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LARGE-ORDER BEHAVIOR OF PERTURBATION THEORY: ON ITS ENERGY DEPENDENCE*

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ABSTRACT

We determine the nature and position of the Borel singularity of instanton-induced amplitudes in theories having explicit dimensional parameters. A strong energy dependence of the strength of the Borel singularity is found. Using the Borel transform, we show that the large-order behavior of the Green's functions has generally nontrivial energy dependence. Specifically, we calculate the large-order behavior of forward scattering amplitudes in several models, including the standard model of electroweak interactions.

1. Introduction

In this talk, we discuss nontrivial energy (momentum) dependence of large-order behavior (LOB) of perturbative series. In the ϕ^4 theory, the large-order behavior of the r-point Green's function $G^{(r)}(p_1, \cdots, p_r)$ takes in general the form[1]

$$\left\{G^{(r)}(p_1, \cdots, p_r)\right\}_n \sim C_r(p_1, \cdots, p_r)\, n!\, n^{\nu-1} R^n \quad \text{for} \quad n \to \infty, \tag{1}$$

where ν and R are constants and C_r is independent of n. The momentum-dependence of the large-order behavior is factored out, and thus trivial. Nontrivial momentum-dependence of LOB could appear in theories having instanton solutions. For example, we show that in the quantum mechanical double-well potential problem, the LOB of the two-point, off-shell Green's function is given by

$$\{G(w, -w)\}_n \sim C(w)\, n!\, n^{(\nu-1+\epsilon)} \left(\frac{1}{2S_0}\right)^n \quad \text{for} \quad n \to \infty. \tag{2}$$

Here $C(w)$ is a function of the energy w, S_0 is the instanton action, and $\epsilon = w/m$, where m is the mass parameter of the theory. Note that the energy dependence in (2) is no longer factored out. In general, we find that the necessary condition for nontrivial energy-dependence of LOB is that the classical fields that determine the Borel singularity must have quasi-zeromodes coupled to the external momenta.

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2. Borel singularity

We consider a theory having a dimensionless coupling g and mass m. An extension to theories hav... more than one mass parameters should be straightforward. The Lagrangian is

$$\mathcal{L} = \frac{1}{g} L(\phi, m) \tag{3}$$

with ϕ representing generic fields. We assume that m is independent of g, and g can be always factored out of L so that L is independent of g. We also assume that there is an instanton solution with the action

$$S_I = \frac{S_0}{g} \tag{4}$$

where S_0 is a constant. Let us consider the perturbative expansion in the coupling g of the two-body forward scattering amplitudes

$$A(-p_1, -p_2 : p_1, p_2, g) = \sum_{n=0}^{\infty} a_n\, g^n. \tag{5}$$

As shown by Crutchfield II,[2] the coefficients of the perturbative series can be determined by the Borel transform, $\tilde{\sigma}(b)$, defined by

$$\tilde{\sigma}(b) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zb}\, \sigma(g)\, d\, z \tag{6}$$

where $z = 2S_0/g$, and $\sigma(g)$ is the forward scattering amplitude in the instanton-anti-instanton background. The Borel transform (6) is defined such that the instanton-anti-instanton singularity of the vacuum tunneling amplitudes is located at $b = 1$. Expanding $\tilde{\sigma}(b)$ at the origin, $b = 0$,

$$\tilde{\sigma}(b) = \sum_{n=0}^{\infty} c_n\, b^n, \tag{7}$$

we can obtain the coefficients a_n by

$$a_n = (n-1)!\, c_{n-1} \left(\frac{1}{2S_0}\right)^n. \tag{8}$$

As will become clear, the leading singularity of the Borel transform arises from the Espinosa-Ringwald type cross section[3] which is an imaginary part of $\sigma(g)$. The Espinosa-Ringwald type cross section is generally given in the form

$$\sigma(g) \sim \left(\frac{1}{g}\right)^{\nu} \exp\left(-\frac{2S_0}{g}\mathcal{F}(x)\right) \sim z^{\nu} \exp\left(-z\mathcal{F}(x)\right), \tag{9}$$

and

$$\mathcal{F}(x) = 1 - U(x) \tag{10}$$

with ν a model-dependent constant, $x = \epsilon/z$, and $\epsilon = E/m$ where E is the c.m. energy.[4] $U(x)$ can be calculated perturbatively in x by doing perturbation about the instanton background, and its first few terms are known in several models.[4-6] In the double-well potential and the two-dimensional abelian Higgs model,

$$U(x) = -\frac{1}{l} x \ln(x) - \frac{k}{l} x \ln\left(\ln \frac{1}{x}\right) + O(x), \tag{11}$$

