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Quasi-Linear Evolution of Compensated Cosmological Perturbations: The Nonlinear Sigma Model

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Abstract

We consider the evolution of perturbations to a flat FRW universe that arise from a “stiff source,” such as a self-ordering cosmic field that forms in a global symmetry-breaking phase transition and evolves via the Kibble mechanism. Although the linear response of the normal matter to the source depends on the details of the source dynamics, we show that the higher-order non-linear perturbative equations reduce to a form identical to those of source-free Newtonian gravity in the small wavelength limit. Consequently, the resulting n -point correlation functions and their spectral counterparts will have a hierarchical contribution arising from this gravitational evolution (as in the source-free case) in addition to that possibly coming from non-Gaussian initial conditions. We apply this formalism to the $O(N)$ nonlinear sigma model at large N and find that observable differences from the case of initially Gaussian perturbations and Newtonian gravity in the bispectrum and higher-order correlations are not expected on scales smaller than about $100 h^{-1}\text{Mpc}$.

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I. INTRODUCTION

This paper attempts to combine several disparate streams of work in the study of cosmological perturbations. Since the early work of Lifschitz [1], Sachs and Wolfe [2] and others, the evolution of large scale matter perturbations to a spatially homogeneous and isotropic expanding universe has been well-studied. More recently, this work has been refined to apply to a more complicated universe containing both radiation (with equation of state $p/\rho = 1/3$) and non-relativistic matter ($p = 0$) constituents, using a gauge-invariant approach [3–5]. In addition, large N -body simulations have been used to examine the fully nonlinear problem of gravitational evolution on scales much smaller than the Hubble radius, where the Newtonian limit of general relativity is sufficient [6]. On the other hand, certain aspects of the full non-linear evolution equations do remain amenable to a more analytic approach. An area of work that has received considerable attention in the past decade is the examination of higher-order corrections to the linear Newtonian equations of motion for the matter in the universe within the framework of perturbation theory [7–10]. These higher-order corrections are intimately related to the evolution of the multi-point correlations of the mass (and therefore, of the galaxies), such as the three-point function, $\xi_3 \propto \langle \delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\delta(\mathbf{x}_3) \rangle$, the skewness, or, in Fourier-space, the bispectrum, $B \propto \langle \delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\delta(\mathbf{k}_3) \rangle$. As increasingly large galaxy surveys become available, observational information on these moments of the galaxy distribution has been extended to scales sufficiently large that this perturbative approach is expected to be reliable. In the case of an initially Gaussian density field for which all reduced higher-order correlations vanish, this “quasi-nonlinear” gravitational evolution leads to a scaling hierarchy of correlation functions: $\bar{\xi}_n \propto \bar{\xi}_2^{n-1}$, where $\bar{\xi}_n$ is the volume-averaged n -point correlation function, and the constant of proportionality depends weakly on the power spectrum or two-point function of the perturbations. Even with an initially non-Gaussian distribution, it may be possible to distinguish the primordial component from that due to gravitational evolution on large scales.

The standard cold dark matter (CDM) model for structure formation invokes a mechanism such as an early epoch of inflation [11] to generate primordial adiabatic density fluctuation. Inflation results in a spatially flat universe overall; quantum fluctuation of the inflaton field lead to an initially scale-invariant spectrum of density perturbations (with details possibly depending upon the specific model of inflation) which evolve thereafter under the influence of gravity. In the simplest models of inflation, this initial distribution of perturbations is Gaussian, so the hierarchical results noted above obtain for the resulting correlation functions. The specific mechanism which drives inflation is inextricably linked to the fundamental particle physics model of the universe. Of course, in addition to providing a mechanism for the generation of perturbations, inflation also has the advantage of solving the horizon and flatness problems which are otherwise left unexplained.

In another class of theories, topological defects (or some other classical field configuration) such as cosmic strings, domain walls or textures act as a continual source for the creation of density perturbations [11]. These structures are the remnants of a symmetry-breaking phase transition of a cosmic field and are thus, like inflation, a cosmological relic of the underlying particle physics. In these scenarios, the field is initially laid down randomly before the phase transition, on scales larger than the Hubble radius at that time. Once the symmetry is broken, the field tries to align itself in order to minimize its energy

density. However, this alignment can only occur coherently on scales where the field has come into causal contact with itself—within the Hubble volume. As the universe expands, the Hubble volume increases and the field orders itself on larger and larger scales—the Kibble mechanism [12]. If the symmetries of the field and its initially random configuration require it, topological defects may result. For example, a broken discrete symmetry like \mathcal{Z}_n produces domain walls, a broken $O(2)$ or $U(1)$ symmetry can result in (gauge or global) cosmic strings, $O(3)$ in monopoles, and global $O(4)$ in global texture. The breaking of a global $O(N)$ symmetry with $N > 4$ does not lead to topological defects, but does result in spatial field gradients and consequently to perturbations to the energy density. Unlike the inflationary scenario, the Kibble mechanism quite generically produces density fluctuations with an initially non-Gaussian distribution. (Of course, these theories do not provide a natural solution to the horizon and flatness problems and in this case it is usually assumed that the universe begins in a perfectly homogeneous and isotropic state with $\Omega = 1$ before the symmetry-breaking phase transition. Thus, in these models one relegates the solution of these puzzles to initial conditions, or to an earlier inflationary epoch driven by a field too weakly-coupled to produce the density perturbations responsible for large-scale structure.)

Within the class of scenarios that create structure via the Kibble mechanism, there is a further conceptual split between global and local (or gauge) symmetries. In a local theory, the large-scale gradient energy of the field is compensated by the gauge field so all the field energy is concentrated in localized defects. Thus, only true topologically stabilized configurations can result. In global theories, on the other hand, non-topological configurations (textures) can result that are nonetheless long-lived because they require energy to “unwind” a configuration by forcing it off its vacuum manifold. In large- N models, these field configurations persist simply because causality constrains them to align only on scales smaller than the Hubble radius, so field gradients persist for approximately a Hubble time. (Also, a new class known as “semi-local” defects has been studied, in which a gauge theory admits defects which are stabilized by the dynamics of theory, not the underlying topology of the symmetry groups [13].)

In this paper, we shall modify and extend aspects of a formalism that has been developed by Veeraraghavan and Stebbins [14] to study the perturbations due to a “stiff source” such as these cosmic field configurations formed by the Kibble mechanism. A “stiff source” evolves in the homogeneous and isotropic background metric of the universe; the back-reaction of the metric perturbations onto the source is considered to be negligible. We shall, however, explicitly account for compensation: the initial response of the matter fields to the stress-energy of the stiff source. We shall extend previous work to allow the perturbations of the matter and radiation fluids in the universe to enter the quasi-nonlinear regime and examine the modifications to the perturbative solution to the equations of motion that result. This, in turn, allows us to study the resulting correlation functions and may modify the scaling hierarchy in such scenarios.

Using this formalism, we shall concentrate on a specific class of models in which the Kibble mechanism is responsible for the initial generation of density perturbations, the nonlinear $O(N)$ sigma model [15–17]. These models arise as the low-energy limit of the breaking of a global $O(N)$ symmetry to $O(N - 1)$. Although the analytic calculations we shall perform rely on the large- N limit, in which there are only spatial gradients, but no topological defects, we can also extract some information about the behavior of this theory

for small N where defects play a role.

The plan of this paper is as follows. In Section II, we develop the equations of motion for the stiff source in the background metric and of the matter fields in the perturbed universe, using the longitudinal gauge, and develop a perturbative expansion about homogeneity and isotropy. In Section III, we apply this formalism to the nonlinear sigma model, and examine the higher-order correlations of the matter. Finally, we present our conclusions. In an appendix, we show some useful results for the distribution of the fields in an $O(N)$ model.

II. COMPENSATED PERTURBATIONS

A. Perturbations in the Longitudinal Gauge

We will consider linear metric fluctuations about a homogeneous FRW background. We work in comoving conformal coordinates with metric signature $(+, -, -, -)$ and assume a spatially flat background metric of

$$ds^2 = a^2 \eta_{\mu\nu} dx^\mu dx^\nu = a^2 (d\eta^2 - \delta_{ij} dx^i dx^j), \quad (1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Here, η is conformal time, related to proper time by $dt = a d\eta$. Throughout, we shall use a prime to denote the derivative with respect to conformal time, and define a conformal expansion rate $\mathcal{H} = a'/a$. The unperturbed Einstein equations in the flat ($\Omega = 1$) Freedman-Robertson-Walker (FRW) universe with mean background density ρ and pressure p and vanishing cosmological constant ($\Lambda = 0$) are

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho; \quad \frac{a''}{a} = \frac{4\pi G}{3} a^2 (\rho - 3p). \quad (2)$$

In a matter-dominated universe ($p = 0$), the scale factor $a \propto \eta^2$ and in a radiation-dominated universe ($p = \rho/3$), $a \propto \eta$. The Hubble constant is $H_0 = 100h \text{ km sec}^{-1} \text{ Mpc}^{-1}$ and the present Hubble radius is $H_0^{-1} = 3000 h^{-1} \text{ Mpc}$. We normalize the conformal time by setting $\eta_0 = 2H_0^{-1} = 6000 h^{-1} \text{ Mpc}$ today. We will write the perturbed metric as

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}). \quad (3)$$

In the longitudinal gauge ($h_{0i} = 0$) to first order in h , $g^{\mu\nu} = a^{-2}(\eta_{\mu\nu} - h_{\mu\nu})$ or $g^{00} = a^2(1 - h_{00})$ and $g^{ij} = -a^{-2}(\delta_{ij} + h_{ij})$.

In the usual longitudinal (often called conformal-Newtonian) gauge analysis, only scalar perturbations are considered, in which case the metric perturbations are determined by two scalar variables (“potentials”), $h_{00} = 2\phi$ and $h = \delta_{ij} h_{ij} = 6\psi$. In this paper, we will in addition allow nonzero vector and tensor perturbations, for which $\tilde{h}_{ij} = h_{ij} - \delta_{ij} h/3$ is nonzero. Most analyses of density perturbations in an expanding universe have used the synchronous gauge ($h_{00} = h_{0i} = 0$) [7] (although a gauge-invariant approach has recently become popular [3–5]). We have chosen the longitudinal gauge in order to more easily compare our results with the perturbative Newtonian equations of motion (and thereby gain insight into higher order correlations which have previously been studied only in the Newtonian limit); the price of this is retaining a nonzero component h_{00} . More importantly, in the longitudinal

gauge we can consider situations in which the density perturbation amplitude $\delta \equiv \delta\rho/\rho$ is large, while the metric perturbations are still small.

A few more words about our approximation scheme are in order. Because we shall only compute the Einstein tensor to first order in the metric perturbation, the perturbed Einstein equations $\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu}$ (where here $\delta T_{\mu\nu}$ represents the perturbation to all the stress-energy in the universe, including matter, radiation and any sources such as those discussed below) can only be considered first order — *in the metric perturbation*. Thus, we must ignore terms like $h_{\mu\nu} \cdot \delta T_{\alpha\beta}$ for consistency. However, this does not determine the form that the stress-energy must take: it may contain v^2 or $v\delta$ terms. (Below, we shall calculate the four-velocity to order v^2 .) Thus, these equations can be valid for “large” values of the density perturbation amplitude $\delta \sim 1$ as long as we still have $v^2, h_{\mu\nu} \ll 1$. Note that this is a gauge-dependent statement. In fact, performing the same exercise in a comoving synchronous gauge, we find that $h' = 2\delta'$ to all orders in the matter perturbation variables for pressureless, nonrelativistic matter, so this scheme would not be successful—we can only use synchronous coordinates until streamlines of the matter flow intersect and caustics form. The advantage of the synchronous formalism is that one can define a comoving gauge where the nonrelativistic matter component has zero velocity for all time (*i.e.*, as long as the coordinate construction is consistent). In the longitudinal gauge we are considering here we still have to calculate velocities. However, the gravitational potential (metric perturbation) in this gauge is in general suppressed compared to the density perturbation by the square of the size of the perturbation relative to the Hubble radius, as in the usual Newtonian Poisson equation: $h \sim \phi \sim (\lambda/H^{-1})^2\delta$ for perturbations of scale λ . (Of course, this equation is itself only applicable for scales smaller than the Hubble radius, beyond which relativistic corrections become important as in *e.g.*, Eq. (18) below.)

