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Anomalous Dimensions of High Twist Operators in QCD at $N \to 1$ and large Q^2

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Abstract

The anomalous dimensions of high-twist operators in deeply inelastic scattering (γ_{2n}) are calculated in the limit when the moment variable $N \to 1$ (or $x_B \to 0$) and at large Q^2 (the double logarithmic approximation) in perturbative QCD. We find that the value of $\gamma_{2n}(N-1)$ in this approximation behaves as $\frac{N_c \alpha_S}{\pi(N-1)} n^2 (1 + \frac{\delta}{3}(n^2-1))$ where $\delta \approx 10^{-2}$. This implies that the contributions of the high-twist operators give rise to an earlier onset of shadowing than was estimated before. The derivation makes use of a Pomeron exchange approximation, with the Pomerons interacting attractively. We find that they behave as a system of fermions.

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1. Introduction

To convey the main goal and result of this paper we start by recalling the principal steps of the theoretical approach to deep-inelastic scattering. We first introduce the moments of the deep-inelastic structure function,

$$M(\omega, r) = \int_0^1 dx_B\, x_B^{N-1}\, x_B G(x_B, q^2) = \int_0^\infty dy\, e^{-\omega y}[x_B G(x_B, q^2)], \qquad (1.1)$$

where $\omega = N - 1$, $y = \ln(1/x_B)$ and $r = \ln(q^2/q_0^2)$, x_B denoting the Bjorken scaling variable, and q_0^2 an IR cut-off. Each moment is given through a Wilson Operator Product Expansion (OPE) in the form

$$\begin{aligned} M(\omega, r) = C_2(\omega, r)\langle p|O^{(2)}|p\rangle + &\frac{1}{Q^2} C_4(\omega, r)\langle p|O^{(4)}|p\rangle + \ldots \\ &+ \frac{1}{Q^{2i-2}} C_{2i}(\omega, r)\langle p|O^{(2i)}|p\rangle + \ldots, \end{aligned} \qquad (1.2)$$

where C_i is the coefficient function and $\langle p|O^{(2i)}|p\rangle$ denotes generically the expectation value of a twist $2i$ operator in a proton state (see [1] for details). In practice, one usually neglects all high twist contributions (i.e. all terms in (1.2) beyond the first), by assuming that they are all supressed at large values of Q^2 due to the factor of Q^{-2i} in front. It is well known from renormalization group arguments that a coefficient function C_{2i} behaves as

$$C_{2i}(\omega, r) \sim e^{\gamma_{2i}(\omega) r}, \qquad (1.3)$$

where γ_{2i} is the anomalous dimension of the twist $2i$ operator [1]. The anomalous dimension of the leading twist contribution can be calculated using the Gribov-Lipatov-Altarelli-Parisi (GLAP) equation and is equal to

$$\gamma_2(\omega) = \frac{N_c \alpha_S}{\pi \omega}. \qquad (1.4)$$

The specific contribution to the anomalous dimensions of high-twist operators originating from the exchange of n 'leading twist ladders' in the t-channel was found in the GLR paper [2]. The result was

$$\gamma_{2n}(\omega) = n\gamma_2(\frac{\omega}{n}). \qquad (1.5)$$

[1] for simplicity we consider here the case of a fixed α_S.

Briefly, to illustrate the above statement, consider the twist four contribution. The two-ladder exchange leads to the following contribution in this case

$$C_4(\omega, r)\langle p|O^{(4)}|p\rangle = \int \frac{d\omega'}{2\pi i} C_2(\omega', r) C_2(\omega - \omega')\langle p|O^{(4)}|p\rangle \sim \int d\omega' e^{\gamma_2(\omega-\omega')r+\gamma_2(\omega')r}. \tag{1.6}$$

This integral has a saddle point at $\omega' = \omega/2$ so $C_4 \sim \exp(2\gamma_2(\frac{\omega}{2})r)$. Thus $\gamma_4 = 2\gamma_2(\frac{\omega}{2})r$.

Recently Bartels on the one hand [3] and Levin, Ryskin and Shuvaev on the other [4] performed the next step in understanding the high twist contribution to (1.1). Both groups calculated the anomalous dimension of the twist four gluon operator, each using quite different techniques. The value of the anomalous dimension was found to be

$$\gamma_4(\omega) = 2\gamma_2(\frac{\omega}{2})[1+\delta] = \frac{4N_c\alpha_S}{\pi\omega}[1+\delta], \tag{1.7}$$

with $\delta = \mathcal{O}((N_c^2-1)^{-2}) \approx 10^{-2}$ small.

The most important lesson to learn from this calculation is the fact that one *cannot* trust the GLAP evolution equation in the region of small ω (or, equivalently, large $\ln(1/x_B)$). Indeed for ω smaller than some ω_{cr} the twist four contribution becomes larger than the leading twist one. The value of ω_{cr} can be found from the equation

$$\gamma_2(\omega_{cr}) = -1 + \gamma_4(\omega_{cr}). \tag{1.8}$$

The same conclusion could be arrived at using the GLR approach but in [3,4] this statement was proved for the whole set of Feynman diagrams instead of just the two-ladder contribution that the GLR approach takes into account.

In this paper we will calculate the anomalous dimension for an arbitrary high twist operator, beyond the result (1.5). The operators in question are so called gluonic Quasi-Partonic Operators (QPO's), which were introduced in [5] and have been studied in detail in [6]. Some of their properties relevant to this paper are the following: 1) the twist of these operators coincides with the number of gluonic fields they contain; 2) under a scale change a QPO can only transform into a QPO or in operators that can be expressed in terms of QPO's using equations of motion. The transformation from other operators into QPO's is possible, but not vice versa. Thus, such a transformation cannot change the value of the anomalous dimension; 3) the evolution equation for such operators looks like a Faddeev-type equation with a pair-like interaction (with the number of particles conserved), the kernels of which are the same as for twist two operators. The analogy with the Faddeev-equation is very important for us since the method by which we will arrive at these anomalous dimensions has also been used in the nonrelativistic three body problem (for which the Faddeev equation was originally derived): we try to extract the resonance-like interaction in the two gluon channel first and subsequently take into account the interaction between such 'resonances'. The so-called Pomeron (colorless ladder for two gluon exchange in the t-channel) plays the rôle of such a resonance in our calculation. In refs. [3,4] it was shown that the value of the twist four anomalous dimension in the double logarithmic approximation is determined mainly by the exchange, and interaction, of two colourless gluon-'ladders' (Pomerons) in the t-channel, and that the interaction between such ladders is small since it is proportional to $1/(N_c^2-1)$. This indicates that our method of taking into account the two gluon interaction first, creating a Pomeron, is right, since it is the correct approach to the problem at least as $N_c \to \infty$. The solution to the problem at finite N_c we present in this paper. The above observations considerably simplify the problem and will enable us to reduce it to solving the Nonlinear Schrodinger Equation for n Pomerons in the t-channel.

The paper is organized as follows: in section 2 we consider in detail the calculation of the twist four anomalous dimension in the two-Pomeron approximation. We can find the value of γ_4 by 'summing' all diagrams in an explicit way and we will use this concrete example to introduce all notations and to illustrate all further steps in finding the value of $\gamma_{2n}(\omega)$. In section 3 we reinterpret the calculation of $\gamma_{2n}(\omega)$ in terms of a two-dimensional theory describing the interaction of n nonrelativistic particles. In section 4 we find the energy of the ground state of this theory, which corresponds precisely to $\gamma_{2n}(\omega)$. In the conclusions we summarize our results and discuss outstanding problems. In Appendix A we discuss some technical details.

2. The anomalous dimension of the twist 4 operator in the two-Pomeron approximation

In this section we will derive the anomalous dimension of the twist 4 operator from diagrams with four gluon exchange in the t-channel. Our method consists of first pairing the gluons up into 'Pomerons', and subsequently calculate the interaction between the latter.

The contributions to the anomalous dimensions of the twist four operator due to Pomeron exchange can be separated into two cases: in one the Pomerons do not interact while being exchanged, in the other they do. We will discuss these cases in turn, but begin by discussing just the two-gluon (one Pomeron) exchange to set the stage. In addition, this exchange is a building block for the four gluon case.

2.1. *The Pomeron in the double log approximation* (DLA).

