Fermi National Accelerator Laboratory

FERMILAB-Pub-94/062-A
March 1994

Stringy Quantum Cosmology of the Bianchi Class A

James E. Lidsey[1]

NASA/Fermilab Astrophysics Center,
Fermi National Accelerator Laboratory, Batavia, IL 60510

Abstract

The quantum cosmology of the string effective action is considered within the context of the Bianchi class A minisuperspace. An exact unified solution is found for all Bianchi types and interpreted physically as a quantum wormhole. The solution is generalized for types VI_0 and VII_0. The Bianchi type IX wavefunction becomes increasingly localized around the isotropic Universe at large three-geometries.

Submitted as an Essay to the Gravity Research Foundation

[1]Electronic address - jim@fnas09.fnal.gov

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

The ongoing experiments investigating the cosmic microwave background radiation indicate that the Universe is very nearly isotropic on very large scales and the origin of this observed isotropy remains a fundamental problem in modern cosmology [1]. The inflationary paradigm offers a partial resolution in the form of the cosmic no hair conjecture [2]; a Universe dominated by vacuum energy should rapidly approach the de Sitter solution regardless of initial conditions and any primordial anisotropies and inhomogeneities are therefore washed out. Such an explanation is not complete, however, since there exist solutions that recollapse before the vacuum energy is able to dominate. This suggests that additional physics is required and further insight might be gained at the Planck epoch where quantum gravity effects are thought to be important.

The superstring theory [3] and the quantum cosmology program [4] are two approaches to the subject of quantum gravity that have been investigated in some detail. If the superstring is indeed the ultimate 'theory of everything', it must explain the observed isotropy at some level. Unfortunately a consistent quantum field theory of the superstring is not currently available and general solutions to the full Wheeler-DeWitt (WDW) equation have yet to be found. However, progress can be made by making a minisuperspace approximation and considering the quantum cosmology of the string effective action [5]. Recently an exact solution for the Bianchi IX minisuperspace was found by Lidsey and remarkably the wavefunction becomes peaked around the isotropic solution as the three-volume of the Universe increases [6]. This represents the first, non-vacuum, exact Bianchi IX solution to the WDW equation and the purpose of this essay is to derive this solution and generalize it within the context of the Bianchi class A models.

The spatially homogeneous models admit a Lie group G_3 of isometries transitive on spacelike three-dimensional orbits. The Euclidean 4-metric is

$$ds^2 = dt^2 + h_{ab}\omega^a\omega^b, \qquad a, b = 1, 2, 3, \tag{1}$$

where the 3-metric on the surfaces of homogeneity is given by

$$h_{ab}(t) = e^{2\alpha(t)}\left(e^{2\beta(t)}\right)_{ab} \tag{2}$$

and the matrix $\beta_{ab} = \mathrm{diag}\left[\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+\right]$ is traceless. The one-forms ω^a satisfy the Maurer-Cartan equation $d\omega^a = \frac{1}{2}C^a{}_{bc}\omega^b \wedge \omega^c$, where $C^a{}_{bc} = m^{ad}\epsilon_{dbc} + \delta^a{}_{[b}\, a_{c]}$ are the structure constants of the Lie algebra of G_3, $a_c \equiv C^a{}_{ac}$ and m_{ab} is symmetric. The Jacobi identity $C^a{}_{b[c}C^b{}_{de]} = 0$ implies that a_b is transverse to m^{ab}, i.e. $m^{ab}a_b = 0$, and the Lie algebra belongs to the Bianchi class A if $a_b = 0$ [7]. This class consists of types I, II, VI_0, VII_0, VIII and IX and the Lie algebra of each type is uniquely determined up to isomorphisms by the rank and signature of m^{ab}.

There are no closed topologies in the Bianchi class B and this may explain why a standard Hamiltonian treatment is not available for this class [8]. Consequently we restrict our attention to the class A. The spatially flat ($k = 0$) and spatially closed

($k = +1$) Friedmann Universes are the isotropic limits ($\beta_+ = \beta_- = 0$) of the Bianchi types $\{\mathrm{I}, \mathrm{VII}_0\}$ and IX respectively.

