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The Borel Singularity of Instanton-induced Amplitudes

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Abstract

The Borel transform of the Espinosa-Ringwald type cross section in theories having explicit mass parameters is considered. The nature and position of the leading singularity in the Borel transform variable b are determined.

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1 Introduction

The Green functions with external particles in the instanton background have been investigated in relation to the anomalous baryon number violation in the standard model [1]. The cross section of two body scattering in the one-instanton sector in the standard model is given by

$$\sigma \sim \exp\left(-\frac{4\pi}{\alpha_w}\mathcal{F}\left(\frac{E}{E_{sph}}\right)\right) \tag{1}$$

where α_w, E_{sph} are the weak coupling and the sphaleron energy respectively. As pointed out by several authors [2], this form of the cross section might suggest a shift of the position of the instanton singularity in the Borel transform variable b as a function of the energy. However, if it does shift, it would contradict 't Hooft's original argument that the singularity position is universal [3]. Therefore, it would be interesting to find out the position and nature of the singularity associated with the instanton-induced cross section.

In this note we consider theories having explicit dimensional parameters, and exclude QCD type interactions that have an infrared problem associated with the instanton size. Then, we can show that independently of the energy the leading singularity is at $b = 1$ as for the vacuum tunnelling amplitude, and the nature of the singularity is determined by the interactions of infinitely far separated instanton and anti-instanton pairs. For the quantum mechanical double-well potential and the two-dimensional abelian Higgs model, the singularity is of branch-point type and its strength depends on the energy.

For the nonabelian Higgs model in three dimensions, the type of the singularity is an essential singularity. For the standard model, the singularity is of branch-point type and its strength is independent of the energy.

2 Singularity in the Borel variable

Let us consider a theory having a dimensionless coupling g and mass m. An extension to theories having more than one mass parameters will be straightforward. The Lagrangian is

$$\mathcal{L} = \frac{1}{g} L(\phi, m) \tag{2}$$

with ϕ representing generic fields. We assume that m is independent of g, and g can be always factored out of L by rescaling ϕ so that L is independent of g. We also assume that there is an exact instanton solution with the action

$$S_I = \frac{S_0}{g} \tag{3}$$

where S_0 is a constant.

We consider the Espinosa-Ringwald type cross section in the valley method approach [4] in which the imaginary part of the forward scattering amplitude is calculated in the instanton-anti-instanton background. In the valley method the instanton-induced cross section is

$$\sigma(g) \sim \left(\frac{S_0}{g}\right)^{\nu} \exp\{ER^{*} - S_{val}(mR^{*})\} \tag{4}$$

where ν is a model-dependent positive constant and S_{val} and E are the valley action and the c.m. energy of the system respectively. R is the distance between the instanton and anti-instanton pair. In the quantum mechnical double-well potential σ is the transition rate between the vacua induced by excited states, and if E is replaced by the chemical potential it becomes the anomalous decay rate of dense baryonic matter[5]. R^* is the saddle point of the exponential part in (4) and satisfies

$$\epsilon - \frac{d}{d(mR)} S_{val}(mR)\Big|_{R=R^*} = 0 \tag{5}$$

where $\epsilon = E/m$. Defining the interaction energy $V(mR)$ of an instanton and anti-instanton pair by

$$S_{val}(mR) = \frac{2S_0}{g}(1 - V(mR)), \tag{6}$$

we can write (5) as

$$x + V'(y(x)) = 0 \tag{7}$$

where $x = \epsilon/z$, with $z = 2S_0/g$ and $y = mR^*$. Note that $V(\infty) = 0$.

The Borel transform of (4) is defined by

$$\tilde{\sigma}(b) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zb}\, \sigma(g)\, dz. \tag{8}$$

