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The Minimal Power Spectrum: Higher Order Contributions

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ABSTRACT

It has been accepted belief for some time that gravity induces a minimal tail $P(k) \sim k^4$ in the power spectrum as $k \to 0$ for distributions with no initial power on large scales. In a recent numerical experiment with initial power confined to a restricted range in k, Melott and Shandarin (1990) found a $k \to 0$ tail that at early stages of evolution behaves as k^4 and grows with time as $a^4(t)$, where $a(t)$ is the cosmological expansion factor, and at late times depends on scale as k^3 and grows with time as $a^2(t)$. They assert that both the early a^4 time dependence and the k^3 scale dependence at late times are anomalous.

I compute several contributions to the power spectrum of higher order than those included in earlier work, and I apply the results to the particular case of initial power restricted to a finite range of k. As expected, in the perturbative regime $P(k) \sim a^4k^4$ from the first correction to linear perturbation theory is the dominant term as $k \to 0$. Numerical show that higher order contributions go as k^4 also. However, perturbation theory alone can not tell whether the $P \sim a^2k^3$ result is "nonperturbative" or a numerical artifact.

Subject Headings: galaxies: clustering — large-scale structure of the universe

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1. Introduction

For some time, it has been part of the conventional wisdom that nonlinear terms induce a minimal power spectrum $P(k) \sim k^4$ as $k \to 0$ for any inhomogeneous density distribution (Zel'dovich 1965; Peebles 1974). Peebles (1980, § 28A) shows that such a term can be understood quite naturally when fluctuations created by moving mass from one position to another in an initially smooth universe. Naively, such a shift would induce Fourier amplitudes $\tilde{\delta}(k) \sim kx_0$, where x_0 is the typical interparticle spacing, but since from momentum conservation the center of mass does not move, this leading term cancels, and the next leading term indeed gives $\tilde{\delta} \sim (kx_0)^2$, or $P(k) \sim k^4$.

Recently, in a high precision numerical experiment with initial power confined to a restricted range $k_1 < k < k_2$, Melott and Shandarin (1990, MS) indeed found at early times a tail $P(k) \sim k^4$ to the power spectrum as $k \to 0$, growing with time as $a^4(t)$. At later times their results changed over to $P(k) \sim k^3$, evolving in time as $a^2(t)$. MS assert that both the a^4 behavior at early times and the k^3 dependence at late times are anomalous. To investigate the possible effect of higher order terms, in the following, I calculate what is expected in such an experiment in perturbation theory, including contributions from higher orders than have been considered previously. In Section 2, I quote some general perturbation theory results and derive expressions for higher-order contributions to $P(k)$. In Section 3, I apply these to the initial conditions taken by MS. Section 4 contains a final discussion. The MS early-time result, $P \sim a^4k^4$, is precisely the behavior expected from the first correction to linear perturbation theory. As $k \to 0$, all higher order contributions are $\mathcal{O}(k^4)$ also. The $P \sim a^2k^3$ result thus must be either "nonperturbative" or a numerical artifact.

2. $P(k)$ in Perturbation Theory

In this section I review the machinery necessary to compute $P(k)$ to arbitrary order in perturbation theory. It is convenient to perform the calculation in k-space. The basic field is the Fourier amplitude

$$\tilde{\delta}(\boldsymbol{k}) = \int d^3x\, \delta(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1}$$

where $\delta(\boldsymbol{x}) = [\rho(\boldsymbol{x}) - \bar{\rho}]/\bar{\rho}$ is the fractional density contrast. The power spectrum is found from the second moment,

$$\langle\, \tilde{\delta}(\boldsymbol{k}_1)\tilde{\delta}(\boldsymbol{k}_2)\,\rangle = [(2\pi)^3 \delta_{\rm D}(\boldsymbol{k}_1 + \boldsymbol{k}_2)]\, P(k), \tag{2}$$

where $k = |\boldsymbol{k}_1| = |\boldsymbol{k}_2|$.