with $l = 2, k = 0$ and $l = 1, k = 1/2$ respectively. For the three-dimensional nonabelian Higgs model,

$$U(x) = 2c\sqrt{x} + \cdots, \tag{12}$$

where c is the 't Hooft-Polyakov magnetic monopole charge, and for the standard model of electroweak interactions

$$U(x) = \frac{1}{2}(3x)^{\frac{4}{3}} + \cdots. \tag{13}$$

Substituting (9) into (6), we have

$$\tilde{\sigma}(b) \sim \frac{1}{2\pi} \int_{a-i\infty}^{a+i\infty} \exp\left[z\left(b - 1 + U(x)\right) + \nu \ln z\right] dz, \tag{14}$$

which may be evaluated by the saddle point approximation. The equation for the saddle point $\bar{z}$, or equivalently $\bar{x} = \epsilon/\bar{z}$ is

$$1 - b = \frac{\nu}{\epsilon}\bar{x} + \left(1 - \bar{x}\frac{\partial}{\partial \bar{x}}\right) U(\bar{x}), \tag{15}$$

and the Borel transform is given by

$$\tilde{\sigma}(b) \sim \frac{1}{\bar{x}\sqrt{1 - \frac{\epsilon}{\nu}\bar{x}U''(\bar{x})}} \exp\left[\bar{z}\left(b - 1 + U(\bar{x})\right) + \nu \ln \bar{z}\right]. \tag{16}$$

Substituting (11),(12), and (13) into (15), it is easy to check that the Borel singularity occurs at $\bar{x} = 0$, or equivalently at $b = 1$. Solving (15) in series of $(1 - b)$, we find the leading singular behavior of the Borel transform for the above models,[6]

$$\begin{aligned}
\tilde{\sigma}(b)_{2D} &\sim (1-b)^{-(\nu+1+\epsilon/l)} \left(\ln \frac{1}{(1-b)}\right)^{-\epsilon k/l}, \\
\tilde{\sigma}(b)_{3D} &\sim (1-b)^{-(2\nu+3/2)} \exp\left(\frac{\epsilon c^2}{1-b}\right), \\
\tilde{\sigma}(b)_{SM} &\sim (1-b)^{-(\nu+1)}, \qquad \text{for} \quad b \to 1.
\end{aligned} \tag{17}$$

Note that the strengths of the Borel singularities in the double-well potential and the three-dimensional nonabelian Higgs model depend strongly on the energy.

3. Large-order behavior

Expanding the Borel transform in (17) around $b = 0$, we can find the large-order behavior of the forward scattering amplitudes. The coefficients c_n can be determined by

$$\begin{aligned} c_{n-1} &= \frac{1}{2\pi i}\int_C \frac{\tilde{\sigma}(b)}{b^n}\,db \\ &= \frac{1}{2\pi i}\int_C \tilde{\sigma}(b)e^{-n\ln b}\,db \end{aligned} \tag{18}$$

where C is a path encircling the origin in the complex b-plane. With $\tilde{\sigma}(b)$ given in (17), equation (18) may be evaluated by the saddle point method for large n. It is easy to check that the saddle point for the double-well potential and the two-dimensional abelian Higgs model is given by[7]

$$\bar{b} = \frac{n}{n+\nu+1+\epsilon/l}\left(1 + k\cdot O\left(\frac{1}{n\ln n}\right)\right), \tag{19}$$

and the perturbative coefficients have the asymptotic form

$$a_n \to C'(E)\,n!\,n^{(\nu-1+\epsilon/l)}\,(\ln n)^{-\epsilon k/l}\left(\frac{1}{2S_0}\right)^n \quad \text{for} \quad n\to\infty, \tag{20}$$

where $C'(E)$ is a function of the energy. A similar calculation for the nonabelian Higgs model in three dimensions gives,

$$\bar{b} = 1 - c\sqrt{\frac{\epsilon}{n}}\left(1 + O\left(\frac{1}{\sqrt{\epsilon n}}\right)\right), \tag{21}$$

and

$$a_n \to C''(E)\,n!\,n^{(\nu-1)}\,e^{2c\sqrt{\epsilon n}}\left(\frac{1}{2S_0}\right)^n \quad \text{for} \quad n\to\infty, \tag{22}$$

where $C''(E)$ is a function of the energy. Note that

$$\bar{b} \to 1 \quad \text{for} \quad n\to\infty \tag{23}$$

in (19) and (21), which implies that the main contribution to the integral in (18) comes from the region near the singularity, where the Borel transforms in (17) are valid. For the standard model, the LOB of the forward scattering amplitudes has the identical form as that of the vacuum amplitude.