We start from the structure function of a two-gluon colorless state (Pomeron) in the DLA of perturbative QCD (pQCD), see Fig.1. We will generically denote this function by $F(x, q^2)$. This is a step-up to our discussion of the four-gluon state later on in this section. We discuss this function for two cases, distinguished by the variable q_t, the transverse momentum along the ladder. In the first $q_t = q_{1t} - q_{2t} = q_{0,1t} - q_{0,2t} = 0$ (see Fig.1 for notation). The two-gluon structure function can be calculated directly from the GLAP [7] evolution equation, with kernel

$$P(z) = \frac{N_c \alpha_S}{\pi z} \equiv \frac{\overline{\alpha}_S}{z}. \tag{2.1}$$

where we have dropped terms that are non-singular as $z \to 0$. In DLA the GLAP evolution equation can be written in the form

$$F(x, \hat{q}_{1t}^2 = \hat{q}_{2t}^2) = \overline{\alpha}_S \int_x^1 \frac{dz}{z} \int_0^{\ln \hat{q}_1^2 = \ln \hat{q}_2^2} dr F(\frac{x}{z}, r). \tag{2.2}$$

where the hat indicates that the momentum has been divided by the corresponding lower cut-off momentum, e.g. $\hat{q}_{1t} \equiv q_{1t}/q_{0,1t}$, etc. The choice of IR cut-off of the GLAP evolution equation is arbitrary. In Fig.1 we depicted the cut-off to lie somewhere along the ladder, but we could have also chosen it be close to the 'blob', in which case we would have $q_{0,1t} \simeq q_{0,2t} \simeq 1/R_h$, very small. Eq. (2.2) can be simplified further by performing two Mellin transforms, one with respect to $y = \ln(1/x)$, as in (1.1), the other with respect to r,

$$F(\omega, f) = \int_0^\infty dr\, e^{-fr} F(\omega, r). \tag{2.3}$$

In ω, f representation eq. (2.2) has the form

$$\omega f = \overline{\alpha}_S. \tag{2.4}$$

We define $G_2(\omega, f)$ by

$$G_2(\omega, f) = \frac{1}{\omega f - \overline{\alpha}_S}. \tag{2.5}$$

It gives the Green function of the GLAP equation in DLA, which satisfies $G_2(x, q_t^2) = \theta\big(\ln(\hat{q}_t^2)\big)$ at $x = 1$.

In the second case we have $q_t \neq 0$, in other words momentum is transferred along the ladder. We will encounter this situation a little further on. Within DLA, the contributions come from two regimes: 1) $q_t^2 << q_{1t}^2 \approx q_{2t}^2$ and 2) $q_t^2 \approx q_{0,1t}^2 >> q_{0,2t}^2$, with $q_{0,2t}^2 \simeq 1/R_h^2$. In spite of the fact that now $q_{1t} \neq q_{2t}$ the two gluon structure function in essence still depends only on one virtuality in these regimes, viz. the smallest one. E.g. for $|q_{2t}| < |q_{1t}|$, we have, for both regimes

$$F(x, \hat{q}_{1t}^2, \hat{q}_{2t}^2) = \frac{(q_{0,1t} \cdot q_{0,2t})}{q_{0,1t}^2} F(x, \hat{q}_{2t}^2) \tag{2.6}$$

For values of q_t other than the ones discussed in this subsection one does not find a large logarithmic contribution. The derivation of and all details related to eq. (2.6) can be found in refs. [2–4,7]. We now go on to discuss the four-gluon structure function.

2.2. *Two-Pomeron case* (2P).

Let us consider now the simplest diagram that gives a contribution to the four-gluon structure function and thus to γ_4, namely the exchange of two 'leading twist' ladders (see Fig.2). In DLA we can neglect the dependence of the deep-inelastic scattering structure function on the momentum q_t (see [2] for the relevant discussion), and the contribution of this diagram can be written in the form

$$\begin{aligned} C_4(\omega, r = \ln(Q^2)) \langle p|O^{(4)}|p\rangle \Big|_{2P} &= \int \frac{d\omega'}{2\pi i} M(\omega', r) M(\omega - \omega', r) < S > \\ &= \int df e^{fr} G_2(\omega', f') G_2(\omega - \omega', f - f') \frac{d\omega' df'}{2\pi i} < S > . \end{aligned} \tag{2.7}$$

where $< S >$ stands for the integration over q_t of the matrix element that describes the emission of 4 gluons from the nucleon. Substituting eq. (2.5) in (2.7) and performing the integrals over ω' and f' by contour integration we get

$$C_4(\omega, r) \langle p|O^{(4)}|p\rangle \Big|_{2P} = \int \frac{df}{2\pi i} \frac{e^{fr}}{\omega f \sqrt{1 - \frac{4\overline{\alpha}_S}{\omega f}}} < S > . \tag{2.8}$$

The contour of integration over f is located to the right of all singularities in this variable. From (2.8) one can easily see that the main contribution comes from $f \to \frac{4\overline{\alpha}_S}{\omega}$ and it is proportional to $\exp(\frac{4\overline{\alpha}_S}{\omega} r)$. Thus the contribution to the value of the anomalous dimension from the diagram in Fig.2 is

$$\gamma_4 = \frac{4\overline{\alpha}_S}{\omega}. \tag{2.9}$$

This will turn out to be the most significant contribution to γ_4.

An alternative method of derivation consists of using the saddle point method for the f' integral, as was done for the ω' integration in (1.6). Starting with (2.7), applying an inverse Mellin transform with respect to ω and performing the ω, ω' integrals by contour integration one obtains

$$C_4(y,r) \approx \int \frac{df\,df'}{f'(f-f')} e^{fr} e^{(\frac{1}{f'}+\frac{1}{f-f'})\overline{\alpha}_S y}. \tag{2.10}$$

One can now do the f' by the saddle point method, yielding the saddle point $f' = f/2$. For n ladders one would obtain

$$f' = \frac{f}{n}. \tag{2.11}$$

Changing variables $f \to 4\overline{\alpha}_S/\omega$ then yields

$$C_4(y,r) \sim \int d\omega \;\; e^{\frac{4\overline{\alpha}_S}{\omega}r+\omega y}, \tag{2.12}$$

and thus we find again $\gamma_4 = 4\overline{\alpha}_S/\omega$.

2.3. *Pomeron interaction* (PI).

However, although (2.9) is the most significant contribution to γ_4, it is not the full answer. As was shown in ref. [3,4] the Pomeron-Pomeron interaction crucially changes the value of the anomalous dimension. In this subsection we want to understand this statement, and to that end let us consider the diagram in Fig.3, which displays this interaction. We will discuss here the contribution to $C_4 < p|O^{(4)}|p >$ from this diagram. All notation is explained in the figure, where all y's are rapidities and all other symbols are transverse momenta. (E.g. k_i is the transverse momentum at rung i in the ladder in the left upper part of the diagram.)

Clearly, there are two distinct parts to this two ladder diagram: the top and bottom half, and we will discuss the contribution from this diagram accordingly. The obvious distinction from Fig.2 is that the two ladders in the top half are 'interchanged' in the bottom half. The main contribution from this diagram in DLA is when all momenta in the top half (above the dashed line) of the diagram are larger than all momenta in the bottom half, yet smaller than Q^2. Note that while there is momentum transferred along the ladders in the top half, that is not the case in the bottom half.

As far as the momenta in the bottom half are concerned, we assume $l_1^2 >> l_2^2$. The case $l_2^2 >> l_1^2$ gives precisely the same answer, so we only discuss the former. The IR cutoff momenta $l_{0,1}, l_{0,2}$ are small, $O(1/R_h)$. Each individual (piece of a) ladder behaves as described in subsection 2.1, either with (top) or without (bottom) momentum transfer along the ladder. In this sense the two-gluon structure function is a building block for the four-gluon structure function presently under study. The contribution of the diagram in Fig.3 is

$$C_4(Y,Q^2)\langle p|O^{(4)}|p\rangle\Big|_{\rm PI} = \tag{2.13}$$
$$\frac{1}{2}\frac{1}{N_c^2-1}2\overline{\alpha}_S\int_0^Y dy_1\int_{l_{0,1}^2}^{Q^2}\frac{dl_1^2}{l_1^2}F(Y-y_1,r_Q-r_{l_1})^2$$
$$\times\Big\{\int_{l_{0,1}^2}^{l_1^2}\frac{dl_1'^2}{l_1'^2}\int_0^{y_1}dy_1'\tilde{F}(y_1',r_{l_1'})\int_{l_{0,2}^2}^{l_1^2}\frac{dl_2^2}{l_2^2}\int_0^{y_1}dy_2\tilde{F}(y_2,r_{l_2})$$
$$+\int_0^{y_1}dy_2\int_0^{y_2}dy_1'\int_{l_{0,1}^2}^{l_1^2}\frac{dl_1'^2}{l_1'^2}\tilde{F}(y_1',r_{l_1'})\int_{l_{0,2}^2}^{l_1^2}\frac{dl_2^2}{l_2^2}\int_0^{y_1}dy_2\tilde{F}(y_2,r_{l_2})\Big\}$$

where $F(y,r)$ is the two-gluon structure function from subsection 2.1, and $\tilde{F}(y,r)=\frac{\partial}{\partial y}\frac{\partial}{\partial r}F(y,r)$. As under (1.1) we define $r_{l_1}=\ln(l_1^2/l_{1,0}^2)$, etc.