The amplitude $\tilde{\delta}(\boldsymbol{k})$ evolves under a set of nonlinear equations describing gravitational instability (*cf.* Peebles 1980, § 9). One approach to such a nonlinear problem, expected to be valid when fluctuations are small, is a systematic solution order by order in perturbation theory in the initial amplitude. To linear order, one finds that fluctuations grow by an overall scale factor, as $\tilde{\delta}(\boldsymbol{k},t) = A(t)\,\tilde{\delta}_0(\boldsymbol{k})$, where $\tilde{\delta}_0$ is the amplitude at some early initial time t_0, $\tilde{\delta}_0(\boldsymbol{k}) = \tilde{\delta}(\boldsymbol{k},t_0)$, and $A(t)$ obeys

$$\frac{d^2A}{dt^2} + \frac{2\dot{a}}{a}\frac{dA}{dt} - 4\pi G\bar{\rho}A = 0 \tag{3}$$

(Peebles 1980). At late times the growing mode dominates. For the standard, (matter-dominated, zero curvature, critical density) cosmological model, this is $A(t) \sim t^{2/3} \sim a(t)$. Thus, to lowest order, $P(k) = A^2(t)P_0(k)$.

Interactions between modes in a nonlinear theory give rise to terms of all orders,

$$\tilde{\delta}(\boldsymbol{k}) = \tilde{\delta}^{(1)}(\boldsymbol{k}) + \tilde{\delta}^{(2)}(\boldsymbol{k}) + \tilde{\delta}^{(3)}(\boldsymbol{k}) + \ldots, \tag{4}$$

where $\tilde{\delta}^{(n)} \sim \mathcal{O}(\delta_0^n A^n)$. For gravitational instability in an expanding universe, $\tilde{\delta}^{(n)}(\boldsymbol{k})$ is a convolution,

$$\tilde{\delta}^{(n)}(\boldsymbol{k}) = \int \frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_n}{(2\pi)^3}\,[(2\pi)^3\delta_{\rm D}(\boldsymbol{k} - \textstyle\sum \boldsymbol{k}_i)]\, G_n^{(s)}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)\,\tilde{\delta}(\boldsymbol{k}_1)\cdots\tilde{\delta}(\boldsymbol{k}_n) \tag{5}$$

(Goroff *et al.* 1986, GGRW), where in the integrals on the right-hand side $\tilde{\delta} = \tilde{\delta}^{(1)} = A(t)\tilde{\delta}_0(\boldsymbol{k})$. The first few of the integral kernels G_n are $G_1 \equiv 1$,

$$G_2^{(s)}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{5}{7} + \frac{1}{2}\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{2}{7}\left(\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2}\right)^2 \tag{6}$$

(Fry 1984; GGRW), and

$$\begin{aligned}
G_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = &\frac{1}{3k_1^2k_2^2k_3^2|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\left[\frac{1}{21}\boldsymbol{k}_1\cdot\boldsymbol{k}_2\,|\boldsymbol{k}_1+\boldsymbol{k}_2|^2 + \frac{1}{14}k_2^2\,\boldsymbol{k}_1\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)\right] \\
&\times\left[7k_3^2(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) + \boldsymbol{k}_3\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)\,|\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3|^2\right] \\
&+\frac{\boldsymbol{k}_1\cdot(\boldsymbol{k}_2+\boldsymbol{k}_3)|\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3|^2}{3k_1^2k_2^2k_3^2|\boldsymbol{k}_2+\boldsymbol{k}_3|^2}\left[\frac{1}{21}\boldsymbol{k}_2\cdot\boldsymbol{k}_3\,|\boldsymbol{k}_2+\boldsymbol{k}_3|^2 + \frac{1}{14}k_3^2\,\boldsymbol{k}_2\cdot(\boldsymbol{k}_2+\boldsymbol{k}_3)\right] \\
&+\frac{\boldsymbol{k}_1\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)}{18k_1^2k_2^2k_3^2}\left[\boldsymbol{k}_2\cdot\boldsymbol{k}_3\,|\boldsymbol{k}_2+\boldsymbol{k}_3|^2 + 5k_3^2\,\boldsymbol{k}_2\cdot(\boldsymbol{k}_2+\boldsymbol{k}_3)\right],
\end{aligned} \tag{7}$$

(GGRW); this expression for G_3 is to be symmetrized over permutations of (123) to obtain $G_3^{(s)}$. GGRW also present an explicit expression for G_4 that I do not reproduce here and a recursion relation that provides the general G_n. A useful general property demonstrated by GGRW is that $G_n \propto k_n^2$, where $\boldsymbol{k}_n$ is the sum of the arguments of G_n, $\boldsymbol{k}_n = \sum \boldsymbol{k}_i$. This property, a consequence of momentum conservation, ensures that $\langle\,\tilde{\delta}^{(n)}\,\rangle = 0$ for all n.