This definition of the Borel transform normalizes the singularity position of the vacuum amplitude to $b = 1$. Substituting (4) into (8)

$$\tilde{\sigma}(b) \sim \int \exp\{z(b - f(x)) + \nu \log z\} dz \tag{9}$$

with

$$f(x) = 1 - x\,y(x) - V(y(x)). \tag{10}$$

To determine the leading singularity in the Borel variable b, (9) may be evaluated by the saddle point approximation. The equation for the saddle point z^* is

$$b - f(x^*(b)) + x^*(b)\,f'(x^*(b)) + \frac{\nu}{\epsilon}\,x^*(b) = 0 \tag{11}$$

where $x^*(b) = \epsilon/z^*(b)$, or

$$b - 1 + \frac{\nu}{\epsilon}\,x^*(b) + V\left(y(x^*(b))\right) = 0 \tag{12}$$

using (10). The Borel transform of σ is then

$$\begin{aligned}\tilde{\sigma}(b) &\sim \frac{(z^*)^{\nu+1}}{\sqrt{\frac{\nu}{\epsilon} + x^*\,f''(x^*)}} \exp\{z^*\,(b - f(x^*))\} \\ &\sim \frac{(z^*)^{\nu+1}}{\sqrt{\frac{\nu}{\epsilon} + x^*\,f''(x^*)}} \exp\left(-\epsilon\,f'(x^*)\right) \\ &\sim \frac{(x^*)^{-(\nu+1)}}{\sqrt{\frac{\nu}{\epsilon} - x^*\,y'(x^*)}} \exp\left(\epsilon\,y(x^*)\right)\end{aligned} \tag{13}$$

using

$$f'(x^*) = -y(x^*) \tag{14}$$

from (7) and (10). An extra factor of z^* in the pre-exponentional part is from the Gaussian determinant. Eq.(13) shows that the leading singularities can arise either at $x^* = 0$ or $y(x^*) = \infty$. However, since in general $V'(\infty) = 0$ and $V(y)$ is monotonically decreasing function, the solution of (7) must satisfy

$$y(0) = \infty, \tag{15}$$

and so the leading singularity can be only at $x^* = 0$. Then (12) immediately tells us that the position of the singularity in the Borel variable b is at $b = 1$ as in the vacuum tunnelling case, and the nature of the singularity is determined by the interaction energy of the infintely far separated instanton and anti-instanton pair. In general, $V(y)$ decreases either exponentially or in powers of $1/y$ as y approaches infinity. In next two sections we discuss the nature of the singularity for the two asymptotic behavior of $V(y)$.

3 Exponentially decreasing potential

Let us assume V is given by

$$V(y) \to \frac{c}{y^k}\, e^{-ny}, \qquad \text{for} \quad y \to \infty \tag{16}$$

where n is a positive integer and c, k are constants. The asymptotic behavior of V in the quantum mechanical double-well potential and the two dimensional abelian Higgs model follows (16) with $n = 2, k = 0$ and $n = 1, k = 1/2$ respectively [6, 7]. In the abelian Higgs model, the mass m must be understood as the minimum of the vector boson and Higgs masses.

Substituting (16) into (7), we find

$$y(x) = -\frac{1}{n}\ln x - \frac{k}{n}\ln(\ln\frac{1}{x}) + O(1)\,, \quad \text{for} \;\; x \to 0. \tag{17}$$

The solution of (12) using (16) and (7) gives

$$x^*(b) \sim 1 - b, \qquad \text{for} \quad b \approx 1 \tag{18}$$

Substitution of (17), (18) into (13) gives

$$\tilde{\sigma}(b) \sim (1-b)^{-(\nu+1+\frac{\epsilon}{n})} \left(\ln \frac{1}{(1-b)} \right)^{-\frac{\epsilon k}{n}} \tag{19}$$

near the singularity. This shows that the singularity is of branch-point type and the strength of the singularity depends on the energy. Note that the strength of the singularity is independent of the details of the potential in the Lagrangian. For example, in the quantum mechanical double-well potential, the nature of the singularity has nothing to do with the shape of the potential outside the classical vacua, and the dependence on the potential appears only through the mass m, which is nothing but the second derivative of the potential at the classical vacua. In this sense, the nature of the singularity is universal. The universality of the singularity originates from the fact that the interaction energies of the infinitely far separated instanton and anti-instanton pairs are determined by the asymptotic form of the instanton solutions, which in turn is determined by the shape of the potential near the classical vacua.