On scales smaller than the Hubble radius, the metric perturbation remains small even when the matter perturbation becomes nonlinear. In particular, we expect this approximation to be valid as long as $h^2 \ll \delta$; with first order quantities and the Poisson equation to relate the potential to the density perturbation, this is equivalent to $(\lambda/H^{-1})^4 \ll 1/\delta$. As expected, our approximation will continue to hold on scales much smaller than the horizon as long as $\delta \lesssim 1$. Conversely, for sufficiently small scales the linear metric perturbation approximation holds for even larger values of the density perturbation. As long as the average source stress-energy is negligible compared to the fluid components and as long as we continue to ignore backreaction, this approximation holds in the same regimes as the normal Newtonian limit.

In the longitudinal gauge, \tilde{h}_{ij} does not contain a scalar part as it does in the synchronous gauge, so in addition to being manifestly traceless, it satisfies $\partial_i\partial_j\tilde{h}_{ij} = 0$. Although the matter variables become more complicated when we allow higher-order terms to enter the equations, the metric perturbation is still usefully decomposed into geometrically distinct parts (*i.e.*, scalar, vector and tensor) [4]. Normally, only the scalar part of the peculiar velocity, which can be expressed as the gradient of some velocity potential, appears in the equations of motion for other scalar quantities like the density perturbation δ . With higher order terms, however, products like $v^i\delta$ or v^iv^j decompose as a vector and symmetric tensor, respectively, with scalar, vector, and (for v^iv^j) tensor parts that are not simply related to the scalar and vector parts of the original v^i . Thus, for example, δ , although a scalar, can

appear in higher-order equations containing the vector perturbations to the metric, and the vector part of v^i can occur in the scalar equations of motion (e.g., Eq. (9a) which contains $v^2 = v^i v^i$).

As in [14] (who instead calculated in synchronous gauge), we assume a universe filled with a multi-component perfect-fluid stress-energy $T^\mu{}_\nu = \sum_n T_{(n)}^\mu{}_\nu$, where n labels the fluid component, as well as a “stiff source” with stress-energy $\Theta_{\mu\nu}$. The source makes a negligible contribution to the spatially-averaged stress-energy and is covariantly conserved with respect to the *background* metric $a^2\eta_{\mu\nu}$:

$$\Theta'_{00} + \mathcal{H}(\Theta_{00} + \Theta) = \partial_i \Theta_{0i}; \quad (a^2 \Theta_{0i})' = a^2 \partial_j \Theta_{ij}. \quad (4)$$

Here $\Theta \equiv \Theta_{ii}$ is the spatial trace of the source stress energy.

Again, we do not consider the back-reaction on the stiff source of the metric perturbations. In principle, we could calculate it and its effects in an iterative, perturbative fashion, writing the full source stress-energy as a sum of a part propagating in the FRW background as in Eq. (4) and another part propagating in the full perturbed metric. This perturbation would, in turn, be fed back into the field equations for the metric perturbation. This would result in new terms of order $(\partial_\mu h_{\alpha\beta})\Theta_{\gamma\delta}$, still at linear order in the metric. Unless either the source stresses or the metric perturbation are large, these terms should be negligible in comparison to those involving the linear fluid perturbations.

The bulk of the universe is filled with several perfect fluids with individual stress-energies

$$T_{(n)}^\mu{}_\nu = (\rho_n + \delta\rho_n + p_n + \delta p_n)u_n^\mu u_{n\nu} - (p_n + \delta p_n)\delta^\mu{}_\nu, \quad (5)$$

where n labels the fluid component, ρ_n and p_n refer to the background density and pressure of each fluid, and $\delta\rho_n$, δp_n to their respective perturbations. The four-velocity of each fluid is $u^\nu = \bar{u}^\nu + \delta u^\nu$, normalized to $u^\nu u_\nu = +1$ with this signature. In the expanding background, the fluid components are at rest, with four-velocity (suppressing the fluid label n)

$$\bar{u}^\mu = (a^{-1}, 0, 0, 0), \quad \bar{u}_\nu = (a, 0, 0, 0) \quad (6)$$

and the velocity perturbations are

$$\begin{aligned} \delta u^0 &= \frac{1}{2}a^{-1}(v^2 - h_{00}), & \delta u_0 &= \frac{1}{2}a(v^2 + h_{00}), \\ \delta u^i &= -a^{-2}\delta u_i = a^{-1}v^i \left(1 + \frac{1}{2}v^2\right) \end{aligned} \quad (7)$$

This defines the peculiar velocity 3-vector, $v^i = u^i/u^0$. We have kept terms of order v^2 , and ignored terms of order $v \cdot h$, for the consistency of our approximation scheme. (Note that the form of the v^2 terms is exactly as one would expect from the usual special-relativistic four velocity expanded to this order.)

B. Equations of Motion for the Matter and Metric Fields

We will assume that each of the perfect fluid components is separately conserved. Except when there is significant energy transfer between the components (*e.g.*, at the epoch of recombination) this is an excellent approximation. To first order in the metric perturbation, the equations of motion for a single fluid component, $\nabla_\mu T_{(n)\nu}^\mu = 0$, become

$$\begin{aligned} & \delta' + v^2 \partial_0 [\delta(1 + \zeta)] + (1 + w + \delta(1 + \zeta)) \partial_0 v^2 + \nabla \cdot [(1 + w + \delta(1 + \zeta)) \mathbf{v}] \\ & + 3\mathcal{H} \left[v^2 (1 + w + \delta(1 + \zeta)) \left(w + \frac{1}{3} \right) + \delta(\zeta - w) + v^2 (1 + w)(w - c^2) \right] \\ & = 3\psi'(1 + w) \end{aligned} \quad (8a)$$

and

$$\begin{aligned} & \mathcal{H} v^i \left[(1 + w + \delta(1 + \zeta))(1 - 3w) + 3(1 + w)(w - c^2) \right] + [\delta(1 + \zeta)]' v^i + \\ & (1 + w + \delta(1 + \zeta)) v^i + \partial_j \left[(1 + w + \delta(1 + \zeta)) v^i v^j \right] + \partial_i (\zeta \delta) = -(1 + w) \partial_i \phi \end{aligned} \quad (8b)$$

where $\delta = \delta\rho/\rho$, $\zeta = \delta p/\delta\rho$, the fluid sound speed $c^2 = \partial p/\partial\rho$, $w = p/\rho$, and we have suppressed the subscript n on all fluid quantities. To linear order, $\zeta_n = c_n^2$ and we shall often take $w_n = \zeta_n = c_n^2$ as well. In the background, the fluid density evolves according to the zeroth-order equations of motion, $\rho'_n = -3(1 + w_n)\rho_n$. Notice that these equations do not depend explicitly upon the stiff source.

The perturbed Einstein equations $\delta G_{\mu\nu} = 8\pi G(\sum_{(n)} \delta T_{(n)\mu\nu} + \Theta_{\mu\nu})$ become

$$\begin{aligned} 4\pi G\tau_{00} &= \nabla^2 \psi \\ &= 3\mathcal{H}(\psi' + \mathcal{H}\phi) + 4\pi G a^2 \sum_n \rho_n \left[\delta_n + v_n^2 (1 + w_n + \delta_n(1 + \zeta_n)) \right] + 4\pi G\Theta_{00} \end{aligned} \quad (9a)$$

$$\begin{aligned} 4\pi G\tau_{0i} &= \nabla \psi' - \frac{1}{4} \partial_j \tilde{h}'_{ij} \\ &= -\mathcal{H}\nabla\phi - 4\pi G a^2 \sum_n \rho_n \mathbf{v}_n (1 + w_n + \delta_n(1 + \zeta_n)) + 4\pi G\Theta_{0i} \end{aligned} \quad (9b)$$

$$\begin{aligned} 4\pi G\tau_{ij} &= \delta_{ij} \left[\psi'' + \frac{1}{2} \nabla^2 (\phi - \psi) \right] - \frac{1}{2} \partial_i \partial_j (\phi - \psi) + \frac{1}{2} {}^{(3)}\tilde{G}_{ij} - \frac{1}{4} \tilde{h}''_{ij} \\ &= \frac{1}{2} \mathcal{H} \tilde{h}'_{ij} - \delta_{ij} \left\{ \left[2\frac{a''}{a} - \mathcal{H}^2 \right] \phi + \mathcal{H}(2\psi' + \phi') \right\} \\ &+ 4\pi G a^2 \sum_n \rho_n \left[\delta_n \zeta_n \delta_{ij} + (1 + w_n + \delta_n(1 + \zeta_n)) v_n^i v_n^j \right] + 4\pi G\Theta_{ij} \end{aligned} \quad (9c)$$

where

$${}^{(3)}\tilde{G}_{ij} = \frac{1}{2} \left[\nabla^2 \tilde{h}_{ij} - \partial_j \partial_k \tilde{h}_{ik} - \partial_i \partial_k \tilde{h}_{jk} \right] \quad (10)$$

is the trace-free part of the spatial Einstein tensor in this gauge. We have written the equations such that the form of the stress-energy pseudotensor $\tau_{\alpha\beta}$ is manifest. By Eqs. (9), the pseudotensor manifestly obeys

$$\partial_\mu \tau^{\mu\nu} = 0; \quad \tau^{\mu\nu} = \eta^{\alpha\mu} \eta^{\beta\nu} \tau_{\alpha\beta}. \quad (11)$$

In effect, the pseudotensor can be written as

$$\tau_{\mu\nu} = T_{\mu\nu} + \Theta_{\mu\nu} + t_{\mu\nu}^{\text{grav}} \quad (12)$$

where $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are the fluid and source stress-energies, respectively, and the remaining term, $t_{\mu\nu}^{\text{grav}}$ represents the “stress-energy of the gravitational field.” As we see in Eqs. (9), this term contains those parts of the Einstein tensor that are either nonlinear in the metric perturbation (although these terms do not appear in our approximation) or those that would vanish for a constant scale factor $a' = 0$.

The pseudotensor is conserved with respect to a Minkowski background; it thereby embodies the concept of a conserved stress-energy for the *entire system* including the “gravitational energy” as well as the matter and source stress-energy [14,18,19], the combined system propagating on a flat, Minkowski background. However, no individual fluid or gravitational components of the pseudotensor can be singled out as being separately conserved in this Minkowski space; moreover, its definition is gauge-dependent. It is useful for setting up the conditions of compensation on scales larger than the Hubble radius: it can describe the flow of energy between matter and “gravity” which only occurs causally, on scales within the Hubble volume. Note also that to this order, the metric perturbation only appears in terms containing factors of \mathcal{H} —in a true Minkowski background (*i.e.*, not expanding), the pseudotensor only differs from the “real” stress-energy tensor at second and higher orders in the metric perturbation [18,19].

The space-space component of the Einstein equations may be rewritten as a trace and traceless part:

$$\begin{aligned} \left[2\frac{a''}{a} - \mathcal{H}^2 \right] \phi + \mathcal{H}(2\psi' + \phi') + \psi'' + \frac{1}{3}\nabla^2(\phi - \psi) \\ = 4\pi G a^2 \sum_n \rho_n \left[\delta_n \zeta_n + \frac{1}{3}(1 + w_n + \delta_n(1 + \zeta_n)) v_n^2 \right] + \frac{4\pi G}{3} \Theta \end{aligned} \quad (13a)$$

$$\begin{aligned} \left[\frac{1}{3}\delta_{ij}\nabla^2 - \partial_i\partial_j \right] (\phi - \psi) + \frac{1}{2} \left[\nabla^2 \tilde{h}_{ij} - \partial_j\partial_k \tilde{h}_{ik} - \partial_i\partial_k \tilde{h}_{jk} - \tilde{h}_{ij}'' - 2\mathcal{H}\tilde{h}_{ij}' \right] \\ = 8\pi G a^2 \sum_n \rho_n (1 + w_n + \delta_n(1 + \zeta_n)) \left(v_n^i v_n^j - \frac{1}{3}\delta_{ij} v_n^2 \right) + 8\pi G \tilde{\Theta}_{ij}. \end{aligned} \quad (13b)$$

Here, Eq. (13a) is the spatial trace of Eq. (9c) ($4\pi G \tau_{kk}$) and Eq. (13b) comes from subtracting off that trace from Eq. (9c) ($\tau_{ij} - \delta_{ij}\tau_{kk}/3$). Also, $\tilde{\Theta}_{ij} = \Theta_{ij} - \delta_{ij}\Theta/3$ is the spatial trace-free part of the source stress energy.