We expect the coefficients in these expressions to sum to the $7!! = 105$ terms from $\langle \tilde{\delta}^8 \rangle$. In P_{26}, 15 of these involve $G_2^{(s)}(k', -k')$, and in P_{44}, 9 involve $G_4^{(s)}(k', -k', k'', -k'')$, both of which vanish. Finally, at $\mathcal{O}(\delta_0^{10})$

$$\begin{aligned}
P_{55} &= 225\, P(k) \left[\int \frac{d^3k'}{(2\pi)^3} \frac{d^3k''}{(2\pi)^3} G_5^{(s)}(k, k', -k', k'', -k'')\, P(k')\, P(k'') \right]^2 \\
&+ 600 \int \frac{d^3k_1'}{(2\pi)^3} \frac{d^3k_2'}{(2\pi)^3} \frac{d^3k_3'}{(2\pi)^3} \frac{d^3k_4'}{(2\pi)^3} G_5^{(s)}(k_1', k_2', k - k_1' - k_2', k_3', -k_3') \\
&\quad \times G_5^{(s)}(k_1', k_2', k - k_1' - k_2', k_4', -k_4')\, P(k_1')\, P(k_2')\, P(k - k_1' - k_2')\, P(k_3')\, P(k_4') \\
&+ 120 \int \frac{d^3k_1'}{(2\pi)^3} \frac{d^3k_2'}{(2\pi)^3} \frac{d^3k_3'}{(2\pi)^3} \frac{d^3k_4'}{(2\pi)^3} G_5^2(k_1', k_2', k_3', k_4', k - \textstyle\sum k_i') \\
&\quad \times P(k_1')\, P(k_2')\, P(k_3')\, P(k_4')\, P(k - \textstyle\sum k_i'). \qquad (21)
\end{aligned}$$

P_{55} contains the $3 \times 5 \times 7 \times 9 = 945$ terms expected from $\langle \tilde{\delta}^{10} \rangle$. Following the results of equations (13), (17), (20), and (21), we can see that the general expression for P_{nn} will contain a term

$$\begin{aligned}
P_{nn} = n! \int \frac{d^3k_1'}{(2\pi)^3} \frac{d^3k_2'}{(2\pi)^3} \cdots \frac{d^3k_{n-1}'}{(2\pi)^3} G_n^2(k_1', k_2', \ldots, k - \textstyle\sum k_i') \\
\times P(k_1')\, P(k_2') \cdots P(k_{n-1}')\, P(k - \textstyle\sum k_i'). \qquad (22)
\end{aligned}$$

In the specific case considered in § 3, terms of this form provide the dominant contribution at large k.

The "minimal" spectrum appears as $k \to 0$. Symmetrizing equation (7) for G_3 over its arguments and averaging over angles, for small k we obtain

$$3G_3^{(s)}(k, k', -k') \to -\frac{61}{630} \frac{k^2}{k'^2}. \qquad (23)$$

Thus,

$$P_{13}(k) \to -\frac{61}{630} k^2 P_{11}(k) \int \frac{d^3k'}{(2\pi)^3} \frac{P_{11}(k')}{k'^2}, \qquad (24)$$

and for $P_0(k) \leq \mathcal{O}(k^4)$ as $k \to 0$, $P_{13} < \mathcal{O}(k^6)$, insignificant at small k. From equation (6), as $k \to 0$

$$G_2(k', k - k') \to \frac{k^2}{14k'^2}(3 - 10\cos^2\theta), \qquad (25)$$

so, averaging over angles, to lowest order

$$P_{22} \to \frac{9}{98} k^4 \int \frac{d^3k'}{(2\pi)^3} \frac{P^2(k')}{k'^4}. \qquad (26)$$

This is the origin of the standard result. From the above we can see that those higher order terms that do not vanish as $k \to 0$ are all expected to depend as k^4 also. As demonstrated by GGRW, it is a property of the G_n that $G_n^{(s)} \sim (\sum k_i)^2$. Thus, we expect that $P_{mn} \sim G_m G_n \to k^4$ as $k \to 0$, to all orders.