4 Power decreasing potential

Now we consider the potential V decreasing in powers of $1/y$,

$$V(y) \to \frac{c^2}{y^k}, \qquad \text{for} \quad y \to \infty, \tag{20}$$

with k a positive number and c a constant. The interaction energy of the instanton and anti-instanton pair in the nonabelian Higgs model in three dimensions has the asymptotic behavior in (20) with $k = 1$. The constant c is then the 't Hooft-Polyakov magnetic monopole charge. A parallel calculation as in sec. 3 gives

$$y(x) \approx \left(\frac{c^2 k}{x}\right)^{\frac{1}{k+1}} \tag{21}$$

and

$$x^*(b) = \frac{c^2 k}{(c^2)^{\frac{k+1}{k}}}(1-b)^{\frac{k+1}{k}} + \cdots \tag{22}$$

near the singularity. Terms neglected in (22) are irrelevant for the leading singularity. Substituting (21), (22) into (13) we get the Borel transform

$$\tilde{\sigma}(b) \sim (1-b)^{-\frac{k+1}{k}(\nu+1-\frac{1}{2(k+1)})} \exp\left(\epsilon\left(\frac{c^2}{1-b}\right)^{\frac{1}{k}}\right) \tag{23}$$

near $b = 1$. Therefore, $b = 1$ is an essential singularity.

5 Constrained instanton case

Until now, we have considered theories that have exact instanton solutions. Thus, our conclusion in the previous sections cannot be directly applied to theories such as the standard model that do not have exact instanton solutions. However, the modification needed is minimal. We illustrate the calculation in the standard model.

As is well known, in the standard model there is no exact instanton solution. However, when a constraint is introduced to restrict the quantum fluctuations to a scale ρ, there is an instanton-like solution, the so called constrained instanton [8]. Because of the extra scale, the valley action now depends on two parameters mR and ρ, where m

is the gauge boson mass. For simplicity, we assume the Weinberg angle vanishes so that $m_w = m_z$.

The valley action can be written as

$$S_{val}(mR, \rho) = \frac{2S_0}{g}(1 - V(\rho/R, mR)) \tag{24}$$

with $S_0 = 8\pi^2$ and $g = g_w^2$, where g_w is the $SU(2)$ gauge coupling. The anomalous cross section is similarly given by (4),(6) with $V(mR^*)$ replaced by $V(\rho^*, mR^*)$. The saddle point equation for R^* and ρ^* corresponding to (7) is then

$$\begin{aligned} x + \frac{\partial}{\partial y} V(w(x), y(x)) &= 0 \\ \frac{\partial}{\partial w} V(w(x), y(x)) &= 0 \end{aligned} \tag{25}$$

where the definition of x, y are same as in sec.2 and $w = \rho^*/R^*$. Truncating the Higgs contribution to the valley action to the leading order, V can be written as [9]

$$V(w, y) = U(w) - \frac{1}{2} w^2 y^2 \tag{26}$$

where U is the valley action of the pure $SU(2)$ gauge theory. Then (25) becomes

$$\begin{aligned} x - w(x)^2 \, y(x) &= 0 \\ U'(w(x)) - w(x) \, y(x)^2 &= 0. \end{aligned} \tag{27}$$

From (27) it is obvious that $y(x)$ can not be divergent for finite x since $U'(w)$ is finite. The saddle point equation for z^* corresponding to (12) is similarly given by

$$b - 1 + \frac{\nu}{\epsilon} x^*(b) + V(w(x^*(b)), y(x^*(b))) = 0, \tag{28}$$

and the Borel transform of the anomalous cross section is exactly given by (13). Therefore, the leading singularity arises at $x^* = 0$. To find out the position of the singularity, we use the asymptotic form of $U(w)$ for $1/w \to \infty$ [9],

$$U(w) \to 6\,w^4, \qquad \text{for} \quad w \to 0. \tag{29}$$

The solution of (27) near $x \approx 0$ is then

$$w \sim y \sim x^{\frac{1}{3}}. \tag{30}$$

From (28) we see that the singularity is at $b = 1$. Solving (28) near $b = 1$

$$x \sim 1 - b, \qquad \text{for} \quad b \sim 1. \tag{31}$$

Substituting (31) into (13) we have

$$\tilde{\sigma}(b) \sim (1 - b)^{-(\nu+1)}, \qquad \text{for} \quad b \sim 1. \tag{32}$$

The position and nature of the singularity are essentially same as those of the vacuum amplitude. Since $y, w \to 0$ as $x \to 0$, the truncation of the Higgs contribution in (26) to the leading order does not affect the nature of the singularity.

