become comparable to one another and this implies that the numerator in the conditional probability approaches zero. This suggests that the conditional probability will become significantly smaller than unity for values of $\epsilon_2 \geq 10^{18}$.

ϵ_2	10^{20}	10^{18}	10^{16}	10^{14}	10^{12}	10^{10}	10^8	10^6
\tilde{P}	0.00	0.20	0.45	0.59	0.72	0.79	0.84	0.85

Table 2: Behaviour of the non-linearly scaled conditional probability distribution \tilde{P} for the tunneling wave function Ψ_V in the R^3 -model of Section 4.2. We specify $\epsilon_1 = 10^{11}$ throughout.