It is reasonable however to ask how quickly this asymptotic behavior might be reached. That is addressed in a particular case in the next section.

3. Application to Top-hat Initial Spectrum

Following the numerical work of Melott and Shandarin (1990, MS), I investigate the behavior of the higher order perturbation theory terms presented in § 2 for a top-hat initial power spectrum,

$$P_0(k) = \begin{cases} C_0 & k_1 < k < k_2, \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

where C_0 is a constant. MS choose $k_2 = 2k_1$; I take $k_1 = 1$, $k_2 = 2$. Perturbatión theory results depend only on the combination $C_0 A^2(t)$; I set $C_0 = 1$. A measure of the amount of evolution or the degree of nonlinearity is the rms density fluctuation for the linearly evolved power spectrum, $\Delta^2 = \langle \delta(x)^2 \rangle$, or

$$\Delta^2 = \int \frac{d^3k}{(2\pi)^3} A^2(t) P_0(k) = \frac{7A^2}{6\pi^2}. \tag{28}$$

The integrals for P_{mn} are evaluated numerically, by a Romberg method for low orders and by Monte Carlo for higher dimensions. The kernels $G_2^{(s)}$ and $G_3^{(s)}$ are given by equations (6) and (7) [symmetrized], while $G_4^{(s)}$ and $G_5^{(s)}$ are evaluated numerically, using the GGRW recursion relation.

Figure 1 shows separate contributions from P_{11}, P_{13}, P_{22}, P_{15}, P_{24}, P_{33}, P_{35}, P_{44}, and P_{55}, plotted for $\Delta^2 = 10^{-2}$. The terms P_{11}, P_{13} (eq. [11]), P_{15} (eq. [14]), and the first term of P_{33} (eq. [17]), ranked by decreasing amplitude, are nonvanishing only for $1 < k < 2$. Note that P_{13} and P_{15} are negative. The leading term for $k < 1$ is P_{22} (eq. [13]), followed by P_{24} (eq. [16]) and P_{33} (eq. [17], second term) at sixth order, and then P_{44} (eq. [20], second term) P_{35} (eq. [19], second term only), and P_{44} (eq. [20], first term) at eighth order, and finally P_{55} (eq. [21], last term only), at tenth order. P_{24} and P_{35} change sign. The leading terms for $k > 2$ are P_{22} for $2 < k < 4$, P_{33} for $4 < k < 6$, P_{44} for $6 < k < 8$, and P_{55} for $8 < k < 10$. For the top-hat spectrum of equation (27), equation (26) gives

$$P_{22} \to \frac{9A^4k^4}{396\pi^2} = \frac{81\pi^2\Delta^4k^4}{4802} \tag{29}$$

as the expected behavior as $k \to 0$. This is plotted as the dot-dash line in Figure 1.

A complete calculation at eighth and tenth order would require G_6 through G_9, beyond the scope of this paper, or at least the strength of the author (computing P_{19} requires from small-scale events occurring at late times, nonlinear contributions may turn out be important. Observations suggest that there appears to be more power on moderately large scales (a few hundred h^{-1} Mpc) than in standard cold dark matter model, yet the microwave background remains remarkably smooth. One way to reconcile these two observations is if power decreases as $k \to 0$ more rapidly than the scale-invariant $n = 1$.