6 Corrections to the Gaussian approximation

We consider in this section the corrections to the Gaussian approximation we have taken in sec.2, and show that they do not change our conclusions on the nature of the singularity. The corrections to the Gaussian approximation can be calculated conveniently in the Feynman diagram technique. Expanding

$$F(z) = z\,(b - f(x)) + \nu \log z \tag{33}$$

in (9) about the saddle point z^*

$$F(z) = \sum_{i=0}^{\infty} \frac{1}{i\,!} F^{(i)}(z^*)\,(z - z^*)^i, \tag{34}$$

where $F^{(i)}$ is the i-th order derivative of F, we have the propagator

$$\left(F^{(2)}(z^*)\right)^{-1} \tag{35}$$

and the vertex functions

$$\frac{1}{i\,!} F^{(i)}(z^*), \qquad i = 3, 4, \cdots \tag{36}$$

for the vertices with i number of legs. We call this vertex the i-th vertex. Now using (14), it can be easily shown that

$$F^{(i)}(z^*) = (x^*)^i \sum_{j=0}^{i-1} c_j \cdot (x^*)^j\, y^{(j)}(x^*). \tag{37}$$

where c_j are constants and $y^{(j)}(x)$ is the j-th order derivative of $y(x)$. By definition, $y^{(0)}(x)$ is supposed to be a constant. The corrections to the Gaussian approximation are given by the connected vacuum bubble diagrams. Let us consider an arbitrary connected bubble diagram, which we call B, having $n_i (\neq 0)$ number of the i-th vertices. Then

$$\frac{1}{2}\sum_i i \cdot n_i = I \tag{38}$$

where I is the number of the internal lines. An evaluation of the bubble diagram B with (35), (36) gives

$$\begin{aligned} B &\sim \frac{\prod_i \left[F^{(i)}(z^*)\right]^{n_i}}{\left[F^{(2)}(z^*)\right]^I} \\ &\sim \frac{\prod_i \left[(x^*)^{i \cdot n_i} \left(\sum_{j=0}^{i-1} c_j \cdot (x^*)^j \, y^{(j)}(x^*)\right)^{n_i}\right]}{\left[(x^*)^2 \left(\frac{\nu}{\epsilon} - x^* \, y'(x^*)\right)\right]^I} \\ &= \frac{\prod_i \left[\sum_{j=0}^{i-1} c_j \cdot (x^*)^j \, y^{(j)}(x^*)\right]^{n_i}}{\left[\frac{\nu}{\epsilon} - x^* \, y'(x^*)\right]^I}. \end{aligned} \tag{39}$$

Now for the exponentially decreasing potential in sec.3, we have from (17)

$$(x^*)^i \, y^{(i)}(x^*) \approx O(1), \qquad \text{for} \quad x^* \to 0 \tag{40}$$

for all i. Thus $B \sim O(1)$ near the singularity, and so the corrections to the Gaussian approximation modify only the overall factor of (13) and (19), and do not affect the nature of the singularity.

For the case of the power decreasing potential in sec.4, we have

$$(x^*)^i \, y^{(i)}(x^*) \sim y(x^*) \sim (x^*)^{-\frac{1}{k+1}} \qquad \text{for} \quad x^* \to 0 \tag{41}$$

from (21). Substituting (41) into (39)

$$\begin{aligned} B &\sim (x^*)^{\frac{1}{k+1}(I - \sum n_i)} \\ &= (x^*)^\beta \end{aligned} \tag{42}$$

where

$$\beta = \frac{1}{k+1}(I - \sum n_i) > 0. \tag{43}$$

Therefore,

$$B \to 0 \qquad \text{as} \quad x^* \to 0 \tag{44}$$

This implies that the Gaussian approximation becomes exact near the singularity. This can be seen directly with (34) by rescaling

$$z - z^* \to \frac{1}{\sqrt{-F^{(2)}(z^*)}}(z - z^*). \tag{45}$$

Then the vertex functions in (36) scale as

$$\frac{1}{i!}F^{(i)}(z^*) \to \frac{1}{i!}\frac{F^{(i)}(z^*)}{[-F^{(2)}(z^*)]^{\frac{i}{2}}}, \tag{46}$$

which can be checked using (37), (41) to approach zero near the singularity. Similarly, it can be shown that for the constrained instanton case, the corrections are of order one near the singularity and do not affect its nature.

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