(Of course, the conservation Eqs. (8) are not independent of the Einstein equations, which imply that the *sum* of the individual fluid stress energies is covariantly conserved via the Bianchi identity. In an approximation that we will consider several times later, a universe with only one fluid in addition to the stiff source, the conservation equations are redundant and can be derived from the Einstein equations by using the conservation formulas for the pseudostress-energy (Eq. (11)) and eliminating the source terms $\Theta_{\mu\nu}$ by using their background equations of motion, Eqs. (4).)

Obviously, these equations (8, 9, 13) cannot be solved analytically for a general equation of state, with general initial data and an arbitrary stiff source. However, we can explore various limits and approximations that are useful in different theories of structure formation.

C. Correspondence with Newtonian Equations of Motion

If we consider a universe filled only with pressureless matter ($w = c^2 = \zeta = 0$) and no stiff source ($\Theta_{\mu\nu} = 0$) we have the conditions of the standard matter-dominated universe. If we further assume the limit of scales small compared to the Hubble radius and small velocities, $v \ll c$, then we have the situation usually described by the Newtonian limit. In the equations of motion for the matter (Eqs. (8)) and the Einstein Eqs. (9,13), we drop terms like $\partial_0 v^2$ in this limit but we keep terms like $\partial_i v^2$ because we assume that the matter component may vary over small scales. Note that we do not assume that the density perturbation δ is small, so we do not ignore terms that contain δ in spatial derivatives, even when multiplied by small quantities like \mathcal{H}^2 or \mathbf{v} . In this case, the spatial Einstein Eqs. (13) only describe the evolution of the scalar components of the metric, ϕ and ψ ; the tensor-mode equations drop out as expected as these components evolve completely independently. The $i \neq j$ Einstein Eq. (13b) then requires $\phi = \psi$. The trace of the $i - j$ Eq. (13a) gives

$$\left(2\frac{a''}{a} - \mathcal{H}^2\right)\phi + 3\mathcal{H}\phi' + \phi'' = 0 \quad (14)$$

with solution $\phi = \text{const}$ (plus a decaying mode proportional to η^{-5}). With these results, the $0 - 0$ Einstein Eq. (9a) and the stress-energy conservation Eqs. (8) (which are not independent from the Einstein equations) become

$$\frac{3}{2}\mathcal{H}^2\delta = \nabla^2\phi, \quad (15a)$$

$$\delta' + \nabla \cdot [(1 + \delta)\mathbf{v}] = 0, \quad (15b)$$

$$\mathcal{H}\mathbf{v} + \mathbf{v}' + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\phi, \quad (15c)$$

where we have used the second equation to eliminate δ' in the final equation.

These equations are just those that come from a purely Newtonian analysis of density perturbations in the fluid limit: the Poisson equation, the continuity equation, and the Euler equation. It should not be surprising that this approximation scheme has yielded these results: in Newtonian theory, the potential $\phi \sim v^2$, so it is suppressed to second order compared to the matter variables when the velocities are small, and ignoring terms like $\phi\delta$ and ϕv should be sufficient to this level of approximation. This approximation can be made more precise by expanding separately in two small parameters: the size of the metric perturbation h and the ratio of the length scale of the perturbation in question to the Hubble length. This expansion generates the Newtonian and post-Newtonian approximations in a cosmological setting [18,20].

If we consider a simple perturbation expansion in both the matter velocity \mathbf{v} and density perturbations δ (with both quantities the same order), and an initially Gaussian density field

(*e.g.*, from inflation), these equations have been shown to produce a scaling hierarchy of correlation functions in the quasi-nonlinear regime [7,8,10]: $\bar{\xi}_n \propto \bar{\xi}_2^{n-1}$, where $\bar{\xi}_n$ is the volume-averaged n -point correlation function or moment of the distribution (*e.g.*, $\bar{\xi}_3$ is the skewness). For the unaveraged multi-point correlation functions $\xi_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$ or their fourier-space spectral counterparts $P_n(\mathbf{k}_1, \dots, \mathbf{k}_n)$, the n -point functions are proportional to sums of appropriate symmetric products of $(n-1)$ two-point functions. For the simple case of the bispectrum, $B_{123} = Q(P_1 P_2 + P_1 P_3 + P_2 P_3)$ as in Eq. (60) below; the trispectrum is a product of three power spectra, $T_{1234} = R_a[P(k_1)P(|\mathbf{k}_1 + \mathbf{k}_2|)P(k_3) + \text{sym.}] + R_b[P(k_1)P(k_2)P(k_3) + \text{sym.}]$, where “sym.” represents the further terms of the same form necessary to keep T symmetric in its arguments. Here, Q , R_a , R_b and their generalizations to higher n -point functions are constants that depend on the configuration of the \mathbf{k} ; and possibly on the initial power spectrum.

For this hierarchical result to apply, two additional assumptions are necessary: the decaying mode component of the solution for the density perturbation must become negligible, and the velocity-field must be curl-free, so it can be expressed as the gradient of some potential; any leftover vortical component of the velocity decays away proportional to a^{-1} , so this assumption is justified at late times.

It is interesting to examine how these equations are modified even in a universe with only a matter component when relativistic effects are included. That is, we no longer make the Newtonian approximation of small velocities and length scales that was used to derive Eqs. (15). Even to *linear* order, these equations have corrections for perturbation wavelengths larger than the Hubble radius ($\lambda \gg H^{-1}$, which we shall hereafter refer to as *superhorizon* scales). The Poisson equation simply becomes the non-relativistic matter ($w = 0$) version of Eq. (9a) with $\phi = \psi$, and the continuity equation gains a potential term (relaxing the assumption $\phi = \text{const}$):

$$\frac{3}{2}\mathcal{H}^2\delta = \nabla^2\phi - 3\mathcal{H}\phi' - 3\mathcal{H}^2\phi \quad (16a)$$

$$3\phi' = \delta' + \nabla \cdot [(1 + \delta)\mathbf{v}] \quad (16b)$$

while the Euler equation is unchanged. The linear solution to these equations is well known, and is presented in the following section, Eqs. (18). The full nonlinear equations also have corrections of order v^2 . However, we expect that when these terms are significant, our approximation that the metric perturbations are still of linear order will break down.

D. Full Equations in the Linear Regime

One useful application of Eqs. (8-13) is to a universe consisting only of pressureless matter, radiation and, possibly, a stiff source, in the linear regime (where now we are considering perturbations linear in everything: δ , v^i and $h_{\mu\nu}$). Because these equations are linear, the full power of previously developed techniques can be applied. In particular, we can use the geometrical decomposition of these equations to study the scalar, vector, and tensor modes separately. Density fluctuations in particular only couple to the scalar mode of the metric perturbation to this order, ϕ and ψ , as well as the scalar parts of the peculiar velocity and, if present, the stiff source. In a universe with non-relativistic matter ($w = c^2 = \zeta = 0$) and

radiation ($w = c^2 = \zeta = 1/3$) fluids, the linear Einstein Eqs. (9a,9b,13) for the scalar modes become

$$\nabla^2 \psi - 3\mathcal{H}(\psi' + \mathcal{H}\phi) - 4\pi G\Theta_{00} = \frac{3}{2}\mathcal{H}^2(\Omega_m \delta_m + \Omega_r \delta_r) \quad (17a)$$

$$\nabla^2 (\psi' + \mathcal{H}\phi) - 4\pi G\partial_i\Theta_{0i} = -\frac{3}{2}\mathcal{H}^2(\Omega_m \nabla \cdot \mathbf{v}_m + \frac{4}{3}\Omega_r \nabla \cdot \mathbf{v}_r) \quad (17b)$$

$$\mathcal{H}(2\psi' + \phi') + \psi'' + \frac{1}{3}\nabla^2(\phi - \psi) = \frac{1}{2}\mathcal{H}^2\Omega_r \delta_r + \frac{4\pi G}{3}\Theta \quad (17c)$$

$$\left[\frac{1}{3}\delta_{ij}\nabla^2 - \partial_i\partial_j\right](\phi - \psi) = 8\pi G\tilde{\Theta}_{ij}^{(s)} = -8\pi G\left[\frac{1}{3}\delta_{ij}\nabla^2 - \partial_i\partial_j\right]s \quad (17d)$$

where Ω_m and Ω_r , respectively, represent the ratio of the matter and radiation densities to the (time-dependent) critical density, with $\Omega_m + \Omega_r = 1$ by assumption. $\tilde{\Theta}_{ij}^{(s)}$ is the scalar, trace-free part of Θ_{ij} , and can be written as above in terms of a second-order linear operator acting on a unique scalar function $s(\eta, \mathbf{x})$, determined, for example, by expanding Θ_{ij} in terms of appropriate harmonic functions. (In particular, $\Theta_{ij} \approx \delta_{ij}\Theta/3$ on superhorizon scales, where the differential operator that projects out the s component approximately vanishes, as do all spatial derivatives.) Thus, the final equation tells us that $\phi - \psi = -8\pi Gs$ up to a spatial constant (and in a universe with no stiff source, $\phi = \psi$ as usual). Even without a stiff source, these equations cannot be solved exactly in the presence of both matter and radiation fluids except in various limits. In a matter-dominated universe ($\Omega_m = 1$), the final two equations give ϕ and ψ in terms of the source, and the first two in turn give the density and velocity.

Because we are again considering linear perturbation theory, the complete solution in the matter-dominated era is just given by the sum of the usual pure matter solution (vanishing Ω_r and $\Theta_{\mu\nu}$) and a ‘‘particular’’ solution generated by the $\Theta_{\mu\nu}$ terms. The source-free matter solution is:

$$\phi_c = \psi_c = C_1(\mathbf{x}) + C_2(\mathbf{x})\eta^{-5} \approx \text{const} \quad (18a)$$

$$\begin{aligned} \delta_c &= \frac{1}{6}\left[\eta^2\nabla^2 C_1 - 12C_1 + \left(\eta^2\nabla^2 C_2 + 18C_2\right)\eta^{-5}\right] \\ &\approx \frac{1}{6}\left(\eta^2\nabla^2 \phi_c - 12\phi_c\right) \end{aligned} \quad (18b)$$

Superhorizon scales can be defined by the condition that $\eta\nabla \rightarrow 0$; in fourier space where $\nabla \rightarrow k \sim 1/\lambda$, this becomes the appropriate $k\eta \rightarrow 0$, since we also have $\eta \sim H^{-1}$. On these scales, $\delta \approx -2\phi \approx \text{const}$, so perturbations do not grow, while on small scales the usual Poisson equation applies. In this case, the equations can be rewritten as the linearized version of the Newtonian case considered above in Section II C.

