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# Bianchi-IX Quantum Cosmology of the Heterotic String

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**Abstract**

The dimensionally reduced effective action of the bosonic sector of the heterotic string in critical dimensions is employed to derive a Wheeler-DeWitt equation for the Bianchi-IX cosmology. An exact solution is found that becomes strongly peaked around the isotropic limit as the volume of the three-geometry increases. In principle the global $O(6,6)$ symmetry of the effective action can be employed to generate new solutions from the one presented here.

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The heterotic string is a candidate for a consistent theory of quantum gravity [1], but a complete quantum field theory of the string does not yet exist. One can gain insight into the full theory, however, by considering the quantum cosmology of the point-like effective action [2]. This field theory limit is most naturally formulated in a higher-dimensional space-time and the question arises as to why only three large spatial dimensions are observed at the present epoch. This has been addressed by Pollock, who claims that physical space-time must be four-dimensional if the complete universe is ten-dimensional [3]. A second problem is why physical space appears highly isotropic and homogeneous on sufficiently large scales [4]. Presumably, if string theory provides a complete description of nature, this observed isotropy should arise naturally within the formalism. Perhaps the quantum cosmology of the field theory can provide some insight?

Motivated by this question we examine the Wheeler-DeWitt (WD) equation [5] derived from the bosonic sector of the heterotic string in critical (ten) dimensions to lowest order in the string tension. Our starting point is the four-dimensional effective action recently derived from this theory by Maharana and Schwarz [6], who employed the Scherk-Schwarz [7] dimensional reduction technique. In this paper the physical universe is assumed to be a Bianchi type IX cosmology. The spatially closed Friedmann-Robertson-Walker (FRW) universe is recovered in the isotropic limit of this model. If a number of reasonable assumptions are made, an exact solution can be found after a suitable conformal transformation on the metric. The wavefunction becomes increasingly peaked around the isotropic FRW cosmology as the volume of the physical space increases.

We briefly review the derivation of the dimensionally reduced effective action presented in Ref. [8]. As a first approximation to a Calabi-Yau space, it is consistent to treat the internal space as a six-torus if the background fields are independent of its coordinates $y^\alpha$, $\alpha = 1, 2 \dots, 6$. The ten-dimensional metric can then be written:

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \left( \begin{array}{cc} g_{\mu\nu} + A^{(1)\gamma}_{\mu} A^{(1)}_{\nu\gamma} & A^{(1)}_{\mu\beta} \\ A^{(1)}_{\nu\alpha} & G_{\alpha\beta} \end{array} \right), \tag{1}$$

where $G_{\alpha\beta}$ is the internal metric and $g_{\mu\nu}(x^\rho)$ is the four-dimensional space-time metric.[1] The effective Euclidean action in four dimensions is

$$S = \int d^4x \sqrt{g} e^{-\phi} \left[ -R - (\nabla\phi)^2 + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{8} \mathrm{Tr} \left( \nabla_\mu M^{-1} \nabla^\mu M \right) \right], \tag{2}$$

where $A^{(2)}_{\mu\alpha} = \hat{B}_{\mu\alpha} + B_{\alpha\beta} A^{(1)\beta}_{\mu}$, $\hat{B}_{\alpha\beta} = B_{\alpha\beta}$, $\phi = \hat{\phi} - \frac{1}{2} \ln \det G$ is the shifted dilaton, $H_{\mu\nu\rho} = \nabla_{[\mu} B_{\nu\rho]}$, and all vector fields $A^i_\mu$ are set to zero. The $6 \times 6$ matrix $M$ is defined:

$$M = \left( \begin{array}{cc} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{array} \right), \qquad M^{-1} = \eta M \eta, \qquad \eta = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right). \tag{3}$$

[1] A hat ˆ denotes quantities in the ten-dimensional space-time. The reader is referred to [6] and [8] for a full treatment.

This action is invariant under a global $O(6,6)$ transformation and a special case of this symmetry is the target space duality transformation associated with an interchange of $M$ and $M^{-1}$.

Assuming the spherically symmetric ansatz allows the $G$ and $B$ matrices to be written in block form:

$$G + B = \mathrm{diag}(\Sigma_1, \Sigma_2, \Sigma_3), \qquad \Sigma_j \equiv \begin{pmatrix} e^{\psi_j} & \sigma_j \\ -\sigma_j & e^{\psi_j} \end{pmatrix}. \tag{4}$$

The action (2) simplifies to

$$S = \int d^4x \sqrt{g} e^{-\phi} \left( -R - (\nabla\phi)^2 + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{2} \sum_{j=1}^{3} \left[ (\nabla\psi_j)^2 + e^{-2\psi_j} (\nabla\sigma_j)^2 \right] \right) \tag{5}$$

and the duality transformed fields are [8]

$$e^{-\bar{\psi}_j} = e^{\psi_j} + e^{-\psi_j}\sigma_j^2, \qquad \bar{\sigma}_j = -\left( e^{\psi_j} + e^{-\psi_j}\sigma_j^2 \right)^{-1} e^{-\psi_j}\sigma_j. \tag{6}$$

The decomposition (4) implies that the action (5) is also invariant under the SL(2,R) transformation [8].