We take as our starting theory the bosonic sector of the effective Euclidean action of the heterotic string in critical (ten) dimensions to zero-order in the inverse string tension [3]. After compactification onto a six-torus the dimensionally reduced, four-dimensional action is equivalent to a scalar-tensor theory if the field strength of the antisymmetric tensor vanishes. In this case the simplest form of the WDW equation is derived by performing a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ on the 4-metric in such a way that the action is rewritten as Einstein gravity minimally coupled to a set of massless scalar fields. If $\Omega^2 = e^{-\phi}$, where ϕ is the shifted dilaton, the transformed Euclidean action becomes

$$S = \int d^4x\sqrt{\tilde{g}}\left\{-\tilde{R} + \frac{1}{2}\left(\tilde{\nabla}\phi\right)^2 + \frac{1}{2}\sum_{j=1}^{3}\left[\left(\tilde{\nabla}\psi_j\right)^2 + e^{-2\psi_j}\left(\tilde{\nabla}\sigma_j\right)^2\right]\right\}, \tag{3}$$

where $\tilde{g} \equiv \det\tilde{g}_{\mu\nu}$, $\{\psi_j, \sigma_j\}$ are scalar fields arising from the compactification and we choose units such that $\hbar = c = 16\pi m_p^{-2} \equiv 1$ in the conformal frame [9]. If we further assume that the dilaton is constant on the surfaces of homogeneity, the transformed world-interval is $d\tilde{s}^2 = d\eta^2 + \tilde{h}_{ab}\omega^a\omega^b$, where

$$\tilde{h}_{ab} = e^{2\tilde{\alpha}}\left(e^{2\beta}\right)_{ab} = e^{-\phi+2\alpha}\left(e^{2\beta}\right)_{ab} \tag{4}$$

is the rescaled 3-metric, $\eta \equiv \int dt\Omega(t)$ and $\tilde{\alpha} \equiv \alpha + \ln\Omega$. Since G_3 is time-independent, a given Bianchi type is symmetric under the action of this conformal transformation and we may therefore consider solutions directly in the conformal frame.

Theory (3) has ten degrees of freedom $q^\mu = (\tilde{\alpha}, \beta_\pm, \phi, \psi_j, \sigma_j)$ with conjugate momenta $p_\mu = \partial S/\partial\dot{q}^\mu$. It is quantized by identifying these momenta with the operators $p_\mu = -i\partial/\partial q^\mu$ and viewing the classical Hamiltonian constraint as a time-independent Schrödinger equation that annihilates the state vector $\Psi(q^\mu)$ for the Universe. It is given by [6]

$$\begin{aligned}\Bigg[e^{-p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}}e^{p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + U\left(\tilde{\alpha}, \beta_\pm\right) - 12\frac{\partial^2}{\partial\phi^2} \\ -12\sum_{j=1}^{3}\left(\frac{\partial^2}{\partial\psi_j^2} + e^{2\psi_j}\frac{\partial^2}{\partial\sigma_j^2}\right)\Bigg]\Psi = 0,\end{aligned} \tag{5}$$

where the superpotential

$$U = \frac{1}{3}\left[(n_{33}\tilde{h}_{33})^2 + (n_{11}\tilde{h}_{11} - n_{22}\tilde{h}_{22})^2 - 2n_{33}\tilde{h}_{33}(n_{11}\tilde{h}_{11} + n_{22}\tilde{h}_{22})\right] \tag{6}$$

is determined by the structure constants of the Lie algebra and the constant p accounts for ambiguities in the operator ordering. The non-zero eigenvalues of n_{ab} are summarized in Table 1.