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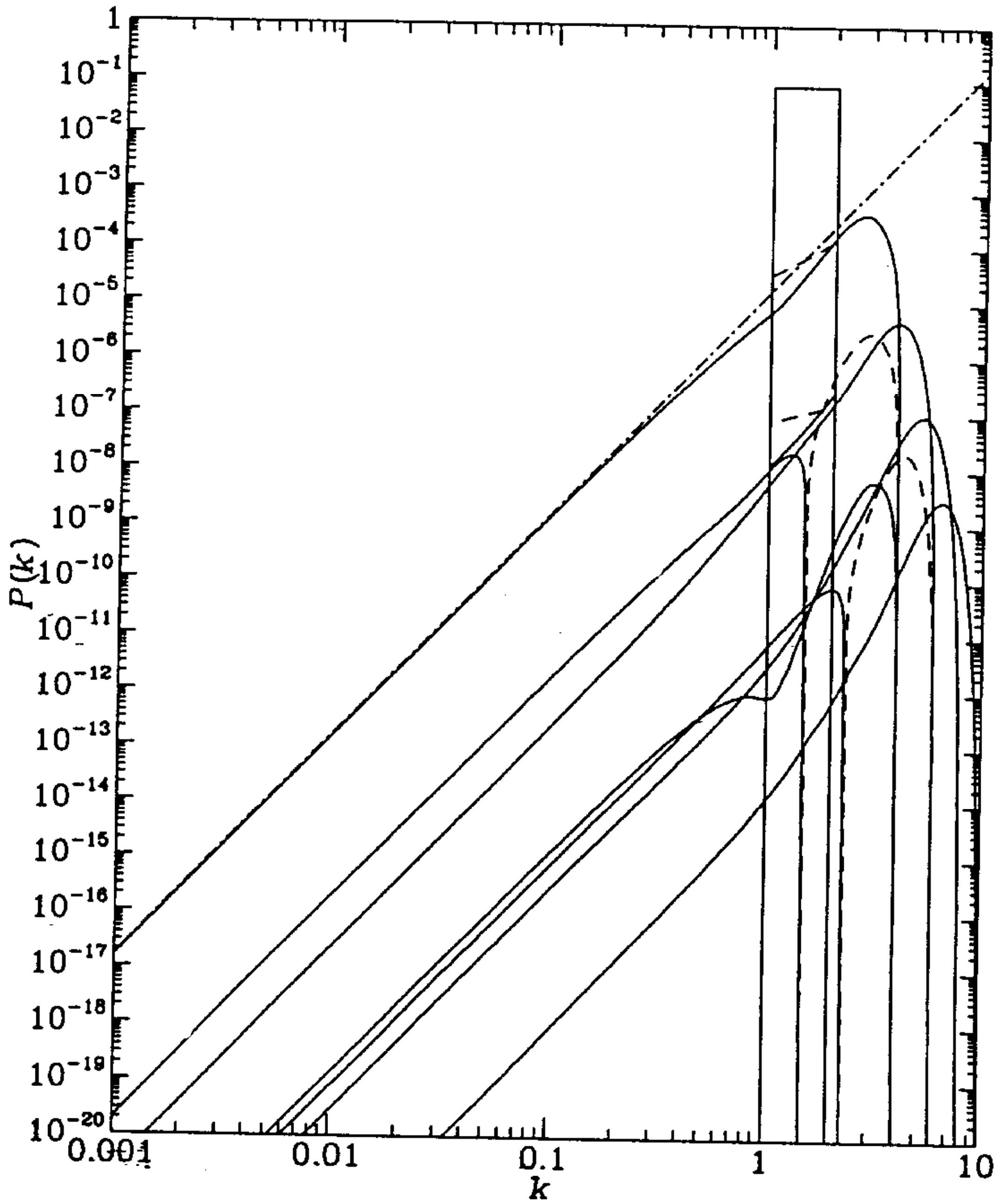
Figure Captions

Fig. 1.—Perturbation terms $P_{mn}(k)$ vs. k at $\Delta^2 = 10^{-2}$. See text for individual identifications. Dashed lines indicate $P_{mn} < 0$. The dot-dash line is the expected asymptotic k^4 behavior from equation (26).

Fig. 2.—Scaled $P_{nn,s}$ (eq. [30]) vs. scaled $k_s = k/2n$ for $n = 2$, 3, 4, and 5 (top to bottom for small k).