The time-dependence of the effective gravitational constant in this theory, as measured by the evolution of the rescaled dilaton field, may be removed after a suitable conformal transformation on the metric $g_{\mu\nu}$ [9]:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \qquad \Omega^2 = e^{-\phi}. \tag{7}$$

In the conformal (Einstein-Hilbert) frame, the Planck mass is constant but the gauge and Yukawa couplings are dynamical variables [10]. The conformally transformed action is

$$\begin{aligned} S = \int d^4x \sqrt{\tilde{g}} \Big\{ -\tilde{R} + \frac{1}{2}\left(\tilde{\nabla}\phi\right)^2 + \frac{1}{12} e^{-2\phi} \tilde{H}_{\mu\nu\rho}\tilde{H}^{\mu\nu\rho} \\ + \frac{1}{2}\sum_{j=1}^{3}\left[ \left(\tilde{\nabla}\psi_j\right)^2 + e^{-2\psi_j}\left(\tilde{\nabla}\sigma_j\right)^2 \right] \Big\}, \end{aligned} \tag{8}$$

where tildes denote quantities in the conformal frame and units are chosen in that frame such that the Planck mass is defined by $\hbar = c = 16\pi m_p^{-2} \equiv 1$.

We now assume that the contribution to the action of the totally antisymmetric three-index field $H_{\mu\nu\rho}$ may be ignored. It is shown at the end of the calculation that this is a consistent assumption. At the classical level the remaining matter sector of the theory behaves as a stiff perfect fluid in which the speed of sound is unity.

Our main interest in this work is the quantum behaviour of the spatially homogeneous, diagonal Bianchi-IX universe. This is an example of a class A Bianchi space

[11]. It admits a group $G$ of isometrics transitive on space-like three-dimensional orbits. The Euclidean metric is:

$$ds^2 = dt^2 + e^{2\alpha(t)}\left(e^{2\beta(t)}\right)_{ij}\epsilon^i\epsilon^j, \tag{9}$$

where $e^{6\alpha}$ is the determinant of the metric on the surfaces $t$ = constant, $\beta_{ij}$ is a $3\times 3$ symmetric, traceless matrix, $\epsilon^i$ are three one-forms in the orbits and obey $d\epsilon^i = C_{ijk}\epsilon^j\wedge\epsilon^k$, and the structure constants $C_{ijk}$ define the Lie algebra of $G$. $G$ does not depend on time $t$ and the conformal transformation is only a function of $t$ if the dilaton is constant on the spaces of homogeneity. Hence (7) does not alter the structure of $G$ and this implies that a given Bianchi type is invariant under (7). In particular, if $g_{\mu\nu}$ is taken to be the diagonal Bianchi-IX, then the conformal metric $\tilde{g}_{\mu\nu}$ will also be a diagonal Bianchi-IX with world-interval:

$$d\tilde{s}^2 = d\eta^2 + e^{2\tilde{\alpha}}\left(e^{2\beta}\right)_{ij}\epsilon^i\epsilon^j, \tag{10}$$

where

$$\eta \equiv \int dt\Omega(t), \qquad e^{\tilde{\alpha}} = \Omega e^{\alpha}. \tag{11}$$

Without loss of generality $\beta$ may be diagonalized and parametrized in terms of two independent components [12]:

$$\beta_{ij} = \mathrm{diag}\left[\beta_+(t)+\sqrt{3}\beta_-(t), \beta_+(t)-\sqrt{3}\beta_-(t), -2\beta_+(t)\right]. \tag{12}$$

Therefore the quantum cosmology of theory (5) may be investigated in the conformal frame. This allows us to apply the well known results from Einstein gravity.

There are ten degrees of freedom $q^i = (\tilde{\alpha}, \beta_\pm, \phi, \psi_j, \sigma_j)$ in this model where $i = (1, 2, \ldots, 10)$. The conjugate momenta are $p_i = \partial S/\partial\dot{q}^i$, where a dot denotes differentiation with respect to conformal time $\eta$. The WD equation is derived by employing the canonical quantization procedure [5]. The classical Hamiltonian constraint $\mathcal{H} = p_i\dot{q}^i - L(q^i) = 0$ is identified as a wave equation acting on the state vector $\Psi(q^i)$ for the universe and the conjugate momenta are identified with the operators [13]:

$$p^2_{\tilde{\alpha}} = -e^{-p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}}e^{p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}} \tag{13}$$

$$p^2_i = -\frac{\partial^2}{\partial q^2_i}, \qquad i \neq 1 \tag{14}$$

and no summation is implied in this equation. The constant $p$ accounts for ambiguities in the operator ordering.

