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# Study of Stochastic Parametric Resonance with Application to Ions Trapped in a Particle Beam

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# Study of Stochastic Parametric Resonance with Application to Ions Trapped in a Particle Beam

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The stochastic parametric resonance equation $\ddot{x}+(\omega_i^2+\omega_i\xi(t))x=0$ with periodic Schottky noise $\xi(t)$ is studied. When the parametric resonance condition is satisfied, the average amplitude of oscillation grows exponentially with time squared. When it is not, the average amplitude can still grow exponentially. The average energy of the oscillators is proportional to the fourth power of the average oscillation amplitude.

The result is applied to ions trapped in a particle beam.

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## I. INTRODUCTION

It is well known[1] that particle beams circulating in a ring can trap oppositely charged particles. In the case of electron and $\bar{p}$ beams, positive ions produced by the collisions of the beam particles with the residual molecules inside the beam pipe could be trapped. The trapped ions oscillate transversely in the potential well produced by the space-charge field of the beam. If the cross section of the beam is a circle of area $A$ with uniform density, the electric field inside the beam is radial with the magnitude

$$E = \frac{\rho r}{2\epsilon_0 AR} , \tag{1}$$

where $2\pi R$ is the machine circumference and $r$ is the radial position relative to the center of the beam. The azimuthal charge density $\rho$ has units of Coulomb/rad. In the first approximation, $\rho$ is independent of time for a coasting beam stored in a storage ring. The ions produced inside the beam are trapped by the harmonic potential with the angular oscillation frequency

$$\omega_i^2 = \frac{q}{2\epsilon_0 ARM}\rho_0 , \tag{2}$$

where q and M are the charge and mass of the ions respectively. The time-averaged azimuthal charge density $\rho_0$ equals $eN/2\pi$, where $e$ is the charge of the beam particles and N is the total number of particles of the beam respectively.

Most of the ions produced are trapped in the potential well. The depth of the potential well for a typical beam is approximately 100 eV, much larger than the approximately 1 eV average ion energy at production. Recent observations[2] at the FNAL $\bar{p}$ Accumulator indicate that ions escape the well at a much faster rate than expected. It is interesting to analyze the interaction of the ions with the beam to see how the ions gain enough energy from the beam to escape the well. In particular, we will analyze the diffusion of the ions due to their interaction with the beam Schottky noise.

An analogy to the Brownian motion may be useful here. Brownian particles diffuse in the fluid they are immersed in. The fluid is not continuous but consists of many molecules. The collision of the molecules and the Brownian particle can be decomposed into two parts (Langevin equation): a time-averaged part which is identified with the friction and a fast fluctuating part which is due to the collision of the Brownian particle with individual molecules. This fast fluctuating part is the noise that causes the Brownian particles to diffuse.

It turns out that the trapped ions can diffuse out of the harmonic potential well through their interaction with the beam Schottky noise. But the diffusion is *not* like the Brownian motion. Since the frequency spread of the $\bar{p}$ beam of FNAL Accumulator is very narrow ($\sigma_f/f \approx 5 \times 10^{-6}$), we will restrict ourself to the monochromatic beam limit.

## II. BEAM SCHOTTKY NOISE

At any point of the storage ring, the azimuthal charge density $\rho$ can be written as[3]

$$\rho(t) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{N} \frac{e}{2\pi} \delta(\omega_j t + \theta_j^0 - 2m\pi) , \tag{3}$$

where $\omega_j$ and $\theta_j^0$ are the angular revolution frequency and initial angular position of the $j^{th}$ beam particle respectively. Equation 3 can be decomposed into a time-averaged part $\rho_0$ and a fast fluctuating part $\phi(t)$. The Schottky noise $\phi(t)$ is due to the fact that beam is not a continuous fluid. Since the beam particles circulate around the storage ring, $\phi(t)$ is a periodic function of the beam revolution period. The Fourier decomposition of $\phi(t)$ is

$$\phi(t) = \frac{e}{\pi} \sum_{m=1}^{\infty} \sum_{j=1}^{N} \cos(m(\omega_j t + \theta_j^0)) . \tag{4}$$