In the presence of a source term, we must add the appropriate particular solution to this homogeneous solution. After setting $\phi - \psi = -8\pi Gs$ as discussed above, the solution for the potential ϕ is

$$\phi = \phi_c + \frac{1}{5}\int \chi\eta d\eta - \frac{1}{5}\eta^{-5}\int \eta^6\chi d\eta; \quad \chi(\eta, \mathbf{x}) = 8\pi G\left(\frac{1}{6}\Theta - 2\mathcal{H}s' - s'' + \frac{1}{3}\nabla^2 s\right). \quad (19)$$

Given these solutions for the scalar potentials, the density fluctuation can be computed immediately. In the presence of a source term, the question of initial conditions for these

fluctuations is crucial. We wish to express the fact that, at the time of “source creation” (e.g., a phase transition) the universe is initially homogeneous, and the matter variables can only respond causally to the source stress-energy—that is, the initial anisotropies of the stiff source stresses must be compensated by the matter fields. Within the Hubble volume, this produces actual perturbations in the fluid component; over superhorizon scales, the matter fields cannot vary coherently due to causality, so the universe must have zero curvature outside the Hubble volume. This isocurvature nature of these perturbations is embodied in the Minkowski-space conservation of the stress-energy pseudotensor $\tau^{\mu\nu}$ whose components were given above. In the initially homogeneous universe, the pseudoenergy $\tau_{00} \equiv \mathcal{E} = 0$ as well as $\Theta_{\mu\nu} = 0$. As is manifest from the Einstein Eqs. (9), the evolution equations for the pseudotensor are given by

$$\tau'_{00} = \partial_i \tau_{0i}; \quad \tau'_{0i} = \partial_j \tau_{ij}. \quad (20)$$

When the phase transition, a Poisson random process, occurs, we expect both of these quantities $\Theta_{\mu\nu}$ and $\tau_{\mu\nu}$ to be initially uncorrelated with themselves and thus gain white noise power spectra on superhorizon scales. That is, we Fourier decompose all quantities as

$$f(\mathbf{k}) = \int d^3x f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (21)$$

and we shall write $f \sim k^n$; this means that the power spectrum $|f_k|^2 \propto k^{2n}$. Thus, we expect $\Theta_{\mu\nu} \sim k^0$ and $\tau_{\mu\nu} \sim k^0$. The $i - j$ components only occur in spatial derivatives, so these white noise spectra should obtain for these components for all time as long as $k\eta \ll 1$ (i.e., while the modes are outside the Hubble radius and out of causal contact with themselves), where we are considering the contribution from logarithmic intervals about a wavenumber k . From the evolution equations, we then see that $\tau_{0i} \sim k$ and $\tau_{00} \sim k^2$, whereas $\Theta_{0i} \sim k$ and $\Theta_{00} \sim k^0$ (since the Θ_{00} equation is dominated by the \mathcal{H} term). Thus, the quantities τ_{00} , τ_{0i} , and Θ_{0i} should remain negligible for $k\eta \ll 1$. A similar argument using the Einstein equations reveals that the density and metric fluctuations should initially have white noise spectra on superhorizon scales, but that velocities should fall as $\mathbf{v} \sim k$.

Therefore, the superhorizon scale density perturbation is simply given from the 0 - 0 Einstein Eq. (9a) by the relation $\tau_{00} = 0$:

$$\sum_n \Omega_n \delta_n = -\frac{2}{3} \mathcal{H}^{-2} \Theta_{00} - 2\mathcal{H}^{-1} \psi' - 2\phi \quad (22)$$

Here, this equation holds for a multi-component universe; below, we shall specialize to a universe dominated by a single component with $p_n/\rho_n = w_n$ and $\Omega_n \approx 1$. On large scales $\psi' \simeq \delta'_n/3(1+w_n)$ (from Eq. (8a)) and $\phi \simeq \text{const}$. For the primary fluid component, this gives the superhorizon evolution equation for δ ,

$$\delta + \frac{2}{3\mathcal{H}(1+w_n)} \delta' = -\frac{2}{3} \mathcal{H}^{-2} \Theta_{00} - 2\phi \quad (23)$$

For a universe with scale factor $a \propto \eta^\alpha$,

$$\delta(\eta, \mathbf{x}) = -\frac{1+w}{\alpha} \eta^{-3\alpha(1+w)/2} \int d\eta \Theta_{00}(\eta, \mathbf{x}) \eta^{3\alpha(1+w)/2+1} - 2\phi(\eta, \mathbf{x}) \quad (24)$$

In a radiation-dominated universe ($w = 1/3$) this gives δ_r . (In order to calculate the superhorizon matter perturbation in this case, we define the entropy perturbation $\sigma = 3\delta_r/4 - \delta_m$. If as causality requires there is no initial entropy perturbation on superhorizon scales, $\delta_m = 3\delta_r/4$ and it is unnecessary to explicitly worry about the details of the several components.)

In particular, if the stiff source obeys a scaling relation $\Theta_{00} \propto \eta^{-2}$ (see Section III), then

$$\delta = -\frac{2}{3\alpha^2}\eta^2\Theta_{00} - 2\phi. \quad (25)$$

Both of these terms are time-independent, so the perturbations do not grow with time outside the Hubble radius. The same result was derived by Davis *et al.* in the synchronous gauge [16]. This leads to the usual scale-invariant Harrison-Zel'dovich power spectrum: $\delta\rho/\rho \approx \text{const}$ at horizon-crossing. In the nonlinear sigma model to be discussed later, the matter-dominated era evolution does obey $\Theta_{00} \propto \eta^{-2}$, but in the radiation era $\Theta_{00} \propto \eta^{-2} \ln(\eta/\eta_1)$, where η_1 is the conformal time of the symmetry-breaking phase transition, so the evolution of the density perturbation is modified by a (divergent) logarithmic term. Thus, the initially white-noise spectrum (constant amplitude on all scales at one time) has been transformed into a scale-invariant spectrum (constant amplitude at horizon-crossing) by the gravitational action of the stiff source.

Compensation thus insures that the pseudoenergy \mathcal{E} vanishes on superhorizon scales for all time. This fact in turn gives the initial condition for the perturbation on a given scale at Horizon crossing. (Note that this is not the case for the usual primordial adiabatic perturbations discussed in CDM models, where the initial perturbation spectrum is considered as a given before the equations are integrated.)

In order to calculate the evolution of density fluctuations well inside the horizon, it is easiest to use the first of Eqs. (20), $\tau'_{00} = \partial_i\tau_{0i}$ and the Einstein equations which define the components of $\tau_{\mu\nu}$, Eqs. (9). Here, it is assumed that the universe contains both radiation and matter fluids. After some algebra to eliminate the matter velocity and the potential ψ , this component of Eq. (20) reduces to

$$\begin{aligned} \mathcal{H}\delta_m'' + \mathcal{H}^2\delta_m' - \frac{3}{2}\mathcal{H}^3\Omega_m\delta_m - 3\mathcal{H}^3(1 - \Omega_m)\delta_r + 3\mathcal{H}^3(\Omega_m - 2)\phi + 3\mathcal{H}^2\phi' \\ = 4\pi G(\partial_i\Theta_{0i} - \Theta'_{00}) \end{aligned} \quad (26)$$

With the initial condition $\mathcal{E} = 0$ due to compensation, in a matter-dominated universe, this becomes simply

$$\frac{3}{2}\mathcal{H}^2(\delta + 2\phi) + \mathcal{H}\delta' = 4\pi G \int^\eta d\eta \partial_i\Theta_{0i} \quad (27)$$

where we have ignored the contribution of Θ_{00} to the total energy density inside the horizon. The solution is

$$\delta = \frac{4\pi G}{2}\eta^{-3} \int^\eta d\eta_1 \eta_1^4 \int^{\eta_1} d\eta_2 \partial_i\Theta_{0i}(\eta_2) - 6\eta^{-3} \int^\eta d\eta \eta^2 \phi + K\eta^{-3}, \quad (28)$$

where we have explicitly included a decaying mode $\propto \eta^{-3}$. A similar equation was derived in the synchronous gauge by Davis *et al.* [16]. In the synchronous gauge, the term involving ϕ is

not present and the equation can be solved directly for the density perturbation. This term, however, is exactly as one would expect in performing the gauge transformation from synchronous gauge to longitudinal gauge for a scalar quantity $\delta\rho$: $\delta_l = \delta_s - 3\mathcal{H}(1+w)a^{-1} \int d\eta a\phi$ [5], where l and s refer to quantities in the longitudinal and synchronous gauge, respectively. In the longitudinal gauge, unfortunately, things are more complicated and we must use our solution for the potential from above subject to the initial condition of $\mathcal{E}(k\eta \rightarrow 0) = 0$ for a complete solution. However, on extreme subhorizon scales, we can in general neglect terms like $\mathcal{H}^2\phi$ in the above equations—the longitudinal and synchronous coordinates nearly coincide. For example, in taking the Newtonian limit of the δG_{00} Einstein equation in a pure CDM universe, we drop such a term in order to derive its nonrelativistic limit, the Poisson equation. Thus, the growing mode solution reduces to solely the first term above. In this case, and also assuming that the source has the power law form $\int d\eta \partial_i \Theta_{0i} \propto \eta^\gamma$ (as in Section III below, the growing mode solution is

$$\delta = \frac{4\pi G}{2(5+\gamma)} \eta^2 \int d\eta \partial_i \Theta_{0i}. \quad (29)$$

Davis *et al.* derived this result for the specific case of $\gamma = 0$ in the synchronous gauge [16], in which case the solution is of the same form as the normal pure-CDM growing mode $\delta \propto a \propto \eta^2$ (which would be the solution here in the case of a nonzero initial value of the pseudoenergy: $\delta \propto \mathcal{E}_0 \eta^2$.)

In a radiation-dominated universe, the situation is more complicated. Note that we are interested in the evolution of *matter* perturbations in this situation. Thus, we take $\Omega_m \approx 0$ in Eq. (26) above, giving

$$\delta_m'' + \mathcal{H}\delta_m' \approx 4\pi G\mathcal{H}^{-1}(\partial_i \Theta_{0i} - \Theta'_{00}) \quad (30)$$

where the terms involving ϕ and δ_r have been ignored well inside the horizon. The solution is

$$\delta = C_1 + C_2 \ln \eta + 4\pi G \int^\eta d\eta_2 \eta_2^{-1} \int^{\eta_2} d\eta_1 \eta_1^2 (\partial_i \Theta_{0i} - \Theta'_{00})_{\eta_1}, \quad (31)$$

which include the usual constant and logarithmic terms, in addition to one due to the source evolution. Again considering the case of $\int d\eta \partial_i \Theta_{0i} \propto \eta^\gamma$, the density perturbation due to the stiff source becomes

$$\delta = 4\pi G \frac{\gamma}{(2+\gamma)^2} \eta^2 \int d\eta \partial_i \Theta_{0i}. \quad (32)$$

In the particular case that $\int d\eta \partial_i \Theta_{0i} \approx \text{const}$, ($\gamma \approx 0$) the response to the stiff source exactly mimics the usual isocurvature perturbation scenario: perturbations only grow inside the horizon and during matter-domination, $\delta \propto a$ (or logarithmically during radiation-domination). If the source varies more rapidly, then corrections to this behavior may become important.

E. Higher Order Equations With a Stiff Source

On superhorizon scales, the linear equations considered in the last section cannot be extended to higher orders in the matter variables within the approximation of linear metric fluctuations. Outside the horizon, large density fluctuations create large fluctuations in the metric, and we would at least need to treat higher orders in $h_{\mu\nu}$ as well as the matter variables (if not solve the full equations numerically). Well inside the horizon, however, the situation is different and the density fluctuation amplitude may be large for a small metric perturbation in the longitudinal gauge. Thus, we may still approximate the initial conditions as a vanishing pseudoenergy, but retain the nonlinear terms on small scales.

Further, on small scales, we may again assume that the usual conditions of the Newtonian limit apply: potentials and velocities are small and slowly-varying. Of course, because of the presence of the source terms, it is important to at least check that these conditions still hold. From the linear solution of Eq. (27), we expect the potential to vary rapidly in regions of spacetime where the source (specifically, its scalar parts Θ and s) is rapidly-varying itself. For the sigma-model discussed below, the time scale of variation is always the Hubble time, so this assumption is justified in that case. For models with topological defects, the regions around the defects themselves would generally have large stresses, and we expect this analysis to fail there.