We will now explain the construction of the contribution in (2.13) piece by piece, starting with the prefactors. First, due to the fact that there is momentum transfer along ladders in the top half of the diagram, and we thus have to employ (2.6), we need to take into account the momentum factor in eq. (2.6) and carefully integrate this factor over the angle between l_{2t} and l_{1t}. It gives an additional factor 1/2 in front of the whole expression. Note that there is no such factor for the bottom half. Second the color factor in front of the diagram of Fig.3 is smaller than for the diagram of Fig.2, by a factor of $1/(N_c^2-1)$, see Figs.4a and 4b. Third, there is a factor of 2 from the bottom half due to the combinatorics of connecting the rungs to the 'vertical lines'. Fourth, the explicit factor of $\overline{\alpha}_S$ means that even at lowest order, we have 1 rung for this contribution. Not having this would mean that the lowest order corresponds to the lowest order for the two-ladder exchange contribution discussed in the previous subsection. Thus we avoid double counting.

We remark that, because we work in DLA, besides the momenta, also all rapidities are strongly ordered and decrease from top to bottom along ladders. The factors on the first line of (2.6) correspond to the two ladders in the top half of Fig.3. The transverse momenta here range from Q^2 down to l_{1t}^2, the momenta at the boundary of the top and bottom half. The expression in curly brackets corresponds to the bottom half of Fig.3, the two terms within these brackets corresponding to the case $y_2<y_1$, and the second to $y_2>y_1$ (we have interchanged integration variables y_2 and y_1 for this term). Eq. (2.13) then is the contribution of Fig.3 summed over all possible values of y_1 and l_1^2.

We now need to perform the integrals in (2.13). This is discussed in the appendix.

The final answer is

$$C_4(\omega,r)\langle p|O^{(4)}|p\rangle\Big|_{\mathrm{PI}} = \delta\cdot\int\frac{e^{fr}df}{2\pi i}\frac{1}{\omega f\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}}}\{\frac{1}{\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}}}-1\} < S > \tag{2.14}$$

where $\delta = 1/(N_c^2-1)$. Next, we will add the contributions from the previous two subsections to all orders in the Pomeron coupling.

2.4. *The value of the anomalous dimension.*

To get the value of the anomalous dimension we need to sum up all diagrams with Pomeron-Pomeron interaction. For two Pomerons is it easy to get the answer, due to the fact that we only need to sum all diagrams of Fig.5. They form a geometric series, the first two terms of which are given in (2.8) and (2.14). Using this, the sum can be written in the form

$$C_4(\omega,r)\langle p|O^{(4)}|p\rangle = \int\frac{e^{fr}df}{2\pi i}\frac{1}{\omega f\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}}(1-\delta[\frac{1}{\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}}}-1])} < S > \tag{2.15}$$

from which we see that there is a pole to the right of $\omega f = 4\overline{\alpha}_S$, namely at

$$1-\delta[\frac{1}{\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}}}-1] = 0, \quad f_0 = \frac{4\overline{\alpha}_S}{\omega}(1+\delta^2). \tag{2.16}$$

Near this singularity the Green function has the form

$$\frac{2\delta}{\omega f - 4\overline{\alpha}_S(1-\delta^2)}. \tag{2.17}$$

Directly from eq. (2.17) we can get the value of the anomalous dimension

$$\gamma_4 = \frac{4\overline{\alpha}_S(1+\delta^2)}{\omega} \tag{2.18}$$

The rather complicated form of the term in brackets in eq.(2.15) originated in the sum of two terms in eq.(2.13). However, none of these complications are essential in the vicinity of the rightmost pole. Indeed we can consider both terms in eq. (2.13) as equal and write down the contributions of the diagram in Fig.3 in the following way

$$\lambda\int\frac{e^{fr}df}{2\pi i}\frac{1}{(\omega f\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}})^2} \tag{2.19}$$

where $\lambda = 4\overline{\alpha}_S/(N_c^2-1) = 4\overline{\alpha}_S\delta$. This expression has a transparent meaning, namely, it describes the interaction between two particles with the Green function of eq.(2.5).

This interaction is attractive, even though $\lambda > 0$. This is explained in the next section, under eq.(3.3). The sum of the diagrams of Fig.5 gives then

$$\frac{1}{\omega f\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}}-\lambda} \tag{2.20}$$

It is easy to check that in the vicinity of $\omega f = 4\overline{\alpha}_S(1+\delta^2)$ this equation gives the same pole as eq.(2.17).

3. The effective two-dimensional theory.

To calculate the anomalous dimensions of higher twist operators we develop here an approach via an effective two-dimensional theory, noting that the rescattering of Pomerons does not change the number of Pomerons (see Fig.6). It means that in fact we are dealing with a quantum mechanical theory. To specify it, let us note that in (ω, f) representation the 'Pomeron' propagator of eqn. (2.5) can be treated as the propagator of a two-dimensional particle which is written in light-cone variables. The variables ω and f play the role of momenta k_+ and k_-, and $\sqrt{\overline{\alpha}_S}$ the role of mass. The total 'momentum' is conserved in interactions. As we showed above, the leading two-Pomeron pole is located near the value of the branchpoint $4\overline{\alpha}_S$, whereas the Pomeron-Pomeron interaction $\lambda = 4\overline{\alpha}_S\delta$ can be considered small compared to the Pomeron mass. This will enable us to use a nonrelativistic approach to the n-Pomeron interaction problem.

To specify this theory let us consider an arbitrary diagram with n Pomerons in the t-channel. In the ω, f representation the propagator for n Pomerons can be written as

$$(\omega - \sum_k \frac{\overline{\alpha}_S}{f_k})^{-1}. \tag{3.1}$$

The sum $\sum_k^n f_k = f$ is conserved throughout all diagrams. Eq. (3.1) can be obtained by integrating eq. (2.5) for each Pomeron in the t-channel separately over ω_k ($\omega_k = \omega_{k,0} = \frac{4\overline{\alpha}_S}{f_k}$). Now let us introduce a new variable Δ_i such that

$$(1+i\Delta_k)\frac{f}{n} = f_k. \tag{3.2}$$

The expansion around f/n looks very natural since we believe that, due to the smallness of the interaction $\lambda = 4\overline{\alpha}_S/(N_c^2-1)$ at large N_c the dominant contribution to f_k in the integral comes still from the saddle point approximation value $f_k = f/n$ (2.11). Assuming that $\Delta_k << 1$ and taking into account that $\sum_k^n \Delta_k = 0$ we can get instead of (3.1) the following expression for the propagator

$$(\omega - \frac{\overline{\alpha}_S n^2}{f} + \frac{\overline{\alpha}_S n}{f} \sum_k^n \Delta_k^2)^{-1}. \tag{3.3}$$

Let us interpret this propagator as $(H - \mathcal{E})^{-1}$ where H is some nonrelativistic Hamiltonian $p^2/2m$ and $\mathcal{E}_n$ one of its eigenvalues. Here $\mathcal{E}_n = -\omega + (\overline{\alpha}_S/f)n^2$ and $m = f/2\overline{\alpha}_S n$. One should note, however the essential difference between the usual quantum mechanical system and the Pomeron one. Due to opposite sign of ω and $\mathcal{E}_n$, the pole corresponding to the bound state of n-particles will be not to the left, as is usual, but to the right of the multiparticle-threshold branchcut, with branchpoint $n\overline{\alpha}_S$. This sign is also the reason that in (2.20) $\lambda > 0$ corresponds to an attractive interaction. As explained in section 2, we need the rightmost singularity in f, which translates into the lowest value of $\mathcal{E}_n$. Thus, our task is to formulate the theory specified by H and determine its groundstate energy, the energy of its n-particle bound state.