For a coasting beam, $\theta_j^0$ is assumed to be uniformly distributed around the ring. If the beam is monochromatic, the above equation can be re-written as

$$\phi(t) = \sum_{m=1}^{\infty} \{a_m \cos(m\omega_0 t) + b_m \sin(m\omega_0 t)\} \ , \tag{5}$$

where $\omega_0$ is the revolution frequency of the beam particles, and

$$\begin{aligned} a_m &= \frac{e}{\pi} \sum_{j=1}^{N} \cos(m\theta_j^0) \\ b_m &= -\frac{e}{\pi} \sum_{j=1}^{N} \sin(m\theta_j^0) \ . \end{aligned} \tag{6}$$

Since $a_m$ and $b_m$ are the sums of $N$ terms and $N$ is large, we can use the Central Limit Theorem of probability to assert that $a_m$ and $b_m$ have Gaussian distributions with zero mean and $\langle a_m a_l \rangle$ and $\langle b_m b_l \rangle$ equal $(\frac{e}{\pi})^2 \frac{N}{2} \delta_{ml}$.

The auto-correlation function of $\phi(t)$ is

$$\langle \phi(t_1)\phi(t_2) \rangle = \sum_{m=1}^{\infty} (\frac{e}{\pi})^2 \frac{N}{2} \cos(m\omega(t_1 - t_2)) \ . \tag{7}$$

## III. EQUATION OF MOTION

The motion of the ions is described by the stochastic differential equation[4]

$$\begin{aligned} \frac{dx_1}{dt} &= \omega_i x_2 \\ \frac{dx_2}{dt} &= -(\omega_i + \xi(t)) x_1 \ , \end{aligned} \tag{8}$$

where $x_1$ and $x_2$ are the position $x$ and the normalized velocity $v/\omega_i$ of the ion respectively, and $\xi(t)$ is the fluctuation due to the beam noise given by

$$\xi(t) = \frac{\omega_i}{\rho_0} \phi(t) \ . \tag{9}$$

It is convenient to write the equations in matrix notation

$$\frac{d\vec{u}}{dt} = (A_0 + \xi(t) B_0)\vec{u} \tag{10}$$

where

$$\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , A_0 = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix} , \text{and } B_0 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} . \tag{11}$$

In the interaction picture, the above equation transforms to

$$\frac{d\vec{w}}{dt} = \xi(t)B(t)\vec{w} \tag{12}$$

where

$$\vec{w} = e^{-A_0 t}\vec{u} \text{ and } B(t) = e^{-A_0 t} B_0 e^{A_0 t} . \tag{13}$$

It can be shown that $e^{A_0 t}$ is the rotation matrix

$$e^{A_0 t} = \begin{pmatrix} \cos(\omega_i t) & \sin(\omega_i t) \\ -\sin(\omega_i t) & \cos(\omega_i t) \end{pmatrix} , \tag{14}$$

and it follows

$$B(t) = \begin{pmatrix} \sin(2\omega_i t)/2 & \sin^2(\omega_i t) \\ -\cos^2(\omega_i t) & -\sin(2\omega_i t)/2 \end{pmatrix} . \tag{15}$$

The formal solution of $\vec{w}(t)$ is

$$\vec{w}(t) = \left(1 + \sum_{i=1}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{i-1}} dt_i \xi(t_1) \ldots \xi(t_i) B(t_1) \ldots B(t_i)\right) \vec{w}(0) \tag{16}$$

$$= T\left(\exp\left(\int_0^t dt_1 \xi(t_1) B(t_1)\right)\right) \vec{w}(0) , \tag{17}$$

where $T(\ldots)$ is the time ordered product operator defined by Eq. 16.