In order to compare with the usual Newtonian equations, the relevant equations are again the covariant conservation equations and the 0 – 0 Einstein equation, now supplemented by the traceless $i - j$ equation to take into account the anisotropic stresses of the stiff stress-energy. On subhorizon scales, these give

$$\frac{3}{2}\mathcal{H}^2\delta = \nabla^2\psi, \quad (33a)$$

$$\delta' + \nabla \cdot [(1 + \delta)\mathbf{v}] = 0, \quad (33b)$$

$$\mathcal{H}\mathbf{v} + \mathbf{v}' + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\phi, \quad (33c)$$

$$\phi - \psi = -8\pi Gs. \quad (33d)$$

(A rapidly-varying potential would add a term $3\psi'$ to the right hand side of Eq. (33b).) The linear solution to these equations for stiff sources originating in a smooth universe is just that derived above. Although these equations do not explicitly contain the term Θ_{0i} that appears in the above solution (Eq. (29)), recall that the source necessarily obeys conservation with respect to the background metric, so the two forms cannot be independent; they are related by the equations of motion for the source, Eqs. (4).

Except for the presence of the s term, these are exactly the same equations as in the Newtonian matter-dominated case without sources, discussed above, Eqs. (15). Note also that s occurs in a “pure source” term—it is a function that is supplied from outside of this set of equations and the set of equations are linear in that function.

This permits a particularly simple derivation of the higher-order equations of motion—*i.e.*, the deviations from the linear solution above. We can write the various quantities as

$$\delta = {}^{(1)}\delta + {}^{(2)}\delta + \dots \quad (34)$$

where the superscript represents an n -th order quantity (i.e., $O[(^{(1)}\delta)^n]$). Once the linear equations have been subtracted off, the resulting equations give differential equations for the higher-order terms in terms of lower-order ones. These equations will have exactly the same structure as in the pure Newtonian case (Section II C above); the only dependence upon the stiff source will be implicitly through the $^{(1)}\delta$ solution. That is, we can write the linear solution as a functional of the stiff source, $^{(1)}\delta = ^{(1)}\delta[s]$. As usual in perturbation theory, the higher-order quantities only explicitly depend on the lower order $^{(n)}\delta$:

$$^{(n)}\delta = ^{(n)}\delta [^{(1)}\delta, \dots, ^{(n-1)}\delta], \quad (35)$$

where the dependence on the source is implicit in $^{(1)}\delta$, and thus in all higher-order terms as well.

In the Newtonian case, it is usually assumed that the initial (linear) fluctuations have a Gaussian distribution; in the presence of a stiff source the distribution of the density fluctuations depend crucially upon the distribution of the source through the linear solution. However, the distribution of a stiff source is not a general property, but depends upon the specific model. As noted above, if $\int d\eta \partial_i \Theta_{0i} \approx \text{const}$, the matter perturbation behaves just as it does under the usual Newtonian evolution. Then, the problem is equivalent to the Newtonian one with the additional possibility of non-Gaussian initial conditions [21].

III. THE NONLINEAR SIGMA MODEL

A. Background Evolution

Although the derivation thus far has been completely general and valid for any stiff source that obeys conservation with respect to the background (except as noted), most sources do not have a simple analytic form that can be “plugged in” to these equations. One notable exception is the $O(N)$ nonlinear sigma model in the limit of large N , which is exactly soluble in an expanding $\Omega = 1$ FRW universe. In order to be reasonably self-contained, we shall follow Davis *et al.* [16] closely in this section (note however the difference in normalizations of the random variables $\alpha_{\mathbf{k}}^a$ and the definition of the power-law exponent α).

First, consider the lagrangian for the $O(N)$ fields ϕ^a , $a = 1 \dots N$:

$$\mathcal{L} = \frac{1}{2} \nabla_\mu \phi^a \nabla^\mu \phi^a - V(\phi). \quad (36)$$

where we raise and lower indices with the background metric $a^2 \eta_{\mu\nu}$. If the $O(N)$ symmetry is broken to $O(N-1)$, the potential will have a nonzero minimum, $V(\phi_0^a) = 0$; the usual example is the broken “phi-fourth” potential, $V(\phi) = \lambda(\phi^2 - \phi_0^2)^2$ with $\phi^2 = \phi^a \phi^a$. With this sort of potential, the ϕ^a will have N massless modes corresponding to angular excitations, and one massive mode corresponding to radial excitations of the N -vector ϕ . In the low-energy or strong-coupling limit, we can represent the potential of the massless modes with a lagrange multiplier term,

$$\mathcal{L} = \frac{1}{2} \nabla_\mu \phi^a \nabla^\mu \phi^a + \frac{1}{2} \lambda (\phi^2 - \phi_0^2) \quad (37)$$

Thus, the N fields ϕ^a are fixed to the $(N - 1)$ -sphere vacuum manifold: $\phi^a \phi^a = \phi_0^2$. The fields begin randomly distributed before the phase transition; as they come inside the Hubble radius, they come into causal contact with one another and order themselves to minimize the gradient energy (the first term of the Lagrangian) on those scales.

The equation of motion is

$$\nabla^\mu \nabla_\mu \phi^a = \frac{\phi^b \nabla^\mu \nabla_\mu \phi^b}{\phi_0^2} \phi^a = - \frac{\nabla_\mu \phi^b \nabla^\mu \phi^b}{\phi_0^2} \phi^a \quad (38)$$

where the second equality follows from the vacuum manifold constraint enforced by the lagrange multiplier [22] (and thus is only strictly true for the nonlinear sigma model, but not in models with a “physical” potential). At large N , we can replace the bilinear $\nabla_\mu \phi^b \nabla^\mu \phi^b$ with its (ensemble or spatial) average and the scaling ansatz

$$\langle \nabla_\mu \phi^b \nabla^\mu \phi^b \rangle = S \frac{\phi_0^2}{a^2 \eta^2}. \quad (39)$$

with a constant S . If this ansatz holds, the “density” of the stiff source is also proportional to η^{-2} and the dynamics are scale-invariant with respect to the horizon size (or Hubble radius). We shall see later that this choice is self-consistent.

Before the phase transition (at temperatures $T \gtrsim \phi_0$), the potential (or lagrange multiplier) term in the lagrangian is irrelevant and we expect $\phi^2 \sim T^2$ due to thermal fluctuations. After the phase transition, the field is pinned to the vacuum manifold. In the large- N limit, the distribution of the individual components ϕ^a becomes a Gaussian peaked around $\phi^a = 0$ with a variance given by $\langle \phi^2 \rangle = \phi_0^2$ (see Appendix). Therefore, we simply Fourier transform,

$$\phi^a(\mathbf{x}, \eta) = \frac{\phi_0}{\sqrt{4\pi N}} \int d^3 k \alpha_{\mathbf{k}}^a \phi_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (40)$$

where the prefactor enforces the constraint $\phi^2 = \phi_0^2$ and the $\alpha_{\mathbf{k}}^a$ are *Gaussian* random variables of mean 0 and normalized to unit variance for later ease of calculation. They are uncorrelated for unequal a and \mathbf{k} ; for example,

$$\langle \alpha_{\mathbf{k}}^a \alpha_{\mathbf{q}}^b \rangle = \delta^{ab} \delta^3(\mathbf{k} + \mathbf{q}), \quad (41)$$

where δ^{ab} and $\delta^3(\mathbf{k})$ are Kronecker and Dirac deltas, respectively, and higher (even) order correlations are sums of appropriate products of this two-point function; averages of odd numbers of the $\alpha_{\mathbf{k}}^a$ vanish.

In a universe with scale factor $a \propto \eta^\alpha$ and the scaling ansatz, Eq. (38) becomes the *linear* equation

$$\phi_k'' + \frac{2\alpha}{\eta} \phi_k' + k^2 \phi_k = - \frac{S}{\eta^2} \phi_k. \quad (42)$$

This has solution

$$\phi_k(\eta) \equiv k^{-3/2} f(k\eta); \quad f(x) = \frac{1}{\mathcal{N}_0^{1/2}} x^{1/2-\alpha} J_{1+\alpha}(x) \quad (43)$$

where the order of the Bessel function (and the value of the constant S) has been chosen so that the solution has a white-noise power spectrum on superhorizon scales, and \mathcal{N}_0 is a normalization constant. The presence of the simple form $f(k\eta)$ indicates the expected scaling nature of the solution: $k\eta$ is the ratio of the horizon ($\eta \propto 1/\mathcal{H}$) to the length scale ($k \sim 1/\lambda$).

Given this solution, we can calculate the stress energy $\Theta_{\mu\nu}$ required for the formalism of Section II. Most generally,

$$\Theta_{\mu\nu} = \frac{2}{\sqrt{-g}} \left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\alpha} \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\alpha g^{\mu\nu})} \right]. \quad (44)$$

In this case, the second term vanishes and the ‘‘potential’’ (lagrange multiplier) term is zero when the vacuum manifold conditions are satisfied. So,

$$\Theta_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^a - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha \phi^a \partial_\beta \phi^a. \quad (45)$$

Particular components are

$$\Theta_{00} = \frac{1}{2} (\phi^{a'})^2 + \frac{1}{2} (\nabla \phi^a)^2 \quad (46)$$

$$\Theta_{0i} = \phi^{a'} \nabla \phi^a \quad (47)$$

$$\Theta_{ij} = \partial_i \phi^a \partial_j \phi^a + \frac{1}{2} \delta_{ij} [(\phi^{a'})^2 - (\nabla \phi^a)^2]. \quad (48)$$

The solutions for the ϕ^a can be inserted into these expressions to calculate the components in this model. We find that

$$\langle \Theta_{00} \rangle \propto \frac{1}{\eta^2} \quad (49)$$

in a matter-dominated universe, so the scaling solution holds. In a radiation-dominated universe, there is also a factor of $\ln(\eta/\eta_1)$, so the scaling is modified by a logarithmic term. We also find that the ‘‘pressure,’’ $\langle \Theta \rangle$ vanishes in a matter-dominated universe and is $(1/3)\langle \Theta_{00} \rangle$ during radiation-domination. We shall calculate Θ_{0i} below when considering density fluctuations.

B. Density Perturbations and Power Spectra

When the sigma-model solution, Eq. (43), and the fourier transform of the expression for the stress-energy from Eq. (47) are inserted into Eq. (29) above for the linear perturbation to the matter density, the result is

$$\delta(\mathbf{k}) = \frac{4\pi G}{10} \eta^2 \int d\eta \partial_i \Theta_{0i}(\mathbf{k}) = \frac{G\phi_0^2}{10N} \eta^2 \int d^3q \mathbf{q} \cdot \mathbf{k} \alpha_{\mathbf{k}-\mathbf{q}}^a \alpha_{\mathbf{q}}^a \int d\eta \phi'_{|\mathbf{k}-\mathbf{q}|}(\eta) \phi_{\mathbf{q}}(\eta) \quad (50)$$

where we have assumed the power law exponent $\gamma = 0$, or $\int d\eta \Theta_{0i} \approx \text{const}$ in Eq. (29), to be justified below. This form enables the computation of n -point power spectra.

In general, a function of the form $\langle \delta(\mathbf{k})^n \rangle$ contains the product of $2n$ Gaussian random variables $\alpha_{\mathbf{k}}^a$; thus even though the odd moments of the field ϕ may vanish, the corresponding moments of the density field may be nonzero, and thus may be non-Gaussian even to linear order.