We can now write down for the n-Pomeron system the following Hamiltonian

$$H = \frac{1}{2m} \sum_k \Delta_k^2 a_k^\dagger a_k + \lambda \sum_k a_k^\dagger a_k^\dagger a_k a_k, \tag{3.4}$$

where $a_k^\dagger, a_k$ are bosonic creation- and annihilation operators. This corresponds to the coordinate space form $\int dx\, \frac{1}{2m} \partial_x \psi^\dagger \partial_x \psi + \lambda : \psi^\dagger \psi^\dagger \psi \psi :$, or, after rescaling x:

$$H = \int dx\, \partial_x \psi^\dagger \partial_x \psi + \bar{\lambda} : \psi^\dagger \psi^\dagger \psi \psi :, \quad \bar{\lambda} = \frac{\lambda}{2m}. \tag{3.5}$$

4. Solution with Bethe Ansatz.

The Hamiltonian (3.5) is well known, and methods have been developed to find its spectrum (8), for certain boundary conditions. Our task is to find the energy of the ground state for the n-Pomeron system in the t-channel, which corresponds to finding the simultaneous eigenfunction and eigenvalue of the Hamiltonian in (3.5), and the number operator

$$N = \int dx\, : \psi^\dagger(x) \psi(x) : . \tag{4.1}$$

In other words, we would like to solve

$$H|\psi_n\rangle = \mathcal{E}_n |\psi_n\rangle \ , \ N|\psi_n\rangle = n|\psi_n\rangle. \tag{4.2}$$

The standard procedure of solution is given in [8], but we repeat here the outline for completeness.

The Bethe ansatz method consists of parametrizing the eigenfunction as

$$|\Psi_n\rangle = \frac{1}{\sqrt{n!}} \int d^n z \chi_n(z_1, \ldots, z_n) \psi^\dagger(z_1) \ldots \psi^\dagger(z_n)|0\rangle, \tag{4.3}$$

where $|0\rangle$ is the Fock vacuum defined by

$$\psi(x)|0\rangle = 0 \; ; \; \langle 0|\psi^\dagger(x) = 0 \; ; \; \langle 0|0\rangle = 1 \tag{4.4}$$

For the function χ_n, eq. (4.2) can be rewritten in the form

$$\mathcal{H}_n \chi_n = \mathcal{E}_n \chi_n, \tag{4.5}$$

where

$$\mathcal{H}_n = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2} + 2\bar{\lambda} \sum_{n \geq k > j \geq 1} \delta(z_k - z_j). \tag{4.6}$$

Eq. (4.5) can be solved now as follows. From (4.6) we see that for the sector $z_1 < z_2 < \ldots < z_n$ χ_n is an eigenfunction of the free Hamiltonian

$$\mathcal{H}_n^0 = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2} \quad , \quad \mathcal{H}_n^0 \chi_n^0 = \mathcal{E}_n \chi_n^0 \tag{4.7}$$

The interaction term in (4.6) contributes only to the boundary conditions

$$\left(\frac{\partial}{\partial z_{j+1}} - \frac{\partial}{\partial z_j}\right)\chi_n = \bar{\lambda}\chi_n \quad , \quad (z_{j+1} = z_j + 0, \; j = 1 \ldots n). \tag{4.8}$$

The eigenfunctions in (4.7) are obvious:

$$\chi_n^0 = \det e^{i\kappa_j z_k}, \tag{4.9}$$

with eigenvalue

$$\mathcal{E}_n = -\sum_{j=1}^{n} \kappa_j^2. \tag{4.10}$$

The eigenfunction χ_n has been found in [8] and is

$$\chi_n = \mathrm{Const}\, \Pi_{j>k}\left(\frac{\partial}{\partial z_{j+1}} - \frac{\partial}{\partial z_j} + \bar{\lambda}\right) \det e^{i\kappa_j z_k}. \tag{4.11}$$

This χ_n satisfies both (4.7) and (4.8). (see [8] for details). The determinant can be written as a sum over all permutations P of the number $1 \ldots n$:

$$\det e^{i\kappa_j z_k} = \sum_{\mathrm{P}} (-)^{[\mathrm{P}]} e^{i \sum_n \kappa_{\mathrm{P}_k} z_k}, \tag{4.12}$$

where [P] is the parity of the permutation. Substituting (4.12) in (4.11) and performing all derivative operations one obtains

$$\chi_n = \text{Const} \sum_{\text{P}} (-)^{[\text{P}]} \Pi_{j>k} (\kappa_{\text{P}_j} - \kappa_{\text{P}_k} - i\bar{\lambda}\varepsilon(z_j - z_k)) e^{i \sum_n \kappa_{\text{P}_k} z_k} \tag{4.13}$$

where $\varepsilon(\kappa)$ is the sign-function. To determine the values of κ_i we have to invoke a boundary condition for our n-particle system. We assume that the ground state for this system occurs when the n-particle wave function falls down exponentially when any $z_i \to \infty$ (i.e. for the n-particle bound state), with the additional condition

$$\sum_{i=1}^{n} \kappa_i = 0. \tag{4.14}$$

which means that we are in the center of mass of the n-particle system. The solution was already found in [9] and is simply:

$$\begin{aligned} \kappa_k^n =& \left(\frac{2k-n-1}{2}\right) i\bar{\lambda} \\ k =& 1, 2, \ldots, n \end{aligned} \tag{4.15}$$

Substituting (4.15) in (4.10) we can calculate

$$\mathcal{E}_n = \sum_{j=1}^{n} \mathcal{E}_n^j = -\sum_{j=1}^{n} (\kappa_j^n)^2 = \frac{\bar{\lambda}^2}{12}(n^3 - n). \tag{4.16}$$

So finally the value of the anomalous dimension for the twist $2n$ operator is

$$\gamma_{2n} = \frac{\overline{\alpha}_S n^2}{\omega}\left(1 + \frac{\delta^2}{3}(n^2 - 1)\right), \tag{4.17}$$

where $\delta = (N_c^2 - 1)^{-1}$. This is our main result.

Let us illustrate (4.15) by giving two examples. First, we discuss the two-Pomeron case (i.e. the anomalous dimension of the twist four operator). Since we have already considered this case separately, it is instructive to see if we obtain the same result as in section 2 from the Bethe ansatz. The eigenfunction (4.13) for the case $n = 2$ is equal to

$$\chi_2 = (\kappa_2 - \kappa_1 - i\bar{\lambda}) e^{i(\kappa_1 z_1 + \kappa_2 z_2)} - (\kappa_1 - \kappa_2 - i\bar{\lambda}) e^{i(\kappa_2 z_1 + \kappa_1 z_2)}\ ,\ z_1 < z_2. \tag{4.18}$$

Using the constraint $\kappa_1 + \kappa_2 = 0$, we can write

$$\chi_2 = (\kappa_2 - \kappa_1 + i\bar{\lambda}) e^{i\kappa_1 (z_2 - z_1)} - (\kappa_1 - \kappa_2 + i\bar{\lambda}) e^{-i\kappa_1 (z_2 - z_1)}\ ,\ z_1 < z_2. \tag{4.19}$$

From this equation we derive that $\kappa_1 - \kappa_2 = 2\kappa_1 = -i\bar{\lambda}$ if we demand exponential fall-off of the wave function as $z_2 - z_1 \to \infty$. Substituting $2\kappa_1 = -i\bar{\lambda}$ in (4.18) and normalizing the wavefunction we find

$$\chi_2 = \frac{\bar{\lambda}}{\sqrt{2|\bar{\lambda}|}}\, e^{-|\bar{\lambda}||z_2 - z_1|/2}. \tag{4.20}$$

With (4.16) and the prescription below (3.3) we recover our old result for $\gamma_4(\omega)$

$$\gamma_4(\omega) = \frac{4\overline{\alpha}_S}{\omega}(1+\delta^2). \tag{4.21}$$