After some algebra the WD equation is found to be

$$\left[e^{-p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}}e^{p\tilde{\alpha}}\frac{\partial}{\partial\tilde{\alpha}} - \frac{\partial^2}{\partial\beta^2_+} - \frac{\partial^2}{\partial\beta^2_-} + U(\tilde{\alpha},\beta_\pm) - 12\frac{\partial^2}{\partial\phi^2} \right.$$
$$\left. -12\sum_{j=1}^{3}\left(\frac{\partial^2}{\partial\psi^2_j} + e^{2\psi_j}\frac{\partial^2}{\partial\sigma^2_j}\right)\right]\Psi = 0, \tag{15}$$

where the 'superpotential' $U$ is determined by the isometry group $G$. For Bianchi-IX it is given by [12]

$$3U = e^{4\tilde{\alpha}}\left[e^{-8\beta_+} - 4e^{-2\beta_+}\cosh 2\sqrt{3}\beta_- + 2e^{4\beta_+}\left(\cosh 4\sqrt{3}\beta_- - 1\right)\right]. \qquad (16)$$

The Bianchi-IX cosmology includes a number of interesting cases. The WD equation for the spatially closed FRW universe is given by Eqs. (15) and (16) when $\beta_+ = \beta_- = 0$ and the momentum operators $\partial^2\Psi/\partial\beta_\pm^2$ are removed. On the other hand the Taub universe [14] is recovered when $\beta_- = 0$ and $\partial^2\Psi/\partial\beta_-^2$ is removed.

An exact separable solution to Eq. (15) may be found. We assume the ansatz $\Psi = XY$, where $X = X(\tilde{\alpha}, \beta_\pm)$ and $Y = Y(\phi, \psi_j, \sigma_j)$, and separate the WD equation:

$$\left[\frac{\partial^2}{\partial\tilde{\alpha}^2} + p\frac{\partial}{\partial\tilde{\alpha}} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2} + U - z^2\right]X = 0 \qquad (17)$$

$$\left[\frac{\partial^2}{\partial\phi^2} + \sum_{j=1}^{3}\left(\frac{\partial^2}{\partial\psi_j^2} + e^{2\psi_j}\frac{\partial^2}{\partial\sigma_j^2}\right) - \frac{z^2}{12}\right]Y = 0, \qquad (18)$$

where $z$ is an arbitrary (possibly imaginary) separation constant. Recently it was shown that the potential (16) satisfies the Hamilton-Jacobi equation [15,16]

$$\left(\frac{\partial\chi}{\partial\tilde{\alpha}}\right)^2 - \left(\frac{\partial\chi}{\partial\beta_+}\right)^2 - \left(\frac{\partial\chi}{\partial\beta_-}\right)^2 + U = 0, \qquad (19)$$

where

$$\chi \equiv \frac{1}{6}e^{2\tilde{\alpha}}\left[e^{-4\beta_+} + 2e^{2\beta_+}\cosh 2\sqrt{3}\beta_-\right]. \qquad (20)$$

This property leads to a very simple analytical solution to Eq. (17). We search for solutions of the form:

$$X = e^{-c\tilde{\alpha}-\chi}, \qquad c = \text{constant}. \qquad (21)$$

Substitution of Eq. (21) into Eq. (17) leads to the condition

$$c^2 - pc - z^2 + 2(p - 2c - 6)\chi = 0, \qquad (22)$$

where Eq. (20) has been employed. This is satisfied when $p = 2(c+3)$ and

$$c = -3 \pm \sqrt{(3+z)(3-z)}. \qquad (23)$$

The constants $c$ and $p$ are real for $|z| \leq 3$. The special case $z = 0$ is equivalent to vacuum Bianchi-IX and the $c = 0$ solution for factor ordering $p = 6$ was found previously in Ref. [16]. We find a new vacuum solution $c = p = -6$. An exact solution for $p = 0$ also exists when $c = \pm z = -3$.

Eq. (18) is solved assuming the ansatz:

$$Y = A_1(\psi_1)A_2(\psi_2)A_3(\psi_3)e^{\pm i\gamma\phi \pm i(\omega_1\sigma_1 + \omega_2\sigma_2 + \omega_3\sigma_3)}, \qquad (24)$$

where $A_j(\psi_j)$ are arbitrary functions and $\{\gamma, \omega_j\}$ are arbitrary constants. Eq. (18) simplifies to three ordinary differential equations:

$$\frac{d^2 A_j}{d\psi_j^2} - \left[\omega_j^2 e^{2\psi_j} + \lambda^2 \cos^2\theta_j\right] A_j = 0, \qquad j = 1, 2, 3, \tag{25}$$

where $\lambda^2 \equiv \gamma^2 + z^2/12$ and the set of constants $\{\theta_j\}$ satisfy the integrability condition:

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1. \tag{26}$$

Eq. (25) reduces to a Bessel equation after the substitution $\psi_j = \ln\xi_j$, whose general solution is a linear combination of modified Bessel functions $Z_{\lambda\cos\theta_j}(\omega_j\xi_j)$ of order $\lambda\cos\theta_j$.