In addition, many of the terms in the expansion of the order n moment of the $\alpha_{\mathbf{k}}^a$ will vanish due to the form of the integrand above. For example, the power spectrum $\langle \delta(\mathbf{k})\delta(\mathbf{k}') \rangle$ contains a term with

$$\mathbf{k} \cdot \mathbf{q} \mathbf{k}' \cdot \mathbf{q}' \delta^{aa'} \delta^{a'a} \delta^3(\mathbf{k}) \delta^3(\mathbf{k}') N^{-2}, \quad (51)$$

where the Kronecker and Dirac deltas result from the mean of four of the $\alpha_{\mathbf{k}}^a$ from the previous equation. This term vanishes due to the presence of $\mathbf{k}' \delta^3(\mathbf{k}') = 0$. Similarly, in any moment, any term which contains a combination of Kronecker deltas that do *not* contract overall to N will also contain a factor like $\mathbf{k}' \delta^3(\mathbf{k}')$ and will thus integrate to zero (this factor is effectively $\langle \delta_{\mathbf{k}} \rangle = 0$); the remaining terms behave like

$$\mathbf{k} \cdot \mathbf{q} \mathbf{k}' \cdot \mathbf{q}' \delta^{a'a} \delta^{a'a} \delta^3(\mathbf{k} + \mathbf{q}') \delta^3(\mathbf{k}' + \mathbf{q}) N^{-2} \quad (52)$$

in the calculation of the (2-point) power spectrum. In this case, the Kronecker deltas contract to one power of N and the Dirac deltas are equivalent to $\delta^3(\mathbf{k} + \mathbf{k}')$ after the integrals have been performed. For an n -point spectrum this works out to $N \times N^{-n} = 1/N^{n-1}$, where the first factor N comes from the contraction of the Kronecker deltas and N^{-n} from the n prefactors of $\delta(\mathbf{k})$ in Eq. (50). Moreover, when all the remaining integrals have been completed, there will still be a delta-function to enforce the requirement that the sum of the \mathbf{k}_i be zero as expected. Therefore, the moment will behave like

$$\langle \delta(\mathbf{k}_1) \cdots \delta(\mathbf{k}_n) \rangle \propto \left(\frac{G\phi_0^2}{10} \eta^2 \right)^n \frac{1}{N^{n-1}} \delta^3 \left(\sum \mathbf{k}_i \right). \quad (53)$$

In fact, we can proceed further without actually doing any calculating. In addition to the factors in the previous equation, the n th moment will contain an integral over d^3q of n quadratics in \mathbf{q} and the \mathbf{k}_i as well as a product of n integrals of the form $\int d\eta \phi'_p \phi_{p'}$, where p, p' are the lengths of linear combinations of \mathbf{q} and the \mathbf{k}_i . As in Davis *et al.* [16], this integral can in general be written as a function of the configuration and overall scale (*i.e.*, $k \equiv k_1$) of the \mathbf{k}_i by writing $\mathbf{q} = k\mathbf{u}$ and $\eta = s/k$, and all of the remaining $\mathbf{k}_i = k(\mathbf{k}_i/k)$. Furthermore, we use the scaling of the ϕ fields,

$$\phi_p(\eta) = p^{-3/2} f(p\eta), \quad \phi'_p(\eta) = p^{-1/2} f'(p\eta) \quad (54)$$

in the η integral. We can then pull out all of the factors of scale to be left with k^{3-n} in front. Thus, the n th moment (factoring out the momentum-conserving δ function) is given by

$$P(\mathbf{k}_1, \dots, \mathbf{k}_{n-1}) = \left(\frac{G\phi_0^2}{10} \eta^2 \right)^n \frac{k^{3-n}}{N^{n-1}} g_n(k\eta, \text{configuration}), \quad (55)$$

where “configuration” refers to the shape (but not the scale) of the n -sided polyhedron defined by the \mathbf{k}_i . (For $n \geq 4$, the irreducible moment P_n differs from the reducible $\langle \delta_1 \cdots \delta_n \rangle$)

by terms that vanish unless two of the wavevectors satisfy $\mathbf{k}_i + \mathbf{k}_j = 0$; for continuous \mathbf{k} this is a set of measure zero, and in any case does not apply to the configurations usually examined when considering data.) For suitably late times and subhorizon scales ($k\eta \rightarrow \infty$), the function g_n should go to a constant, dependent only on the configuration of the figure.

In particular we find that the subhorizon power spectrum is Harrison-Zel'dovich, $P(k) \propto k$, the bispectrum $B(\mathbf{k}_1, \mathbf{k}_2)$ is independent of scale, and all higher moments go as a negative power of k ; that is, they decrease with decreasing scale. However, this calculation is only valid for scales which have entered the horizon in the matter-dominated era. Thus, the spectrum of perturbations on scales smaller than the horizon at matter-radiation equality cannot be calculated from these formulae.

Thus far, we have neglected the fact that we must take the large- N limit of these quantities. If we normalize them to the (two-point) power spectrum $P(k)$ through a quantity such as J_3 (the volume integral of the two-point correlation function) or σ_8 , we get a finite numerical value for the combination $G\phi_0^2/\sqrt{N} \approx 3 \times 10^{-5} \sigma_8$ [16] (where σ_8^2 is the variance of the mass distribution in spheres of $8 h^{-1} \text{Mpc}$; the variance in the number density of galaxies on that scale is measured to be $\sigma_8(\text{gal}) = 1$ [23]). The scaling of this value, $\phi_0^2 \propto \sqrt{N}$ is a consequence of the analytic calculations reproduced in this section for large N . However, the numerical value comes from various simulations. Comparison of the normalization (to σ_8) for specific values of $N \leq 6$ in simulations (which explicitly include topological-defect field configurations not present for large N), indicates that this value for ϕ_0 may be as much as a factor of 10 too high. Although the calculations reproduced in this section show that we should expect $\phi_0^2 \propto \sqrt{N}$ in the large- N limit, for these low values of N that have been numerically investigated, such scaling has not yet been reached [24]. In any case, none of these values are expected to be known to better than about 50% [17] (The low- N cases are in the process of being solved by using exact Green's functions rather than direct integration of the equations of motion presented here. This is expected to give more accurate results [25].)

Thus, the n -point spectra work out to be

$$P_n = \left(\frac{G\phi_0^2}{10\sqrt{N}} \eta^2 \right)^n g_n \frac{k^{3-n}}{N^{n/2-1}}. \quad (56)$$

In the large- N limit, these quantities decrease with N for all $n \geq 3$. However, the leading term in $1/N$ causes a non-Gaussian distribution to this order, and we expect these calculations to be at least approximately valid for N greater than a few, since the character of the sigma-field is approximately the same in the absence of topological defects such as strings ($N = 2$) or textures ($N = 4$).

The asymptotic value of g_2 for large k (small scales) is given by

$$g_2 = \int d^3u \left[\mathbf{u} \cdot \hat{\mathbf{k}} (2\mathbf{u} \cdot \hat{\mathbf{k}} - 1) \right] I(|\hat{\mathbf{k}} - \mathbf{u}|, \mathbf{u}; k\eta) \quad (57)$$

as in [16], where

$$I(a, b; x) = \int^x ds \frac{f(as)f'(bs)}{a^{3/2}b^{1/2}}. \quad (58)$$

We have already assumed that $k\eta \gg 1$ so we may integrate by parts and eliminate surface terms to consolidate the various integrals $I(a, b; k\eta) \equiv I(a, b)$ that appear (see below and Fig. (1)). The corresponding value for the bispectrum, g_3 , is

$$g_3(\mathbf{v}) = \int d^3u \mathbf{u} \cdot \hat{\mathbf{k}} I(|\hat{\mathbf{k}} - \mathbf{u}|, u) \left[(1 + 2\mathbf{v} \cdot \hat{\mathbf{k}} - 2\mathbf{u} \cdot \hat{\mathbf{k}} + v^2 - 2\mathbf{u} \cdot \mathbf{v})(2\mathbf{u} \cdot \mathbf{v} - 2\mathbf{v} \cdot \hat{\mathbf{k}} - v^2) \times \right. \\ \left. I(|\hat{\mathbf{k}} + \mathbf{v} - \mathbf{u}|, |\mathbf{u} - \hat{\mathbf{k}}|) I(u, |\hat{\mathbf{k}} + \mathbf{v} - \mathbf{u}|) + \right. \\ \left. (v^2 + 2\mathbf{u} \cdot \mathbf{v})(2\mathbf{u} \cdot \hat{\mathbf{k}} + 2\mathbf{u} \cdot \mathbf{v} + v^2 - 1) I(u, |\mathbf{u} + \mathbf{v}|) I(|\mathbf{u} - \hat{\mathbf{k}}|, |\mathbf{v} + \mathbf{u}|) \right] \quad (59)$$

where $\mathbf{v} = \mathbf{k}_2/k_1$ embodies the dependence upon the configuration of the wavevectors. For an equilateral triangle, $k_2 = k_1 = k$ and $\mathbf{u} \cdot \mathbf{v} = -1/2$.

Note that these quantities do have a residual scale dependence in the form of the upper limit of the innermost integrals I . In Fig. (1), we plot numerical integrations of g_2 and g_3 as a function of k where we integrate from horizon crossing ($k\eta = 1$) to the present epoch ($k\eta = k\eta_0$) at each wavenumber. Both of these quantities reach their asymptotic values ($g_2 \approx 12, g_3 \approx 1.6$) by $k^{-1} \approx 10^3 h^{-1} \text{Mpc}$, well outside of the range of our ability to reliably measure higher-order correlation functions. This quick approach to constant values indicates that, as suspected, $\int d\eta \partial_i \Theta_{0i} \approx \text{const}$, so we are justified in assuming the power-law exponent $\gamma \approx 0$ in Eqs. (29) and (32) above—the situation mimics a simple growing mode. Physically, this is simple to understand: as a given mode enters the horizon, it comes into causal contact fairly quickly, on the order of a Hubble time. After that, the field is roughly static, so its momentum density Θ_{0i} is small, and the matter perturbations are only generated on scales near the horizon. Subsequently, they evolve under the influence of small-scale Newtonian gravity alone. (However, it is not hard to imagine some other kind of stiff source with “ordering physics” that continues to be active on smaller scales which would result in density fluctuations that do not mimic the Newtonian growing-mode result.)

One traditional way to characterize the bispectrum is through the ratio

$$Q \equiv \frac{B(k_1, k_2)}{P_1 P_2 + P_1 P_3 + P_2 P_3} \quad (60)$$

which for an equilateral triangle becomes $Q = B(k)/3P(k)^2$. The usual hierarchical result from the Newtonian analysis with Gaussian initial conditions is $Q = \text{const}$ from second-order perturbation theory. With the non-Gaussian initial conditions from the sigma model, we instead have $Q \propto k^{-2}$ from *linear* theory.

The question remains, then, on what scales do we expect these linearly evolved moments to dominate over higher-order effects? From Section II E above, the leading nonlinear contribution to the higher moments is

$$P_{n, \text{nl}}(k) = Q_n P_2^{n-1} = Q_n \left(\frac{G\phi_0^2}{10\sqrt{N}\eta^2} \right)^{2(n-1)} (g_2 k)^{n-1}, \quad (61)$$

where this equation is actually only correct for $n = 3$ with equilateral triangles, where $Q_3 = 3Q$. (For $n > 3$, the right-hand side of this equation should be a sum over the different “tree-level” graphs connecting the labelings of the wavevectors, each term with a

corresponding constant $Q_{n;a}$ [8,10,26]. However, for analyzing the scale-dependencies here, the form above shall suffice.)

Consider the quantities $d_n = P_n/P_2^{n/2}$ which normalize the n -point spectra to $|\delta(\mathbf{k})|^n$. We see that the linear contribution is, of course, constant,

$$d_{n,\text{lin}} = \frac{g_n}{g_2^{n/2}} k^{3(1-n/2)} N^{1-n/2} \quad (62)$$

whereas the nonlinear contribution grows with time,

$$d_{n,\text{nl}} = Q_n \left(\frac{G\phi_0^2}{10\sqrt{N}} g_2 k \eta^2 \right)^{n/2-1} \quad (63)$$

At a given wavenumber, the nonlinear contribution grows with time with respect to the linear contribution. A similar result for spatial correlation functions was derived for $n = 3$ in [21], where they also included the effects of a primordial four-point function (whose contribution to the skewness also grows with time).

The ratio of the nonlinear contribution to that of the linear evolution is

$$\frac{P_{n,\text{nl}}}{P_{n,\text{lin}}} = Q_n \left[\frac{G\phi_0^2}{10\sqrt{N}} \right]^{n-2} N^{n/2-1} \frac{g_2^{n-1}}{g_n} (k\eta)^{2n-4} \quad (64)$$

which goes as a positive power of k for $n > 2$, so the nonlinear terms become more important on small scales, as expected. This is a function of $k\eta$, the ratio of the length involved to the horizon size, so the scaling nature of the solution is preserved.