We have shown that $\xi(t)$ is Gaussian with zero mean in section II. We will take advantage of the fact that in this case the first cumulant and the cumulants above second order of $\xi(t)$ are zero. The second order cumulant is the same as the auto-correlation function $\langle\xi(t_1)\xi(t_2)\rangle$. Taking the average of Eq. 16 and expanding $\langle\xi(t_1)\xi(t_2)\ldots\xi(t_i)\rangle$ in cumulants, we find that all odd $i^{th}$ terms are zero. We find Eq. 17 reduces to

$$\langle \vec{w}(t) \rangle = T \left( \exp \left( \int_0^t dt_1 \int_0^t dt_2 \langle \xi(t_1)\xi(t_2) \rangle B(t_1) B(t_2) \right) \right) \vec{w}(0) \ . \tag{18}$$

Explicitly, when we expand the time ordered exponential of Eq. 18, each term is

$$\int_0^t dt_1 \ldots \int_0^{t_{i-1}} dt_i \left( \sum \langle \xi(t_{k_1})\xi(t_{k_2}) \rangle \ldots \langle \xi(t_{k_{i-1}})\xi(t_{k_i}) \rangle \right) B(t_1) \ldots B(t_i) \ , \tag{19}$$

where the sum is over all possible partitions of $\{1, 2 \ldots i\}$, with $i$ even, into subsets of 2 numbers. There are a total $i!/(2^{i/2}(i/2)!)$ combinations.

## IV. THE MONOCHROMATIC BEAM CASE

From Eq. 7 we obtain the correlation function to be

$$\langle \xi(t)\xi(t-\tau) \rangle = \left( \frac{2\omega_i^2}{N} \right) \left( -\frac{1}{2} + \frac{\pi}{\omega_0} \sum_{m=-\infty}^{\infty} \delta(\tau - \frac{2m\pi}{\omega_0}) \right) \ , \tag{20}$$

where we have transformed Eq. 7 to its $\delta$ function representation.

The first order term of Eq. 18 is

$$\begin{aligned}
&\int_0^t dt_1 \int_0^{t_1} dt_2 \langle \xi(t_1)\xi(t_2) \rangle B(t_1) B(t_2) = \\
&- \left( \frac{\omega_i^2}{N} \right) \int_0^t dt_1 \int_0^{t_1} dt_2 B(t_1) B(t_2) \\
&+ \left( \frac{2\omega_i^2}{N} \right) \sum_{m=0}^{m<t\omega_0/2\pi} \frac{\pi}{\omega_0} \int_0^t dt_1 B(t_1) B(t_1 - 2m\pi/\omega_0) \ .
\end{aligned} \tag{21}$$

We find,

$$B(t_1)B(t_2) = \sin(\omega_i(t_1 - t_2)) \begin{pmatrix} -\sin(\omega_i t_1)\cos(\omega_i t_2) & -\sin(\omega_i t_1)\sin(\omega_i t_2) \\ \cos(\omega_i t_1)\cos(\omega_i t_2) & \cos(\omega_i t_1)\sin(\omega_i t_2) \end{pmatrix} \ . \tag{22}$$

The integrals are:

$$\begin{aligned}
&\int_0^t dt_1 \int_0^{t_1} dt_2 B(t_1) B(t_2) = \\
&\frac{-1}{16\omega_i^2} \begin{pmatrix} 2\omega_i^2 t^2 - 2\omega_i t \sin(2\omega_i t) - \cos(2\omega_i t) + 1 & 4\omega_i t + 2\omega_i t \cos(2\omega_i t) - 3\sin(2\omega_i t) \\ 2\omega_i t \cos(2\omega_i t) - \sin(2\omega_i t) & 2\omega_i^2 t^2 + 2\omega_i t \sin(2\omega_i t) + 3\cos(2\omega_i t) - 3 \end{pmatrix}
\end{aligned} \tag{23}$$

and

$$
\int_0^t dt_1 B(t_1)B(t_1 - t_m) =
$$

$$
\frac{t-t_m}{4}\begin{pmatrix} -2\sin^2(\omega_i t_m) & -\sin(2\omega_i t_m) \\ \sin(2\omega_i t_m) & -2\sin^2(\omega_i t_m)) \end{pmatrix} +
$$