Next we discuss the three-Pomeron case (i.e. the anomalous dimension of the twist six operator). We can write the wavefunction $\chi_3(z_1, z_2, z_3)$ according to (4.13), as follows

$$\begin{aligned}
\chi_3 \quad \propto \quad & e^{ix_3\kappa_{31}+ix_2\kappa_{21}}(\kappa_{32}-i\bar{\lambda})(\kappa_{31}-i\bar{\lambda})(\kappa_{21}-i\bar{\lambda}) \\
& + e^{-ix_3\kappa_{21}+ix_2\kappa_{32}}(\kappa_{31}+i\bar{\lambda})(\kappa_{21}+i\bar{\lambda})(\kappa_{32}-i\bar{\lambda}) \\
& + e^{-ix_3\kappa_{32}-ix_2\kappa_{31}}(\kappa_{21}-i\bar{\lambda})(\kappa_{32}+i\bar{\lambda})(\kappa_{31}+i\bar{\lambda}) \\
& + e^{ix_3\kappa_{21}+ix_2\kappa_{31}}(\kappa_{32}+i\bar{\lambda})(\kappa_{21}-i\bar{\lambda})(\kappa_{31}-i\bar{\lambda}) \\
& + e^{ix_3\kappa_{32}-ix_2\kappa_{21}}(\kappa_{31}-i\bar{\lambda})(\kappa_{21}-i\bar{\lambda})(\kappa_{21}+i\bar{\lambda}) \\
& + e^{-ix_3\kappa_{31}-ix_2\kappa_{32}}(\kappa_{21}+i\bar{\lambda})(\kappa_{31}+i\bar{\lambda})(\kappa_{32}+i\bar{\lambda})
\end{aligned} \tag{4.22}$$

where $\kappa_{ij} = \kappa_i - \kappa_j$. Here we have factored out the center of mass coordinate $r = (z_1 + z_2 + z_3)/3$, ordered the imaginary parts of the κ's ($\Im\kappa_3 > \Im\kappa_2 > \Im\kappa_1$) and used the relative coordinates $x_i = z_i - r$, $i = 1, 2, 3$. We also employed the relation $\kappa_1 + \kappa_2 + \kappa_3 = 0$.

From this example and the previous one we can easily understand why (4.15) is the solution. Indeed, let us assume that $x_3 >> x_2$. The terms with a plus sign for the x_3 give a rising exponent. Thus the coefficient in front of such a term must be zero. The solution (4.15) for the case $n = 3$ accomplishes just that. The wavefunction for this case is then

$$\chi_3 = \bar{\lambda}\, e^{2|x_3|\bar{\lambda}+|x_2|\bar{\lambda}} \quad , \quad \bar{\lambda} < 0. \tag{4.23}$$

The above generalizes to arbitrary n [9]. We conclude this section with the observation of a remarkable feature of the solution of the n-Pomeron exchange amplitude: although we started out with a completely bosonic system, we find a non-degenerate momentum spectrum, implying that in this sense the Pomerons behave as fermions. Fig.7 shows the one particle levels in our system of interacting bosons (Pomerons) and it is easy to see that the direction of motion plays a role in the spin of the fermion since each level has two Pomerons moving in different directions. We believe that the understanding of this property is another important result of this paper. We will discuss it further in the conclusions.

5. Conclusions.

Let us repeat our main assumptions which led us to the value of the anomalous dimension of eq. (4.17).

1) We assumed that only Pomeron-Pomeron interactions contribute to the value of γ_{2n}. The experience from the exact solution of the next-to-leading twist anomalous dimension taught us that other color states in the t-channel for four gluons, due to diagrams such as depicted in Fig.8, lead simply to a renormalization of λ. Such a renormalization can easily be accounted for by replacing λ with $\tilde{\lambda}$, taking the latter from [3,4].

2) The problem of the contribution of color states in the system of 6 or more gluons, which would induce a direct interaction between 3 and more Pomerons (see Fig.9) is still an open one.

3)We would like to stress again that we can trust the answer of (4.17) only if $\delta n^2/3 << 1$, which reflects our assumption that $\Delta_k << 1$.

Our calculated value of γ_{2n} leads to the following new insights.

Firstly, we cannot trust the GLAP evolution equation in the region of small x_B (or $\omega \to 0$), since the high-twist contributions rapidly become more important in the Wilson OPE than the leading twist one. Recall that the GLAP evolution equation can be used only to calculate the anomalous dimension of the leading twist contribution.

Secondly, to obtain the correct evolution equation in the region of small x_B we need to sum all high-twist contributions in the Wilson OPE. The nonlinear GLR equation is an example of an evolution equation that does take into account all high-twist contributions, but it should be noted that this equation assumed for the anomalous dimensions the simple value $\gamma_{2n} = \overline{\alpha}_S n^2/\omega$. Thus the result of the present paper shows that the GLR equation cannot be valid and could only have some numerical accuracy related to small values of δ^2. In other words, only in the limit $N_c \to \infty$ does the GLR equation correctly take into account the high-twist contributions which induce shadowing corrections.

Thirdly, the calculated value of γ_{2n} shows us that the theory with a large number of colors ($N_c \to \infty$) cannot be a good approximation to reality, since the high-twist contributions with $n > N_c$ are larger than the ones with $n < N_c$.

Finally, we found a remarkable regularity in the bosonic system of $2n$ gluons inside of the parton cascade: they behave in a certain sense as fermions. The direction of motion plays the role of spin here, and obeys Fermi statistics. The above property is the direct consequence of the two-dimensional character of our DLA theory. The nonrelativistic, fermionic nature of our system allows us to understand why the bosonic system does not collapse, in spite of the attractive forces acting betweens the bosons (Pomerons). This result, perhaps, indicates a way to a more systematic statistical description of the parton cascade in a situation with a large parton density.

Some open problems spawned by these considerations are the following. First, it should be understood what the restricted kinematical region is where the nonlinear GLR equation is sufficiently accurate in estimating the value of shadowing corrections. Second, we need a generalization of the GLR equation which includes the correct behavior of γ_{2n} when $\omega \to 0$. And finally, the calculated value of γ_{2n} shows that deviations from the GLAP equation in deep-inelastic scattering and other hard hadronic processes should enter earlier than was estimated in the framework of the GLR equation. This is perhaps encouraging in the search for experimental signals of shadowing corrections.

Appendix A

In this appendix we derive (2.14), starting from (2.13). We start by transforming this expression to ω, f space by applying $\int_{-\infty}^{\infty} dY \exp(-\omega Y) \int_{-\infty}^{\infty} dr_Q \exp(-f r_Q)$. Since the complete integral vanishes for negative Y and r_Q, we have extended the integrals down to $-\infty$ to facilitate our calculations. We now focus on the overall factor $F(Y - y_1, r_Q - r_{l_1})^2$ first, and use the representation

$$F(y,r) = \int \frac{d\omega' df'}{(2\pi i)^2} G_2(\omega', f') e^{\omega' y + f' r}, \tag{A.1}$$

with G_2 given in (2.5). Performing the Y and r_Q integrals (after a change of variables) we obtain, similarly to (2.7) ,(2.8), the factor

$$\frac{\overline{\alpha}_S}{(N_c^2-1)} \frac{1}{\omega f \sqrt{1 - \frac{4\overline{\alpha}_S}{\omega f}}}. \tag{A.2}$$

The remainder then takes the form

$$\int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dr_{l_1} e^{-\omega y_1 - f r_{l_1}} \int \frac{d\omega_1 df_1}{(2\pi i)^2} \int \frac{d\omega_2 df_2}{(2\pi i)^2}$$

$$(1 + \frac{\omega_2}{\omega_1}) \frac{e^{\omega_1 y_1 + f_1 r_{l_1}}}{\omega_1 f_1 - \overline{\alpha}_S} \frac{e^{\omega_2 y_1 + f_2 r_{l_1}}}{\omega_2 f_2 - \overline{\alpha}_S} \qquad \text{(A.3).}$$

Performing the integrals over y_1 and r_{l_1} yields a product of δ-functions: $\delta(\omega-\omega_1-\omega_2)\delta(f-f_1-f_2)$. Next one performs the ω_2 integral by closing around the pole $\overline{\alpha}_S/f_2$. Then using the δ-functions for the ω_1 and f_1 integrations one is left with

$$\int \frac{df_2}{2\pi i}\frac{\omega f_2}{\omega f_2-\overline{\alpha}_S}\frac{1}{\omega f_2(f-f_2)-\overline{\alpha}_S f}. \tag{A.4}$$

This integrand has poles in f_2 at $f_2^0=\overline{\alpha}_S/\omega$ and $f_2^{\pm}=f(1\pm\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}})/2$. The contour of f_2 runs to the right of f_2^0, but we can close it either to the left or to the right, since there is no longer an exponent to guide us. It is easy to see that in the limit of large ω the pole f_2^- migrates to f_2^0, so that we best close the contour on to the right, picking up only the f_2^+ pole. After some straightforward algebra we find that the result for the integral in (A.4) is

$$\frac{1}{2}\frac{1}{\overline{\alpha}_S}\Big(\frac{1}{\sqrt{1-\frac{4\overline{\alpha}_S}{\omega f}}}-1\Big). \tag{A.5}$$

Combining this with the result for the overall factor in (A.2), and multiplying by 2 to take into account the case $l_2^2>>l_1^2$, we arrive at (2.14).