Finally, the equivalent solution in the original frame is deduced by substituting for $\alpha$ with Eq. (11). The full solution, modulo a constant of proportionality, is therefore

$$\Psi = \exp\left[-\frac{1}{6}\left(e^{-4\beta_+} + 2e^{2\beta_+}\cosh 2\sqrt{3}\beta_-\right)e^{2\alpha-\phi}\right] e^{-c(\alpha-\phi/2)} e^{\pm i\gamma\phi \pm i(\omega_1\sigma_1+\omega_2\sigma_2+\omega_3\sigma_3)}$$
$$\times Z_{\lambda\cos\theta_1}\left(\omega_1 e^{\psi_1}\right) Z_{\lambda\cos\theta_2}\left(\omega_2 e^{\psi_2}\right) Z_{\lambda\cos\theta_3}\left(\omega_3 e^{\psi_3}\right). \tag{27}$$

The $O(6,6)$ global symmetry generates, from a given solution to the WD equation, a generally inequivalent class of solution. As an example we may employ the duality transformed fields (6) to generate the class of solution:

$$\Psi = e^{-c\tilde{\alpha}-\chi} e^{\pm i\gamma\phi} \prod_{j=1}^{3} \left\{ Z_{\lambda\cos\theta_j}\left[\omega_j\left(e^{\psi_j} + e^{-\psi_j}\sigma_j^2\right)^{-1}\right]\right.$$
$$\left.\times \exp\left[\mp i\omega_j\left(e^{\psi_j} + e^{-\psi_j}\sigma_j^2\right)^{-1} e^{-\psi_j}\sigma_j\right]\right\}. \tag{28}$$

The $X$ component of the wave function is invariant under this duality transformation.

It was first noticed in [16] that the function $e^{-\chi}$ with $\phi = 0$ is strongly peaked around $\beta_+ = \beta_- = 0$ in the $(\beta_-, \beta_+)$ plane for large values of $e^\alpha$. When the spatial metric degenerates, however, $-\alpha$ diverges and $\chi$ becomes vanishingly small. Solutions (27) and (28) exhibit similar features for given values of $\{\phi, \psi_j, \sigma_j\}$. If one adopts the proposal of Hartle and Hawking [13] and interprets $|\Psi|^2$ as an unnormalized probability density, it follows that all values of $\beta_\pm$ are equally likely near the singularity. However, the probability density becomes progressively more peaked around the isotropic FRW solution as $\alpha$ increases [16]. This implies that there is a progressively higher probability of finding this universe in the isotropic state as its three-volume increases.

A number of assumptions were made in this calculation however. Firstly we ignored the contribution from the $H_{\mu\nu\rho}$ field. When $c \geq 0$ we find from Eq. (27) that $|\Psi|^2$ is maximized in the $\phi$ direction as the dilaton diverges. Hence the conformal weighting $e^{-2\phi}$ on the $\tilde{H}^2$ term of the action (8) suggests this is a consistent assumption if the interpretation of $|\Psi|^2$ as a probabilistic measure is valid.

A further assumption implicit in the calculation was that the fermions and gauge supermultiplet could be treated as constant background fields. This is not necessarily consistent since fermions should be included due to supersymmetry. The effects of including the fermionic degrees of freedom can be significant in certain circumstances [17]. Moreover, for vacuum Bianchi-IX the form (19) of the potential implies that the Hamiltonian leading to the WD equation is the bosonic part of a supersymmetric Hamiltonian [15,18]. Since it is reasonable to suppose that this symmetry should also be preserved at the quantum level, the supersymmetric partners should be included.

Finally, it is not clear that these solutions are directly relevant to the universe in which we live. They are Euclidean for all values of the scale factor and can not therefore be interpreted as Lorentzian four-geometries [19]. However it is possible that they may be viewed as quantum wormholes, since they are regular when the three-surface degenerates and decay exponentially fast at infinity [20].

The symmetry leading to Eq. (19) appears to hold for all Bianchi class A spaces [21]. Thus solutions similar to the ones presented here may be found in other Bianchi types. It will also be interesting to further explore the consequences of the $O(6,6)$ and SL(2,R) symmetries of the action.

In conclusion we have found a solution to the WD equation based on the dimensionally reduced string effective action when the four-dimensional physical space-time is viewed as the anisotropic Bianchi-IX model. This solution is peaked around the isotropic limit at large three-geometries.

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