The scale of this crossover from the domination of the linear to the nonlinear evolution of the fields depends on the values of the g_n for a given configuration of wavevectors and N (although the powers of N explicitly cancel in the above expression, recall that the normalization of ϕ_0 is such that factor in brackets above is constant). Crossover thus occurs at scale

$$k_{\text{nl}}^{-1} = \eta \left(Q_n \frac{g_2^{n-1}}{g_n} \right)^{1/(2n-4)} N^{1/4} \left(\frac{G\phi_0^2}{10\sqrt{N}} \right)^{1/2}. \quad (65)$$

Actually, the full evolution of the three-point function depends on the four-point function (or trispectrum $T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$) as well [21], which we do not explicitly calculate here, but we expect this crossover scale to be independent of it, since from above we have $T \propto k^{-1}$ so the trispectrum falls on small scales, at least for those greater than that of matter-radiation equality. Moreover, the value of n only enters these equations through the prefactor involving the Q_n and g_n , which are not expected to vary greatly with n . So, for increasing n , the n -dependent part of the first factor in parentheses will be come less important, and the crossover scale becomes approximately

$$k_{\text{nl}}^{-1} \approx \eta g_2^{1/2} N^{1/4} \left(\frac{G\phi_0^2}{10\sqrt{N}} \right)^{1/2}; \quad (\text{large } n). \quad (66)$$

If we naively plug the asymptotic values of $g_2 \approx 12$ and $g_3 \approx 1.6$ into Eq. (65), with the hierarchical value for equilateral triangles of $Q_3 = 3Q = 34/7$ we find

$$k_{\text{nl}}^{-1} \approx 219 N^{1/4} \sigma_8^{1/2} h^{-1} \text{Mpc}; \quad (n = 3) \quad (67)$$

for $G\phi_0^2/\sqrt{N} \approx 3 \times 10^{-5} \sigma_8$. If we let this quantity be a factor of 10 smaller as indicated above, then the crossover scale is reduced to $k_{\text{nl}}^{-1} \approx 69 N^{1/4} \sigma_8^{1/2} h^{-1} \text{Mpc}$. To calculate the scale at which the primordial four-point function or trispectrum becomes significant, we will use the hierarchical pattern $T \approx 16RP^3$ for tetrahedral configurations, with $R \sim Q^2$, [8] so $Q_4 \approx 16Q^2 \sim 5$ (although observations [26] are typically $R \approx 1$.) Thus,

$$k_{\text{nl}}^{-1} \approx (17 - 54) g_4^{-1/4} N^{1/4} \sigma_8^{1/2} h^{-1} \text{Mpc}; \quad (n = 4) \quad (68)$$

spanning the range of possible normalizations of ϕ_0 , where we have left the integral g_4 unevaluated, but expect it to be of order unity.

Even this scale, unfortunately, is just beyond the largest considered in surveys for higher-order power spectra, which have gone out to $k^{-1} \approx 20 h^{-1} \text{Mpc}$ [26]. We expect only the *nonlinear* contribution to the three-point function to be important on the smallest observable scales. If instead we use the second, large- n form of k_{nl} , we find

$$k_{\text{nl}}^{-1} \sim (11 - 36) N^{1/4} \sigma_8^{1/2} h^{-1} \text{Mpc}; \quad (\text{large } n) \quad (69)$$

This is more possibly in the range of observed and observable scales, but still (for $N \gtrsim 6$ where we expect to believe these results) barely on the edge of current observations of the bispectrum and trispectrum.

Thus, the usual Newtonian analysis of higher-order correlation should suffice there. Unfortunately, that means that it will be more difficult to distinguish the nonlinear sigma model from simple Gaussian theories on small scales using the mass distribution alone. However, as discussed in [17], the large-scale normalization of the nonlinear sigma model to the COBE results may require an unreasonably large bias in order to match the galaxy distribution.

More problematic is the calculation on scales that entered the horizon during the radiation-dominated period. A complete calculation requires the numerical integration of the equations of motion of the sigma model fields and the radiation and matter fluids as in [17]. They present a possible parametrization of the transfer function which takes the power spectrum from $P(k) \propto k$ on large scales to its small-scale form (usually $P(k) \propto k^{-3}$, possibly modulated by the logarithmic growth of modes in the radiation-dominated universe). However, the equivalent form has not been calculated for the three-point function. However, we do not expect the linear evolution to dominate on small scales. Higher order or fully nonlinear effects tend to dominate on the smallest scales. In order for the linear correlations to be observable below the scale of matter-radiation equality, the linear bispectrum would have to increase relative to the nonlinear contribution on these scales. Moreover, because of the form of Eqs. (33a), the relative importance of higher order effects with respect to linear effects is again expected to be the same as in the purely Newtonian case.

C. Comparison with Observations

As we have seen, the nonlinear sigma model is not expected to give non-Gaussian results for higher-order correlation functions on as-yet observed scales. So far, all observations are

consistent with Newtonian hierarchical results with Gaussian initial conditions. Baumgart and Fry [26] actually analyze the n -point power spectra for two galaxy surveys and find that $Q(k) = B/3P^2 \simeq 4/7$ (for equilateral triangles) for $k^{-1} \lesssim 10 h^{-1}\text{Mpc}$. This is exactly the value expected from the Newtonian theory with Gaussian initial conditions. They also calculate $R(k) = T/16P^3 \approx 1$ (for regular tetrahedral configurations), a result which is weakly consistent with the hierarchical expectation.

More recent results of analyses of the IRAS dataset are also consistent with the hierarchical expectations. Bouchet *et al.* [27] have calculated the volume-averaged n -point functions $\bar{\xi}_n$ for $n \leq 5$ and again find that the hierarchical form holds: $\bar{\xi}_n \propto \bar{\xi}_2^{n-1}$. Moreover, Nusser *et al.* [28] have reconstructed the *initial* one-point probability density function for the smoothed IRAS density field (using an algorithm based on Newtonian gravity and the Zel'dovich approximation) and found it to be consistent with a Gaussian distribution on scales smaller than about $70 h^{-1}\text{Mpc}$.

It may be possible to examine the n -point functions at still larger scales using two-dimensional angular correlations [29]. However, an angular analysis introduces errors due to uncertainties in the luminosity function of galaxies and consequently the survey's selection function, especially on the large scales we are most interested in. Nonetheless, Gaztañaga [30] has estimated the n -point angular correlation functions for the APM dataset and found that the results are consistent with an initially Gaussian distribution for scales out to about $30 h^{-1}\text{Mpc}$; however, this is still below the scales required to see any possible effects of a sigma-model field.

Precise measurements of the bispectrum and trispectrum out to scales of $k^{-1} \approx 100 h^{-1}\text{Mpc}$ should be sufficient to begin to more explicitly test this model. We should expect the next generation of survey results for higher-order correlations to at least begin to probe the scales on which primordial non-Gaussianity from a sigma-model source should be observable, at least for moderate values of N . That we do not yet observe a marked increase in the three-, four-, or five-point functions (or their spectral counterparts) on the largest scales implies that the density field on those scales remains dominated by Newtonian evolution. If this observational trend continues, then any crossover to linear evolution will be on still larger scales, and we will be able to make stronger statements about the normalization or existence of any sigma-model field (*e.g.*, using Eq. (65)).

We re-emphasize the fact that observations of the scaling hierarchy of correlation function do not imply Gaussian initial conditions, even on the scale of the observations. As we have seen here, the correlation functions on small scales are dominated by the contributions from the initially Gaussian component of the density field, which evolves nonlinearly into the quasi-Gaussian scaling hierarchy. This behavior is expected to be generic to any scenario with non-Gaussian initial conditions, as the nonlinear evolution should in any case dominate on small scales.

Other observational ramifications of the sigma-model may be more damning even today. As Pen *et al.* [17] point out, normalization of the microwave distortions produced in these models to the COBE observations require what may be an uncomfortably high bias, $b = 1/\sigma_8 \approx 2/h$. Moreover, the detailed fit of the power spectrum of mass fluctuations (determined from numerical simulations) to the measured QDOT galaxy distribution seems to require a bias of order 6 on scales of $20 h^{-1}\text{Mpc}$.

IV. CONCLUSION

We have developed a formalism to examine the quasi-nonlinear behavior of perturbations in a universe with a “stiff” source. In particular, we have shown that their evolution is formally very similar to those encountered in a purely Newtonian analysis with an initial perturbation spectrum, but without a source. In the particular case of the nonlinear sigma model, we find that the behavior of higher correlation functions of the density field on extreme subhorizon scales is exactly the same as that which occurs with primordial *adiabatic* perturbations. On larger, currently unobservable, scales, two effects occur that might differentiate the sigma model. First, the sigma model (at least at moderate N), has an initially non-Gaussian distribution, which might be observed on scales of several hundred megaparsecs. Furthermore, we find that the density perturbations are actually being created on scales comparable to the horizon, where the self-ordering physics of the source is occurring. Unfortunately, the prospects of observing effects on these scales are slim. However, it may be possible to invent models in which the equivalent self-ordering occurs on smaller scales as well. Unfortunately, the chief candidates—defect theories like cosmic strings—are ill-suited for this analysis, because the gravitational effects of defects generally occur on too small a scale. Therefore, a quasi-nonlinear analysis is insufficient and full-blown numerical simulations must be performed.

There are several possible extensions to the work we have presented here. We have concentrated on scalar perturbations, because those result in the readily-observable density fluctuations, even when higher-order terms in the matter variables are present. By considering vector and tensor perturbations, we can go beyond this and calculate quantities like the full velocity field (including any possible vortical part), as well as the tensor perturbations (gravitational radiation) from the stiff source.

If we consider a universe filled with both matter and radiation fluids, it should be possible to use this formalism to make accurate calculations of microwave anisotropies in these theories. If we have also calculated the gravitational radiation, then we can separate out the scalar and tensor components [31].

Finally, it should be possible to extend the cosmological post-Newtonian suite of approximations [20] to include the possibility of a stiff source. Because this involves going to a higher order in the metric perturbation, things become considerable more complicated, but such approximations are often useful even when perturbations have become extremely nonlinear.

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APPENDIX: DISTRIBUTION OF THE FIELDS IN THE $O(N)$ MODEL

In the nonlinear $O(N)$ sigma model, the fields ϕ^a are constrained to lie upon the vacuum manifold, an $(N - 1)$ -sphere of radius ϕ_0 , where they will be distributed uniformly in the absence of any further symmetry breaking. Thus, the probability density will be proportional to the solid angle on the $(N - 1)$ -sphere.