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FIGURE CAPTIONS

Figure 1.

The two-gluon structure function $F(x, q_1^2, q_2^2)$. q_{1t} and q_{2t} are the transverse components of the gluon momenta q_1 and q_2, whereas $q_{0,1t}$ and $q_{0,2t}$ are cut-off momenta.

Figure 2.

The two-ladder contribution. Q^2 is the photon mass, q_0^2 is an IR cut-off. q_t is the transverse momentum along the ladder.

Figure 3.

A twist-four contribution to the gluon structure function. Here k_i, m_i, q_t, l_1 and l_2 are all transverse momenta, all 'y' 's are rapidities, and the IR cut-off momenta $l_{0,1}$, $l_{0,2}$ are both of order $1/R_h$.

Figure 4.

Fig. 4b shows that at the same order in α_S the two-ladder contribution has one power of $N_c^2 - 1$ more than the Pomeron interaction contribution in Fig. 4a. Again, the Pomerons are represented by gluon ladders. The wiggly lines also represent gluons.

Figure 5.

A ladder of Pomeron-interactions, whose contribution is given in (2.15).

Figure 6.

Pomeron interactions. Note the total number of Pomerons is always constant.

Figure 7.

One particle levels for Pomerons in the t-channel for fixed n (here n=8). Note that there are two states per energy level, and that thus the direction of motion behaves as a spin quantum number.

Figure 8.

Interaction of non-singlet ladders. This leads to a renormalization of the coupling λ.

Figure 9.

Three Pomeron interaction. This has not been taking into account in this paper.