Fig. 3.—$P(k)$ from the sum of perturbation terms vs. k for $\Delta^2 = 1/512^2$, $1/128^2$, $1/32^2$, $1/8^2$, $1/2^2$, and 2^2 (bottom to top).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10^-18
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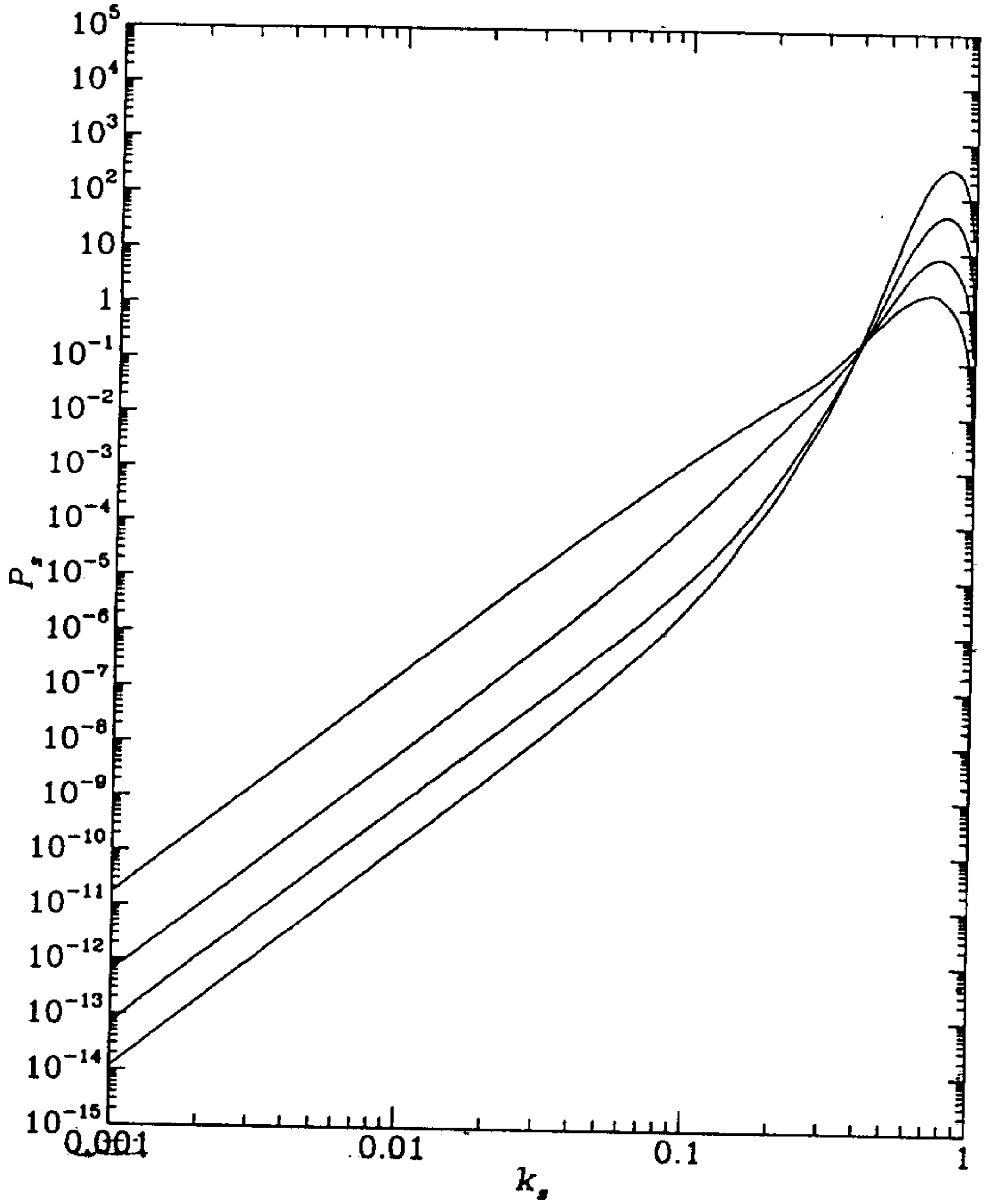
P(k)
0.001
0.01
0.1
1
10
k