4. Lipatov method

In previous section, we have derived the nontrivial energy-dependence of LOB in several models using the Borel transform. Those results may be checked using the Lipatov method[8] of calculating the large-order behavior. For simplicity, we consider the two-point, off-shell Green's function in the double-well potential

$$G(w,-w,g) = \int D\phi\, e^{-S(\phi)}\tilde{\phi}(w)\tilde{\phi}(-w). \tag{24}$$

Scaling the field

$$\phi \to \frac{1}{g}\,\phi, \tag{25}$$

and writing ϕ as a sum of the valley configuration[9] ϕ_{val} of instanton-antiinstanton pairs and the fluctuations η about the valley,

$$G(w,-w,g) \sim \left(\frac{1}{g}\right)^{\nu} \int D\eta\, dR\, e^{-\frac{1}{g}S(\phi_{val}(R)+\eta)}(\tilde{\phi}_{val}(w)+\tilde{\eta}(w))(\tilde{\phi}_{val}(-w)+\tilde{\eta}(-w)), \tag{26}$$

where R is the distance between the instanton-anti-instanton pairs. Writing

$$G(w,-w,g) \equiv \sum a_n(w)\, g^n, \tag{27}$$

and naively applying the residue theorem, we have

$$\begin{aligned} a_n(w) &= \frac{1}{2\pi i}\oint dg \frac{G(w,-w,g)}{g^{n+1}} \\ &\sim \int dR\, dg\, \exp\left[-\frac{1}{g}S_{val}(R) - (n+\nu+1)\ln g\right]\, \tilde{\phi}_{val}(w)\tilde{\phi}_{val}(-w). \end{aligned} \tag{28}$$

In the last step, we have integrated out the fluctuations by the Gaussian approximation, and

$$S_{val}(R) = S(\phi_{val}(R)) \equiv 2S_0\,[1 - V(mR)] \tag{29}$$

is the valley action. For large R,

$$\tilde{\phi}_{val}(w) \approx \tilde{\phi}_{\bar{I}}(w)\, e^{iwR/2} + \tilde{\phi}_I(w)\, e^{-iwR/2}, \tag{30}$$

where $\tilde{\phi}_{\bar{I},I}$ are antiinstanton and instanton configurations respectively. Analytic continuing the Green's function to Minkowski space by $w \to -iw$, we have

$$\begin{aligned} a_n(w) &\sim \int dR\, dg \exp\left[-\frac{1}{g}S_{val}(R) - (n+\nu+1)\ln g\right] \times \\ &\qquad \left\{\tilde{\phi}_{\bar{I}}(w)\tilde{\phi}_{\bar{I}}(-w) + \tilde{\phi}_{\bar{I}}(w)\tilde{\phi}_I(-w)e^{wR} + \tilde{\phi}_{\bar{I}}(-w)\tilde{\phi}_I(w)e^{-wR} + \tilde{\phi}_I(w)\tilde{\phi}_I(-w)\right\} \\ &\sim \int dR\, dg\, \exp\left[wR - \frac{2S_0}{g}(1 - V(mR)) - (n+\nu+1)\ln g\right], \end{aligned} \tag{31}$$

since for large n, main contribution to the integral comes from the region $R \to \infty$. Now Eq. (31) may be evaluated by the saddle point approximation to obtain

$$a_n(w) \sim \frac{1}{\sqrt{F''}} e^{F(R^*,g^*)}, \tag{32}$$

where

$$F(R,g) = wR - \frac{2S_0}{g}(1 - V(mR)) - (n+\nu+1)\ln g, \tag{33}$$

and

$$F'' = \left| \begin{pmatrix} \frac{\partial^2 F}{\partial g^2} & \frac{\partial^2 F}{\partial g \partial R} \\ \frac{\partial^2 F}{\partial g \partial R} & \frac{\partial^2 F}{\partial R^2} \end{pmatrix} \right| . \tag{34}$$

The saddle points are determined by

$$\begin{aligned} w + \frac{2S_0}{g} V'(R) \Big|_{\{R=R^*, g=g^*\}} &= 0 \\ \frac{2S_0}{g^*}(1 - V(R^*)) &= n + \nu + 1. \end{aligned} \tag{35}$$

For the double-well potential, the potential energy of instanton-antiinstanton pairs is given by

$$V(mR) \sim e^{-2mR}. \tag{36}$$

With this potential, it is easy to work out (34), (35), to obtain

$$a_n(w) \to n!\, n^{(\nu-1+w/m)} \left(\frac{1}{2S_0}\right)^n , \tag{37}$$

which is exactly the result obtained from the Borel transform method.

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