$$
\frac{1}{8\omega_i}\begin{pmatrix} \sin(2\omega_i t) - \sin(2\omega_i (t-t_m)) - \sin(2\omega_i t_m) & -2\sin^2(\omega_i t_m) - \cos(2\omega_i t) + \cos(2\omega_i(t-t_m)) \\ -2\sin^2(\omega_i t_m) - \cos(2\omega_i t) + \cos(2\omega_i (t-t_m)) & -\sin(2\omega_i t) + \sin(2\omega_i(t-t_m)) + \sin(2\omega_i t_m) \end{pmatrix} \tag{24}
$$

where $t_m = 2m\pi/\omega_0$.

### A. Parametric Resonance Case

Substituting Eq. 23 and 24 into Eq. 21, we find the dominant term to be

$$
\left(\frac{2\omega_i^2}{N}\right)\left[\frac{\pi}{4\omega_0}\sum_{m=0}^{m<t\omega_0/2\pi}(t-t_m)\cos(2\omega_i t_m)\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{25}
$$

When the parametric resonance[5] condition $2\omega_i/\omega_0 = n$, $n = 1, 2, \ldots$ is satisfied, $\cos(2\omega_i t_m) = 1$. Equation 25 is proportional to $t^2$ and equals $\omega_i^2 t^2/8N$ times the unit matrix.

For the resonant case, keeping only the dominant $t^2$ term, we get

$$
B(t_1)B(t_2) = -\frac{1}{2}\sin^2(\omega_i(t_1 - t_2))\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{26}
$$

In this case, $B(t_1)$ and $B(t_2)$ commute and the time ordered product operator in Eq. 18 can be dropped. We find

$$
\langle \vec{w}(t) \rangle \ = \exp(\frac{\omega_i^2 t^2}{8N})\vec{w}(0) . \tag{27}
$$

Transforming back to the original picture, we see that $\langle \vec{u}(t) \rangle$ is a rotating vector with its length proportional to $\exp(\omega_i^2 t^2/8N)$.

The second moments of the ion distribution are calculated to be

$$\begin{pmatrix} \langle x^2\rangle \\ \langle (v/\omega_i)^2\rangle \\ \langle x(v/\omega_i)\rangle \end{pmatrix} = \frac{1}{2}\exp(\frac{\omega_i^2 t^2}{4N}) \begin{pmatrix} \cos(2\omega_i t) & -\cos(2\omega_i t) & 2\sin(2\omega_i t) \\ -\cos(2\omega_i t) & \cos(2\omega_i t) & -2\sin(2\omega_i t) \\ -\sin(2\omega_i t) & \sin(2\omega_i t) & 2\cos(2\omega_i t) \end{pmatrix} \begin{pmatrix} \langle x^2\rangle_0 \\ \langle (v/\omega_i)^2\rangle_0 \\ \langle x(v/\omega_i)\rangle_0 \end{pmatrix}$$
$$+\frac{1}{2}\exp(\frac{\omega_i^2 t^2}{2N}) \begin{pmatrix} 1\,1\,0 \\ 1\,1\,0 \\ 0\,0\,0 \end{pmatrix} \begin{pmatrix} \langle x^2\rangle_0 \\ \langle (v/\omega_i)^2\rangle_0 \\ \langle x(v/\omega_i)\rangle_0 \end{pmatrix} \quad . \tag{28}$$

The energy of the ions grows proportional to the *fourth* power of the average amplitude, i.e.,

$$\begin{aligned} \langle E(t)\rangle &= \frac{M}{2}(\langle v^2\rangle + \omega_i^2\langle x^2\rangle) \\ &= \exp(\frac{\omega_i^2 t^2}{2N})\langle E(0)\rangle \ . \end{aligned} \tag{29}$$

The diffusion of the ions causes the energy to be proportional not to the square of the average amplitude but to the fourth power.

It is interesting to compare these results to those for the parametric resonance[5]. The average amplitude grows exponentially with time squared, faster than the case of typical parametric resonance where the amplitude is proportional to $\exp(2\omega_i t/n)$. Also note that both the amplitude and the energy are independent of $n$. For the typical parametric resonance, the resonance is weaker for higher $n$. This is due to the fact that the Schottky noise contains all frequencies which are integral multiples of $\omega_0$.