Type	n^{11}	n^{22}	n^{33}	$\lvert S\rvert$
I	0	0	0	0
II	0	0	1	$\frac{1}{6}e^{2\tilde{\alpha}-4\beta_+}$
VI_0	-1	1	0	$\frac{1}{3}e^{2\tilde{\alpha}+2\beta_+}\sinh 2\sqrt{3}\beta_-$
VII_0	1	1	0	$\frac{1}{3}e^{2\tilde{\alpha}+2\beta_+}\cosh 2\sqrt{3}\beta_-$
VIII	1	1	-1	$\frac{1}{6}e^{2\tilde{\alpha}}\left[-e^{-4\beta_+}+2e^{2\beta_+}\cosh 2\sqrt{3}\beta_-\right]$
IX	1	1	1	$\frac{1}{6}e^{2\tilde{\alpha}}\left[e^{-4\beta_+}+2e^{2\beta_+}\cosh 2\sqrt{3}\beta_-\right]$

Table 1: The non-zero eigenvalues of n^{ab} for the Bianchi class A are shown for each Bianchi type. This matrix has the same rank and signature for a given Bianchi type as the matrix determining the structure constants of the Lie algebra of G_3. The function $S = \pm\frac{1}{6}n^{ab}\tilde{h}_{ab}$, where summation over upper and lower indices is implied, is a solution of the Euclidean Hamilton-Jacobi equation (7). It is interpreted physically as the conformally transformed Euclidean action for the classical vacuum theory. The reader is referred to the text for details.

Although solving this equation appears to be a formidable task, one gains valuable insight from the Hamiltonian $\mathcal{H}_g$ of the vacuum theory. At the classical level the Universe may be thought of as a zero-energy point particle moving in a time-dependent potential well with $0 = \mathcal{H}_g \propto G^{\lambda\kappa}p_\lambda p_\kappa + U(q^\lambda)$, where $G^{\lambda\kappa}$ is the $(2+1)$-Minkowski space-time metric, $\tilde{\alpha}$ plays the role of 'time' and $\{\lambda,\kappa\}$ represent the $\tilde{\alpha},\beta_\pm$ minisuperspace coordinates. This implies that the vacuum Bianchi A minisuperspace can be supersymmetrized [10] by solving the Euclidean Hamilton-Jacobi equation

$$U = G^{\lambda\kappa}\frac{\partial S}{\partial q^\lambda}\frac{\partial S}{\partial q^\kappa} \tag{7}$$

and introducing the fermionic variables $\chi^\kappa, \bar{\chi}^\lambda$ defined by the spinor algebra

$$\chi^\kappa\chi^\lambda + \chi^\lambda\chi^\kappa = \bar{\chi}^\kappa\bar{\chi}^\lambda + \bar{\chi}^\lambda\bar{\chi}^\kappa = \bar{\chi}^\kappa\chi^\lambda + \chi^\kappa\bar{\chi}^\lambda - G^{\kappa\lambda} = 0. \tag{8}$$

The Hamiltonian is then equivalent to $\mathcal{H}_g \propto Q\bar{Q} + \bar{Q}Q$, where the supercharges

$$Q = \chi^\kappa\left(p_\kappa + i\frac{\partial S}{\partial q^\kappa}\right), \qquad \bar{Q} = \bar{\chi}^\kappa\left(p_\kappa - i\frac{\partial S}{\partial q^\kappa}\right) \tag{9}$$

satisfy $Q^2 = 0 = \bar{Q}^2$, and after quantization one obtains the 'square roots' of the WDW equation, i.e. $Q\Psi = 0 = \bar{Q}\Psi$. It follows that a solution to the standard WDW equation may be found by solving these square roots and restricting one's attention to the bosonic sector of the wavefunction. It is straightforward to show that, modulo a constant of proportionality, the bosonic component of the wavefunction annihilated by the supersymmetric quantum constraints is [10]

$$\Psi_{\text{bosonic}} = e^{-S}. \tag{10}$$

We find that a unified solution to Eq. (7) is

$$S = \pm\frac{1}{6}n^{ab}\tilde{h}_{ab} \tag{11}$$

where summation over upper and lower indices is implied and the full expressions for each Bianchi type are presented in Table 1. The elegance of (10) and (11) motivates us to separate the matter and gravitational sectors of the WDW equation (5) with the ansatz $\Psi = X(\tilde{\alpha}, \beta_\pm)Y(\phi, \psi_j, \sigma_j)$, where $X = W(\tilde{\alpha})e^{-S(\tilde{\alpha},\beta_\pm)}$. This implies that