We choose $N - 1$ polar coordinates $(\varphi, \theta_1, \dots, \theta_{N-2})$ such that the solid angle is

$$d\Omega_N = d\varphi \sin \theta_1 d\theta_1 \cdots \sin^i \theta_i d\theta_i \cdots \sin^{N-2} \theta_{N-2} d\theta_{N-2}. \quad (\text{A1})$$

Thus, the distribution of fields in polar coordinates is given by

$$p(\varphi, \{\theta_i\}) d\varphi d\theta_1 \cdots d\theta_{N-2} \propto d\Omega_N. \quad (\text{A2})$$

We are concerned, however, with the distribution of the ϕ^a , the Cartesian components of the fields. In this coordinate system, there is always one component given by $z \equiv \phi^a/\phi_0 = \cos \theta_{N-2}$, and, due to the symmetry of the system, we can calculate the distribution of any single Cartesian component:

$$\begin{aligned} p(z) &= \int p(\varphi, \{\theta_i\}) d\varphi d\theta_1 \cdots d\theta_{N-2} \delta(z - \cos \theta_{N-2}) \\ &= A' \int d\Omega_N \delta(z - \cos \theta_{N-2}) \\ &= A' \int d\varphi \int d\theta_1 \sin \theta_1 \cdots \int d\theta_{N-3} \sin^{N-3} \theta_{N-3} \int d\theta_{N-2} \sin^{N-2} \delta(z - \cos \theta_{N-2}) \\ &= 2\pi A' \int d\theta_1 \sin \theta_1 \cdots \int d\theta_{N-3} \sin^{N-3} \theta_{N-3} \int_{-1}^1 dy (1 - y^2)^{(N-3)/2} \delta(z - y) \\ &= A(1 - z^2)^{(N-3)/2} \end{aligned} \quad (\text{A3})$$

where A, A' are constants determined by the requirement that $\int dz p(z) = 1$, and we have made the change of variables $y = \cos \theta_{N-2}$. This gives

$$p(z) dz = \frac{1}{\sqrt{\pi}} \frac{\Gamma(N/2)}{\Gamma(N/2 - 1/2)} (1 - z^2)^{(N-3)/2} dz. \quad (\text{A4})$$

This distribution has mean $\langle z \rangle = 0$ and variance $\langle z^2 \rangle = 1/N$ (which can be calculated directly or seen from the constraint $\sum \phi^a \phi^a = \phi_0^2$). The higher moments are given by

$$\langle z^m \rangle = \frac{1}{\sqrt{\pi}} \frac{\Gamma(N/2) \Gamma(m/2 + 1/2)}{\Gamma(N/2 + m/2)} \quad (m \text{ even}), \quad (\text{A5})$$

which behave as $O(1/N^{m/2})$ for large N ; the odd- m moments vanish due to the symmetry of the distribution. For large N ,

$$p(z) \rightarrow \sqrt{\frac{N}{2\pi}} (1 - z^2)^{N/2}. \quad (\text{A6})$$

We wish to compare this to a Gaussian distribution with some width σ ,

$$g(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-z^2/2\sigma^2); \quad \langle z^m \rangle = \frac{(2\sigma^2)^{m/2}}{\sqrt{\pi}} \Gamma(m/2 + 1/2) \quad (m \text{ even}). \quad (\text{A7})$$

For small z^2 and large N ,

$$p(z) \rightarrow \sqrt{\frac{N}{2\pi}} \left(1 - \frac{Nz^2}{2}\right), \quad g(z) \rightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \left(1 - \frac{z^2}{2\sigma^2}\right), \quad (\text{A8})$$

which implies $\sigma^2 = 1/N$ to first order in $1/N$, as expected from $\langle z^2 \rangle = 1/N$ and the expression for the higher moments of the Gaussian distribution at large N . For $z \approx 1$, both $p(z), g(z) \rightarrow 0$ for large N or small σ (although the ratio $p(z)/g(z)$ can be quite large for a finite value of N ; this is because the Gaussian is normalized over the interval $(-\infty, +\infty)$). Moreover, in the limit of $N \rightarrow \infty$, both $p(z)$ and $g(z)$ approach the Dirac δ -function. In Figure 2 we show $p(z)$ and $g(z)$ for various values of N . Note that for $N \lesssim 5$, the departures from Gaussianity are significant.

Thus, the distribution of the Cartesian components of the field approaches that of a Gaussian with $\sigma^2 = 1/N$. For large N , then, the components of the field become more and more strongly peaked about $z = 0$. Note that this distribution is applicable only on superhorizon scales, where the ordering dynamics of the field are unimportant. Within the horizon, the fields will continually organize in order to minimize their gradient energy as new scales enter the causally-connected region.

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FIGURES

FIG. 1. The integrals $g_2(k)$ (solid) and $g_3(k)$ (dashed) which are the prefactors in the calculation of the linear contribution to the power spectrum $P = P_2 \propto g_2 k$ and bispectrum (for equilateral triangles) $B = P_3 \propto g_3 k^0$ respectively (Eq. (55)). We have integrated from horizon-crossing ($k\eta = 1$) to the present day ($k\eta = k\eta_0$). In each case the asymptotic value is reached by $k^{-1} \approx 1000 h^{-1} \text{Mpc}$. Note that we have already assumed $k\eta \gg 1$ in writing down the expressions for the g_n , so the details of the approach to the asymptotic values ($g_2 \rightarrow 12.2$ and $g_3 \rightarrow 1.6$) should only be taken as indicative.

FIG. 2. A comparison of $p(z = \phi^a/\phi_0)$ (solid), the actual distribution of fields ϕ^a in the $O(N)$ sigma model, with $g(z)$ (dashed), the corresponding Gaussian distribution of mean 0 and variance $1/N$, for $N = 3, 5, 10$ and 50 .

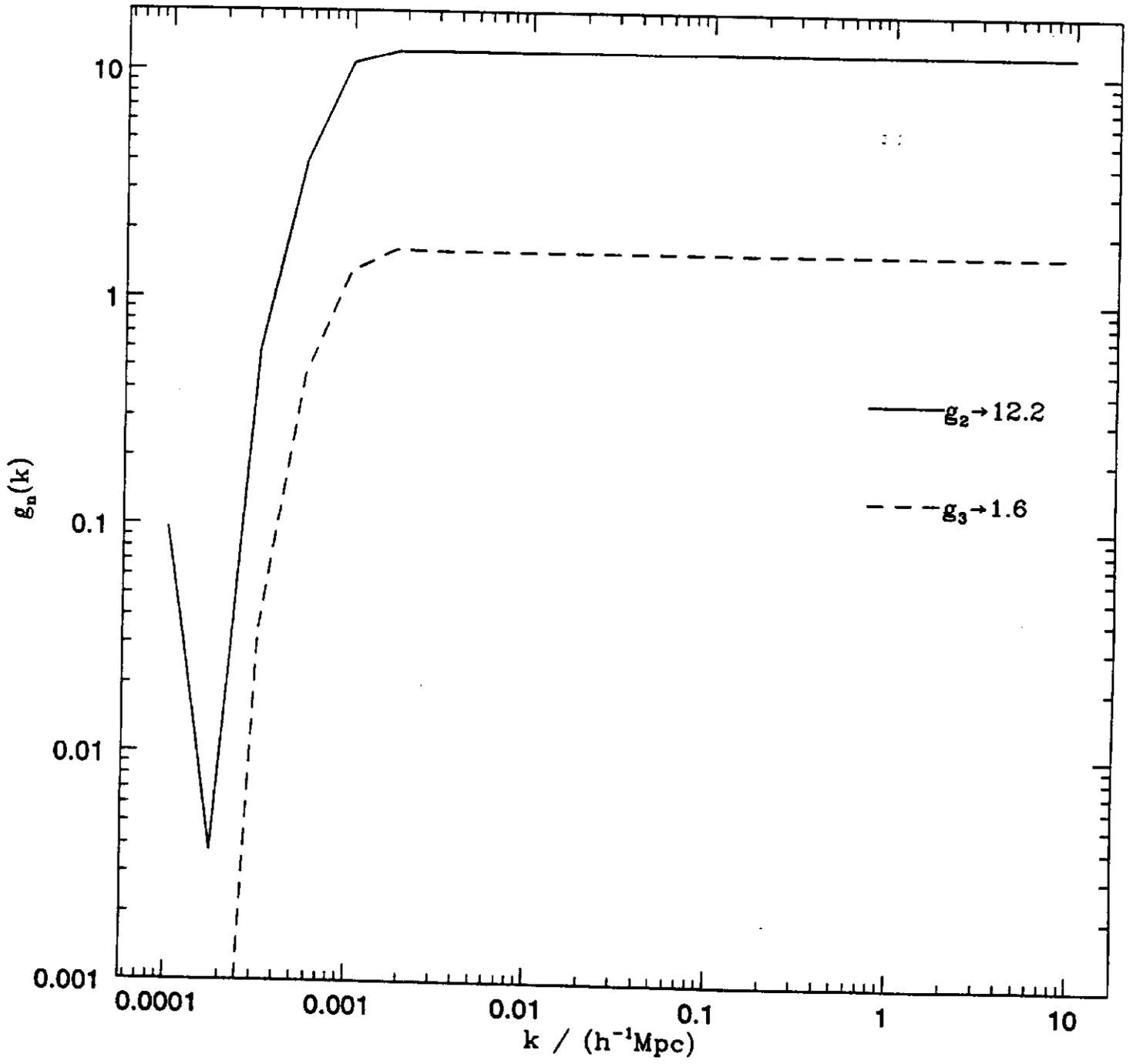


FIGURE 1.

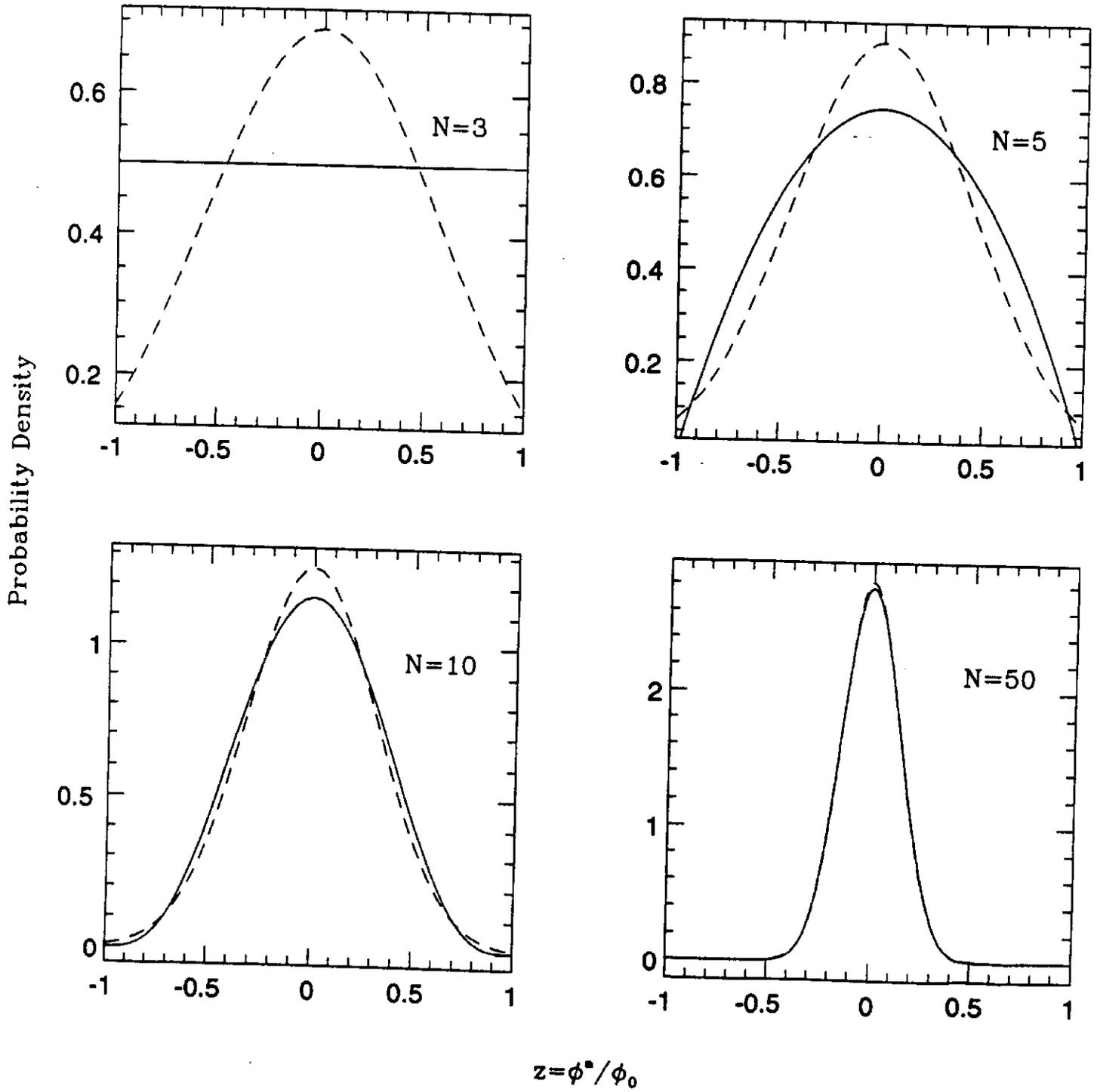


FIGURE 2.