### B. Non-resonant Case

If the resonant condition is not satisfied, Eq. 25 is

$$\left(\frac{2\omega_i^2}{N}\right)\left[\frac{\pi}{4\omega_0}\left(\frac{t}{2}+\frac{\pi}{\omega_0}\frac{\sin^2(m'a/2)}{\sin^2(a/2)}\right)\right]\begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} \ , \tag{30}$$

where $a = 2\pi\frac{2\omega_i}{\omega_0}$ and $m'$ equals the largest integer $\leq t\omega_0/2\pi$. The second term of Eq. 30 is oscillatory when the resonant condition is not satisfied. When $a$ approaches the resonant condition, this term approaches the $t^2$ term of the resonant case.

The first term is more interesting. It is linear in t and it is independent of the resonant condition. The ions *can* absorb energy from the beam even though the resonant condition is not satisfied.

We will work out the limit where the off-diagonal terms are small and $a$ is in the middle of two resonances($a = (n+\frac{1}{2})2\pi$). The dominant off-diagonal terms are

$$\left(\frac{2\omega_i^2}{N}\right)\left[\frac{t}{16\omega_i} - \frac{\pi}{8\omega_0}\left(t\cot(a/2) - \frac{\pi}{\omega_0}\frac{\sin(m'a)}{\sin^2(a/2)}\right)\right]\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \qquad (31)$$

Notice that the $\sin(m'a)$ term cancels the $t\cot(a/2)$ term at resonances. When $a$ is in the middle of two resonances($a = (n+\frac{1}{2})2\pi$), Eq. 31 reduces to $\omega_i t/8N$.

From Eq. 30 and 31, we find the ratio of diagonal term to off-diagonal term to be $a/2$. As long as $a/2$ is large, we can keep only the diagonal terms. Following the same procedure as in section IV A, we get

$$\langle\vec{w}(t)\rangle = \exp\left(\frac{a\omega_i t}{16N}\right)\vec{w}(0) . \qquad (32)$$

Similarly, the energy of the ions grows proportional to the fourth power of the average amplitude, i.e.,

$$\langle E(t)\rangle = \exp\left(\frac{a\omega_i t}{4N}\right)\langle E(0)\rangle . \qquad (33)$$

Notice that in this case the average amplitude and the energy depend on $2\omega_i/\omega_0$. We recall that in the resonant case, they are independent of $2\omega_i/\omega_0$.

## V. DISCUSSION

It is shown that trapped ions gain energy through their interaction with the beam Schottky noise. If the resonant condition is satisfied, the average amplitude $\langle\vec{u}(t)\rangle$ grows like $\exp(\omega_i^2 t^2/8N)$ and the size of the distribution diffuses like $\exp(\omega_i^2 t^2/2N)$.

In the non-resonant case, the ions still absorb energy from the beam. In the case $2\omega_i/\omega_0$ is in the middle of two resonances, if $a/2$ is large, the average amplitude grows like $\exp(a\omega_i t/4N)$. It is interesting to compare with cases where the auto-correlation time is short, e.g., white noise. It has been shown[4] that the energy rate of growth is proportional to the spectral power density of the noise at the frequency of $2\omega_i$, i.e., the resonant condition has to be satisfied for the ions to absorb energy. In our case, where the correlation time is infinite, the ions absorb energy even when the resonant condition is not satisfied.

In the case of ions trapped in the $\bar{p}$ beam in FNAL Accumulator, $n$ is 5 and $a/2$ is 16. The ions trapped in the beam will diffuse out of the beam by the mechanism described in this paper. In reality, the trapping potential of a real beam is non-linear. In analogy to the betatron resonances of storage rings, the resonant condition for the $k^{th}$ non-linear resonance could be $k\omega_i/\omega_0 = n,\ k \geq 3$. It is possible ions could also escape the beam by this mechanism.

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