$$\left[\frac{\partial^2}{\partial\tilde{\alpha}^2} + p\frac{\partial}{\partial\tilde{\alpha}} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + U - z^2\right] X = 0 \tag{12}$$

and

$$\left[\frac{\partial^2}{\partial\phi^2} + \sum_{j=1}^{3}\left(\frac{\partial^2}{\partial\psi_j^2} + e^{2\psi_j}\frac{\partial^2}{\partial\sigma_j^2}\right) - \frac{z^2}{12}\right] Y = 0, \tag{13}$$

where z is an arbitrary separation constant that can be interpreted physically as the total momentum eigenvalue of the matter sector. This separation is possible for any theory that is equivalent at the classical level to Einstein gravity minimally coupled to a stiff perfect fluid.

It follows from Eq. (11) that $\partial S/\partial\tilde{\alpha} = 2S$ and $G^{\kappa\lambda}\partial^2 S/\partial q^\kappa \partial q^\lambda = 12S$ for all Bianchi types. We therefore deduce with the help of these identities that

$$X = e^{(3-p/2)\tilde{\alpha}-S} \tag{14}$$

is a solution to Eq. (12) provided we choose $p^2 = 4(9 - z^2)$. It only remains to solve Eq. (13) and this is achieved with the separable ansatz

$$Y = A_1(\psi_1)A_2(\psi_2)A_3(\psi_3)e^{\pm i\gamma\phi \pm i(\omega_1\sigma_1 + \omega_2\sigma_2 + \omega_3\sigma_3)} \tag{15}$$

where A_j satisfy the wave equation of Liouville quantum mechanics and $\{\gamma, \omega_j\}$ are separation constants [6]. A reduction to the Bessel equation follows after the change of variables $\psi_j = \ln \xi_j$ and this yields the general solutions $A_j = Z_{\pm\Lambda\cos\theta_j}(\omega_j e^{\psi_j})$, where Z denotes some linear combination of modified Bessel functions of the first and second kinds, $\Lambda^2 = \gamma^2 + z^2/12$ and the constants θ_j are solutions to the constraint equation $\sum_{j=1}^3 \cos^2\theta_j = 1$.

Finally the solution in the original frame follows from Eq. (4). We therefore arrive at the unified set of solutions

$$\Psi = \exp\left[\left(3-\frac{p}{2}\right)\alpha - \frac{1}{6}e^{-\phi}\left|n^{ab}h_{ab}\right| + \left(\frac{1}{4}(p-6) \pm i\gamma\right)\phi\right] \times \prod_{j=1}^{3}\left[Z_{\pm\Lambda\cos\theta_j}\left(\omega_j e^{\psi_j}\right)e^{\pm i\omega_j\sigma_j}\right]. \qquad (16)$$

These solutions cannot be interpreted as Lorentzian four-geometries because the wavefunction is Euclidean for all values of the scale factor. On the other hand, they remain regular, in the sense that they do not oscillate an infinite number of times, when the spatial metric degenerates ($\alpha = -\infty$) and, with the exception of the type I solution, they are exponentially damped at large α. Consequently they satisfy the Hawking-Page boundary conditions and may therefore be interpreted as quantum wormhole solutions [11].

The function W behaves as a variable amplitude for the gravitational component of the wavefunction. The type VI_0 and VII_0 solutions may be generalized in such a way that this amplitude becomes a function of both the $\tilde{\alpha}$ and β_+ variables. We find that a second solution to Eq. (12) is

$$X = \exp\left[\frac{1}{24}\left((p^2 - 12p + 4z^2 + 36)\alpha + (p^2 - 36 + 4z^2)\beta_+\right) - S\right] \qquad (17)$$

and these solutions do not depend on a specific choice of factor ordering.