Figure 1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P_s
0.001
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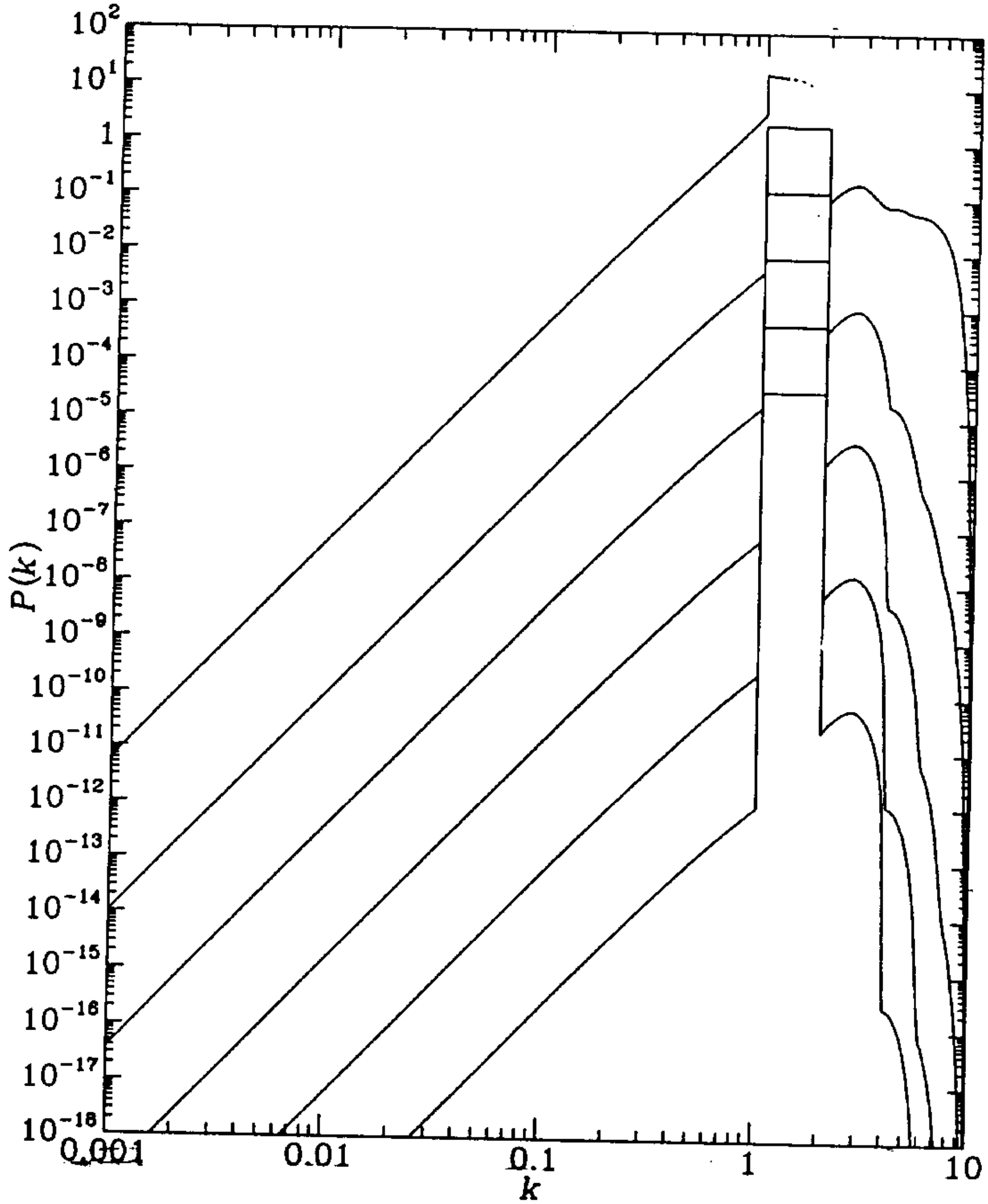
1
k_s

Figure 2



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10^{-15}
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10^{-18}
$P(k)$
0.001
0.01
0.1
1
10
k

Figure 3