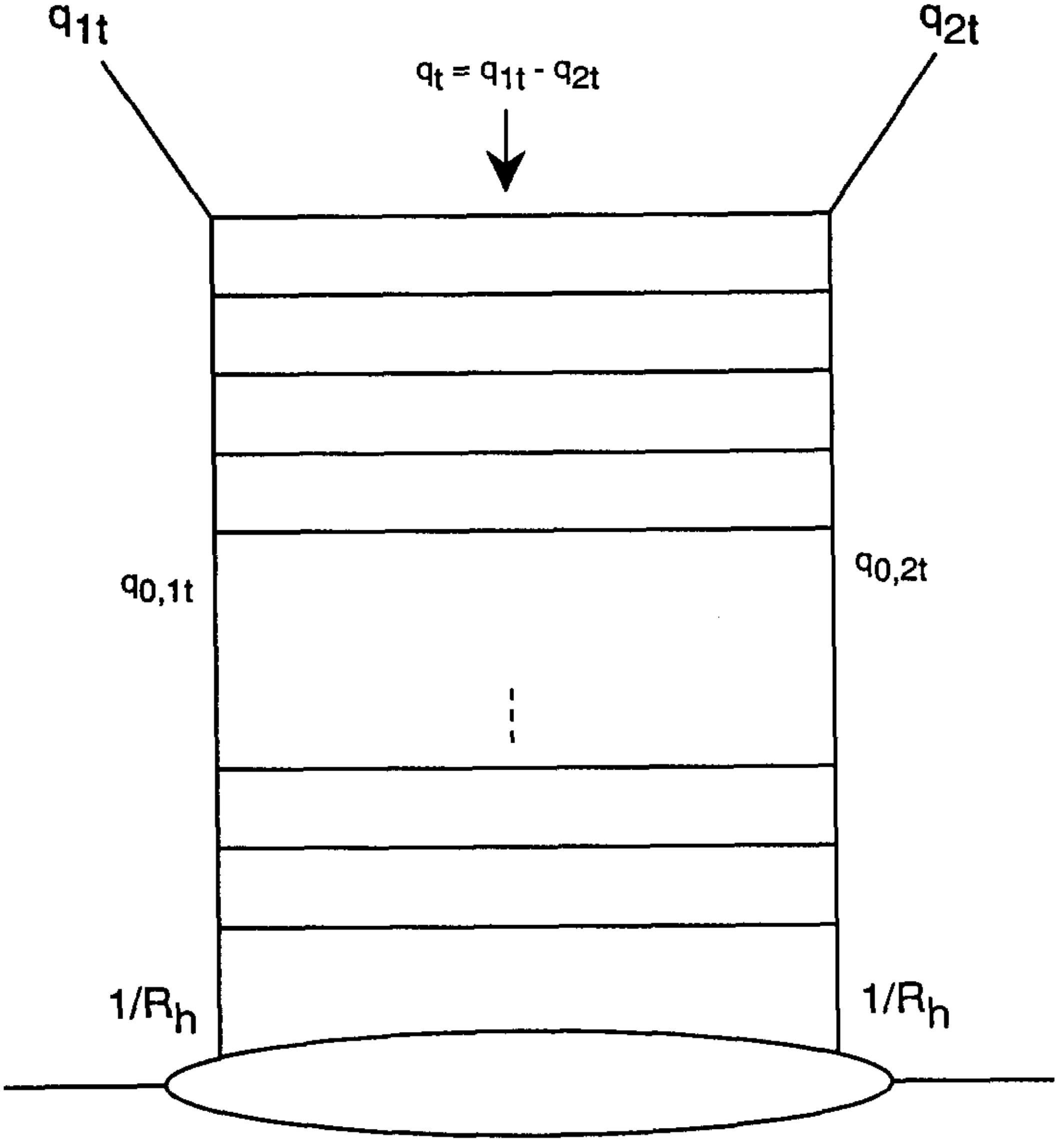


FIG. 1

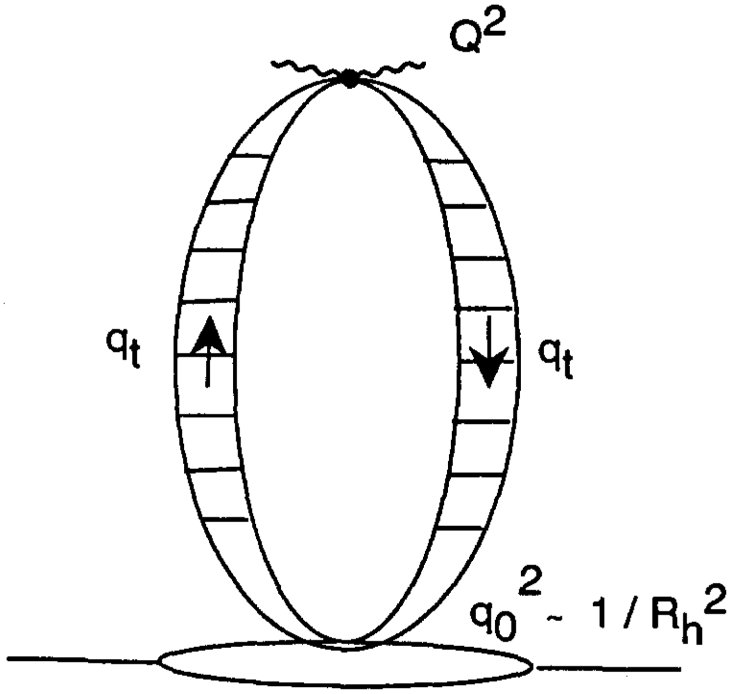


FIG. 2

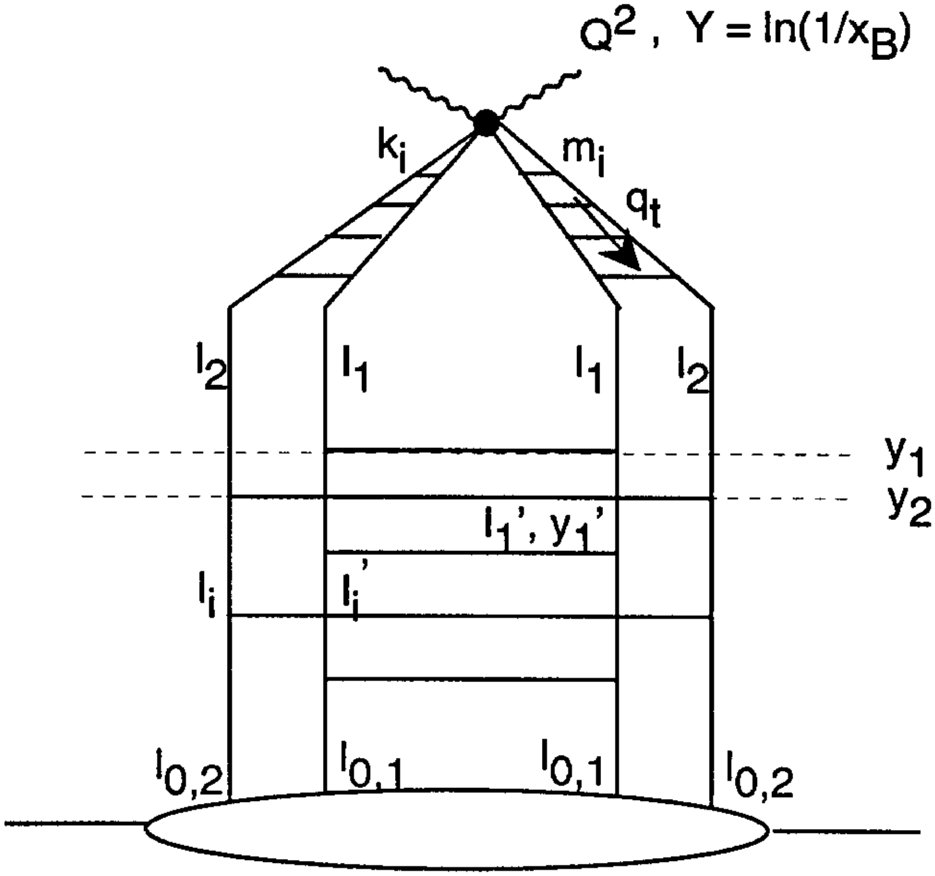


FIG. 3

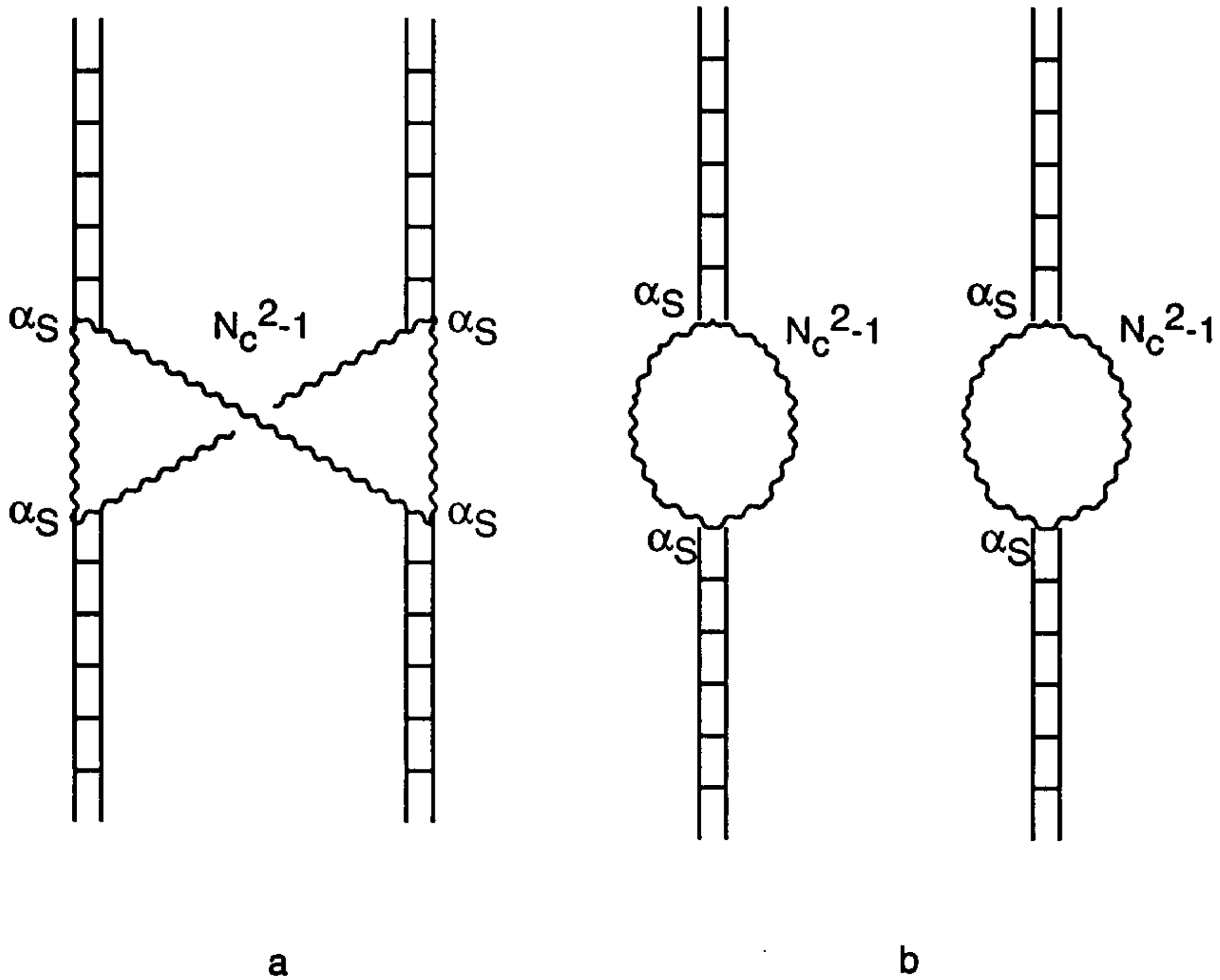


FIG. 4

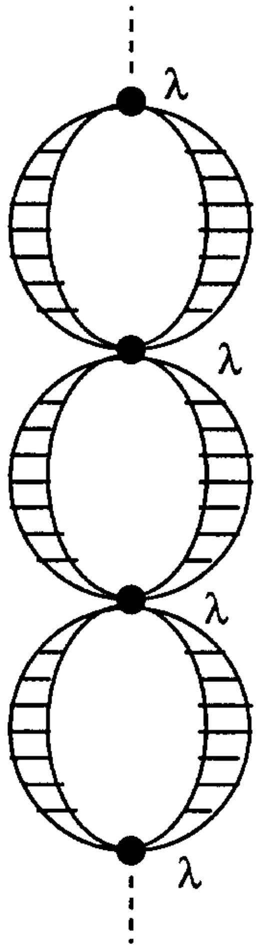


FIG. 5

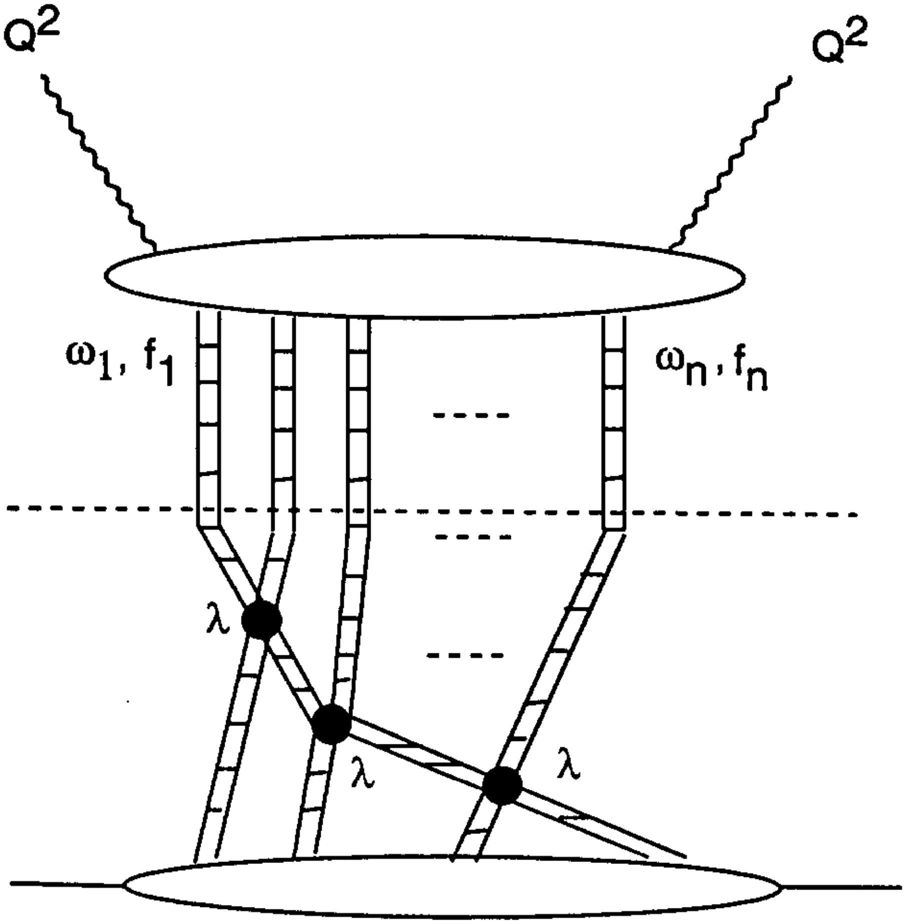