The problem of extracting physical predictions in quantum cosmology from the wavefunction of the Universe is an unresolved one. However, it is reasonable to suppose that a strong peak in the wavefunction represents a prediction in some sense. This is the case, for example, if one adopts the Hartle-Hawking proposal and interprets $|\Psi|^2$ as an unnormalized probability density [12]. When the three-surface degenerates, S becomes vanishingly small for all Bianchi types, and the wavefunction exhibits no peak in the (β_+, β_-) plane. As shown in Figure 1, however, the Bianchi IX wavefunction becomes strongly localized around the isotropic solution $\beta_+ = \beta_- = 0$ as the scale factor increases. This suggests that there is a progressively higher probability of finding this Universe in the isotropic state as α increases [13]. Unfortunately the same is not true for the Bianchi type VII_0. We see from Figure 2 that the wavefunction is indeed peaked around $\beta_- = 0$, but there is no local maximum for finite β_+. Consequently, it is not clear whether this solution selects the spatially flat Friedmann Universe.

In conclusion we have found a unified exact solution to the WDW equation for the Bianchi class A minisuperspace with a matter sector motivated by the string effective action. If such a solution is to have any direct cosmological relevance, it is necessary to find a mechanism that leads to the emergence of the classical domain. In principle, such a domain could be reached if one or more of the scalar fields were to acquire an effective mass once the scale factor became sufficiently large. In the type IX example the Universe would then tunnel from the Euclidean regime into a highly isotropic Lorentzian state and this may point towards a possible non-inflationary resolution of the isotropy problem.

Acknowledgments The author is supported by a Science and Engineering Research Council (SERC) UK postdoctoral research fellowship and is supported at Fermilab by the DOE and NASA under Grant NAGW-2381.

References

[1] G. F. Smoot *et al.*, *Astrophys. J. Lett.* **396**, L1 (1992).

[2] G. W. Gibbons and S. W. Hawking, *Phys. Rev.* **D15**, 2738 (1977).

[3] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1988).

[4] B. S. DeWitt, *Phys. Rev.* **160**, 1113 (1967); J. A. Wheeler, *Battelle Rencontres* (Benjamin, New York, 1968).

[5] K. Enqvist, S. Mohanty and D. V. Nanopoulos, *Phys. Lett.* **B192**, 327 (1987); D. Birmingham and C. Torre, *Phys. Lett.* **B194**, 49 (1987); P. F. Gonzalez-Diaz, *Phys. Rev.* **D38**, 2951 (1988); H. Luckock, I. G. Moss and D. Toms, *Nucl. Phys.* **B297**, 748 (1988); K. Enqvist, S. Mohanty and D. V. Nanopoulos, *Int. J. Mod. Phys.* **A4**, 873 (1989); M. D. Pollock, *Nucl. Phys.* **B315**, 528 1989; M. D. Pollock, *Nucl. Phys.* **B324**, 187 (1989); M. D. Pollock, *Int. J. Mod. Phys.* **A7**, 4149 (1992); J. Wang, *Phys. Rev.* **D45**, 412 (1992); S. Capozziello and R. de Ritis, *Int. J. Mod. Phys.* **D2**, 373 (1993).

[6] J. E. Lidsey, *Phys. Rev.* **D49**, R599 (1994); In press, *Class. Quantum Grav.* (1994).

[7] G. F. R. Ellis and M. A. H. MacCallum, *Commun. Math. Phys.* **12**, 108 (1969).

[8] A. Ashtekar and J. Samuel, *Class. Quantum Grav.* **8**, 2191 (1991); Y. Fujiwara, H. Ishihara and H. Kodama, *Class. Quantum Grav.* **10**, 859 (1993).

[9] For details of this derivation see J. E. Lidsey, *Class. Quantum Grav.* **9**, 149 (1992); S. P. Khastgir and J. Maharana, *Phys. Lett.* **B301**, 191 (1993); J. Maharana and J. H. Schwarz, *Nucl. Phys.* **B390**, 3 (1993).

[10] R. Graham, *Phys. Rev. Lett.* **67**, 1381 (1991); O. Obregón, J. Socorro, and J. Benítez, *Phys. Rev.* **D47**, 4471 (1993); P. D. D'Eath, S. W. Hawking, and O. Obregón, *Phys. Lett.* **B302**, 183 (1993); M. Asano, M. Tanimoto and N. Yoshino, *Phys. Lett.* **B314**, 308 (1993); O. Obregón, J. Pullin and M. P. Ryan, *Phys. Rev.* **D48**, 5642 (1993); J. Bene and R. Graham, *Phys. Rev.* **D49**, 799 (1994); R. Capovilla and J. Guven, Preprint gr-qc/9402025, (1994).

[11] S. W. Hawking and D. N. Page, *Phys. Rev.* **D42**, 2655 (1990).

[12] J. Hartle and S. W. Hawking, *Phys. Rev.* **D28**, 2960 (1983).

[13] V. Moncrief and M. P. Ryan, *Phys. Rev.* **D44**, 2375 (1991).

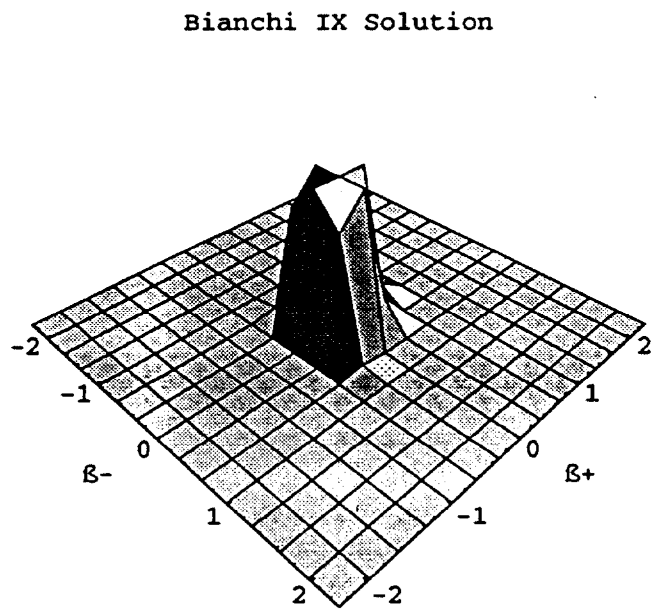

Figure 1: The gravitational component of the Bianchi type IX wavefunction (15) is plotted when $\tilde{\alpha} = 0$. The wavefunction is localized around the point $\beta_+ = \beta_- = 0$ in the (β_+, β_-) plane. This point corresponds to the isotropic Friedmann solution. The peaked becomes more pronounced as the scale factor increases.

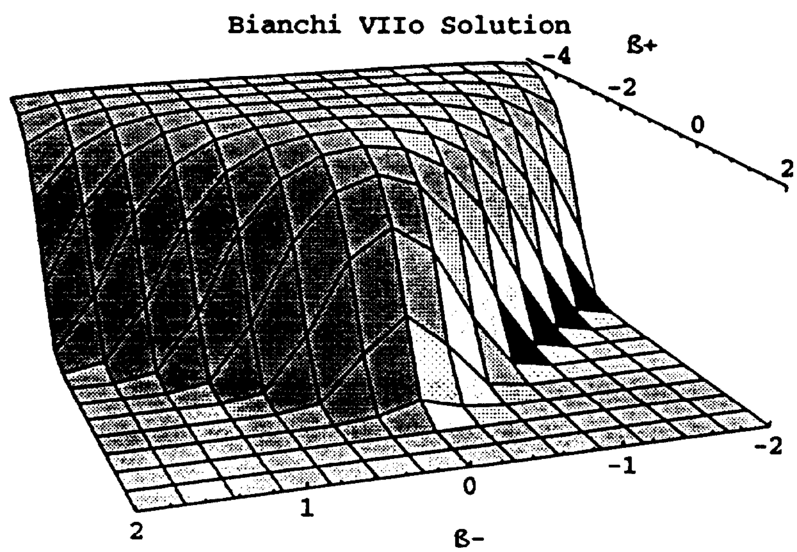

Figure 2: The equivalent solution to Figure 1 for the Bianchi type VII_0 cosmology. The wavefunction is peaked around $\beta_- = 0$, but there is no local maximum for finite β_+.