FIG. 6

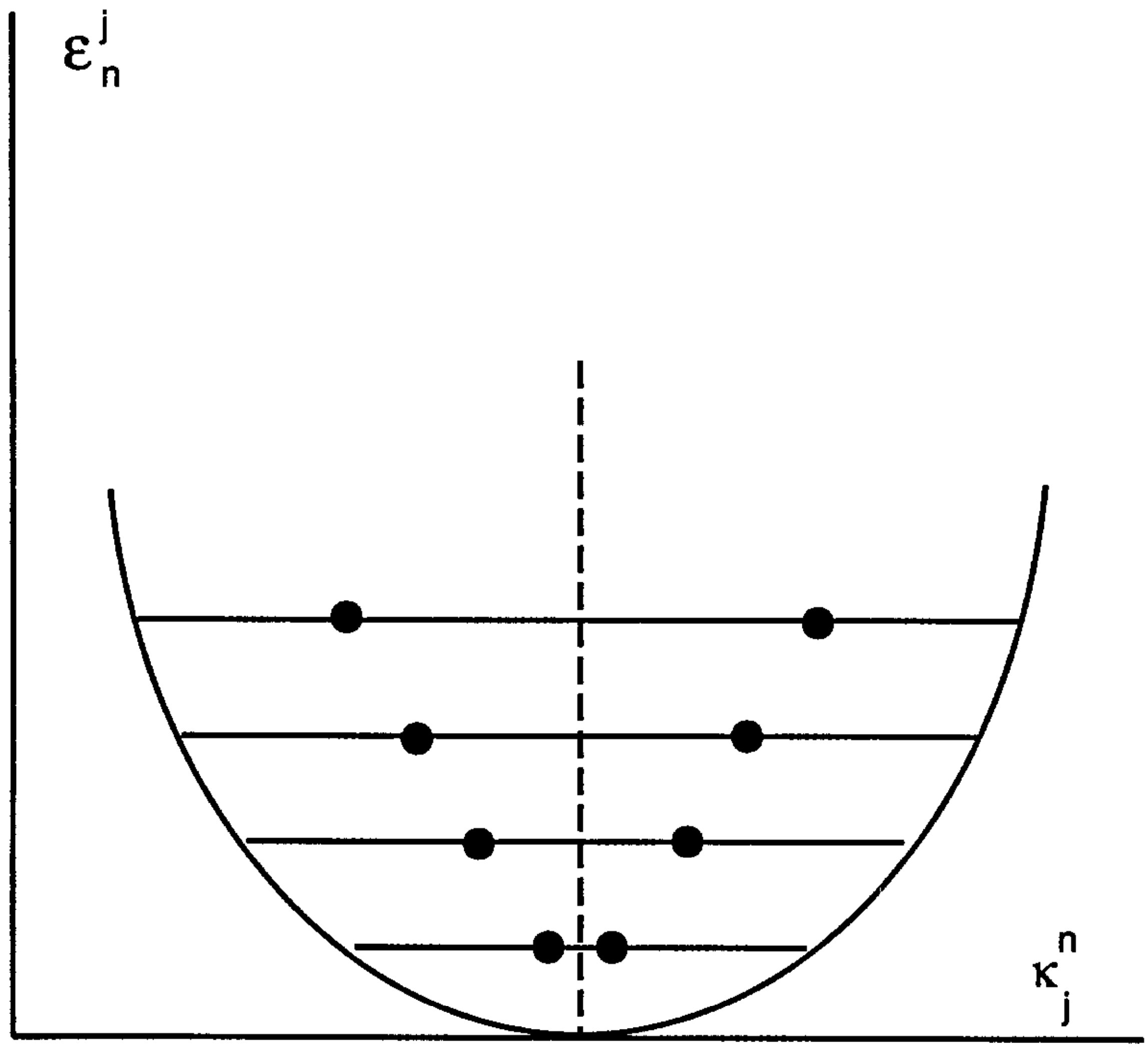
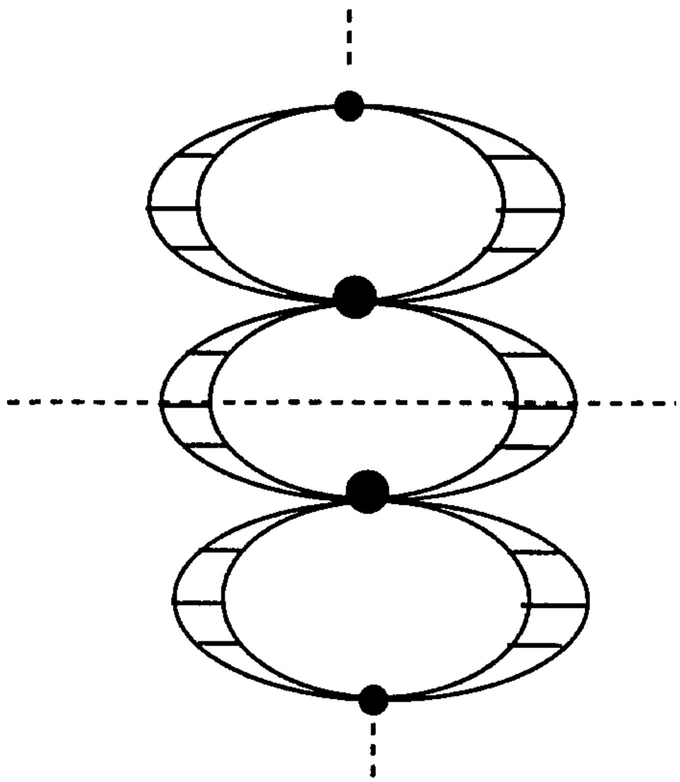


FIG. 7



$8_a \times 8_a + 8_s \times 8_s +$
$(10 + \underline{10}) \times (10 + \underline{10}) +$
27×27

FIG. 8

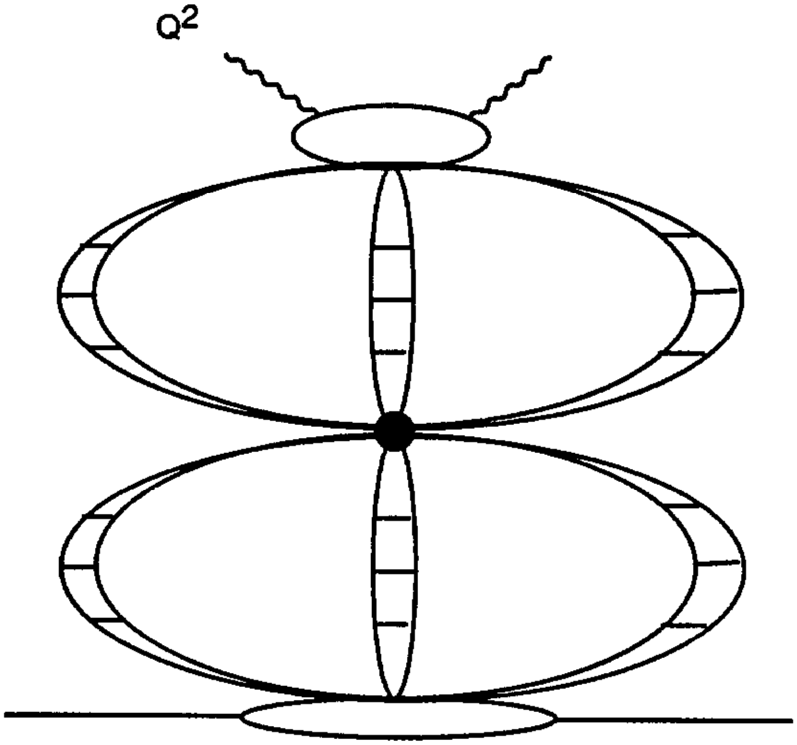


FIG. 9