

# Solutions to the Wheeler-DeWitt Equation Inspired by the String Effective Action

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## Abstract

The Wheeler-DeWitt equation is derived from the bosonic sector of the heterotic string effective action assuming a toroidal compactification. The spatially closed, higher dimensional Friedmann-Robertson-Walker (FRW) cosmology is investigated and a suitable change of variables rewrites the equation in a canonical form. Real- and imaginary-phase exact solutions are found and a method of successive approximations is employed to find more general power series solutions. The quantum cosmology of the Bianchi IX universe is also investigated and a class of exact solutions is found.

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# 1 Introduction

Superstring theory remains a promising candidate for a complete theory of quantum gravity [1]. If string theory is indeed correct, it should describe the complete quantum history of the universe. However a consistent second quantization of the superstring is not currently available and to make progress one must consider the low-energy limit of the effective field theory. One approach adopted by a number of authors has been to investigate the quantum cosmology of the superstring by solving the corresponding Wheeler-DeWitt (WDW) equation for the effective action [2,3].

String theory and the WDW equation may be related at a fundamental level. It has been shown that the effective action,  $S$ , may be written as an integral over a spatial hypersurface that does not depend on cosmic time [4]. This is significant because the WDW equation may be viewed as a time-independent Schrödinger equation and its solution in the WKB approximation is determined by the phase factor  $\exp(iS)$ . Consistency therefore suggests that the action should not depend on cosmic time and the above approach appears well motivated.

Recently the Scherk-Schwarz [5] dimensional reduction technique has been employed to construct a string effective action in which  $d$  internal dimensions are compactified onto a torus [6]. In this paper we search for solutions to the WDW equation derived from this action. Broadly speaking a singular, oscillating solution may be interpreted at the semiclassical level in terms of Lorentzian four-geometries, whereas an exponential solution can represent a quantum wormhole if appropriate boundary conditions are satisfied. Hawking and Page [7] have proposed that the wavefunction  $\Psi$  should be exponentially damped at infinity and obey a suitable regularity condition when the spatial metric degenerates (i.e. its radius  $a$  vanishes) [8]. Presently the precise form of such a condition is not clear, but it appears that the wavefunction should vary as  $\Psi \propto a^p$ , where the exponent  $p$  is a constant. The sign of  $p$  is associated with the choice of factor ordering and may be negative [9].

The derivation of the string effective action is summarized in Sect. 2 and the corresponding WDW equation for the spatially closed,  $(D + 1)$ -dimensional Friedmann-Robertson-Walker (FRW)

space-time is derived. Singular and non-singular solutions to this equation are found in Sect. 3. The dimensionally reduced action is invariant under a global  $O(d, d)$  transformation and in principle this symmetry may be employed to generate new solutions. Such a procedure is illustrated within the context of the duality symmetry of string theory [10]. The WDW equation is rewritten in a simplified canonical form after a suitable redefinition of the independent variables and this allows a method of successive approximations to be developed in Sect. 4. Additional power series solutions are found. Some of these series can be written in closed form and are exponentially damped at infinity. The more general Bianchi IX (mixmaster) cosmology is investigated in Sect. 5 and an exact solution is found that is sharply peaked around the FRW universe at large three-geometries.

Units are chosen such that  $\hbar = c = 16\pi G = 1$ .

## 2 The Effective Action and the Wheeler-DeWitt Equation

We first summarize the derivation of the string effective action presented in [6]. At the tree level, the Euclidean action of the gauge singlet, bosonic sector of the  $\hat{D}$ -dimensional heterotic string is<sup>1</sup>

$$S_{\hat{g}} = \int_{\mathcal{M}} d^{\hat{D}}x \sqrt{\hat{g}} e^{-\hat{\phi}} \left[ -\hat{R}(\hat{g}) - (\hat{\nabla}\hat{\phi})^2 + \frac{1}{12} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right], \quad (2.1)$$

where  $\hat{\phi}$  is the dilaton field and  $\hat{H}$  is the totally antisymmetric 3-index field [11]. In the technique of dimensional reduction the universe is viewed as the product space  $\mathcal{M} = \mathcal{J} \times \mathcal{K}$ , where the  $(D+1)$ -dimensional space-time  $\mathcal{J}(x^\rho)$  has metric  $g_{\mu\nu}(x^\rho)$  and the  $d$ -dimensional internal space  $\mathcal{K}(y^\alpha)$  must be Ricci flat if the matter fields are independent of its coordinates  $y^\alpha$ . This is the case for the Calabi-Yau spaces often considered in string theory, but for the purposes of the present work it is sufficient to assume  $\mathcal{K}$  is a torus, i.e.  $\mathcal{K} = S^1 \times S^1 \times \dots \times S^1$ . The complete metric is then

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + A_{\mu}^{(1)\gamma} A_{\nu\gamma}^{(1)} & A_{\mu\beta}^{(1)} \\ A_{\nu\alpha}^{(1)} & G_{\alpha\beta} \end{pmatrix}, \quad (2.2)$$

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<sup>1</sup>A hat denotes quantities in the  $\hat{D}$ -dimensional spacetime  $\mathcal{M}$ , where  $\hat{D} = 1 + D + d$ .

where  $G_{\alpha\beta}$  is the metric on  $\mathcal{K}$ . The effective action in  $(D + 1)$ -dimensions is

$$S = \int d^{D+1}x \sqrt{g} e^{-\phi} \left[ -R - (\nabla\phi)^2 + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{8} \text{Tr} \left( \nabla_\mu M^{-1} \nabla^\mu M \right) + \frac{1}{4} \mathcal{F}_{\mu\nu}^i \left( M^{-1} \right)_{ij} \mathcal{F}^{\mu\nu j} \right], \quad (2.3)$$

where

$$A_{\mu\alpha}^{(2)} = \hat{B}_{\mu\alpha} + B_{\alpha\beta} A_\mu^{(1)\beta} \quad (2.4)$$

$$\phi = \hat{\phi} - \frac{1}{2} \ln \det G \quad (2.5)$$

$$H_{\mu\nu\rho} = \nabla_\mu B_{\nu\rho} - \frac{1}{2} \mathcal{A}_\mu^i \eta_{ij} \mathcal{F}_{\nu\rho}^j + (\text{cyc. perms.}) \quad (2.6)$$

$$\mathcal{F}_{\mu\nu}^i = \nabla_\mu \mathcal{A}_\nu^i - \nabla_\nu \mathcal{A}_\mu^i \quad (2.7)$$

and the  $2d \times 2d$  matrices  $M$ ,  $\eta$  and  $M^{-1}$  are defined by

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M^{-1} = \eta M \eta, \quad (2.8)$$

respectively. It follows from the definition of  $M$  that  $M^T \eta M = \eta$  and this implies  $M \in O(d, d)$ . Hence, since  $g_{\mu\nu}$  and  $\phi$  are also invariant under the action of this group, the dimensionally reduced action is symmetric under a global  $O(d, d)$  transformation.

We follow the approach of [6] by choosing  $\phi = \text{constant}$  with  $H_{\mu\nu\rho} = 0$  and  $\mathcal{A}_\mu^i = 0$ . We assume a line element for  $\mathcal{J}$  of the form

$$ds^2 = n^2(t) dt^2 + a^2(t) d\Omega_D^2, \quad (2.9)$$

where  $n(t)$  defines the lapse function,  $d\Omega_D^2$  is the line element of the unit  $D$ -sphere and  $a(t)$  is the scale factor. This assumption of spherical symmetry implies that all variables are functions of  $t$  only.

Thus the  $d \times d$  matrix  $G + B$  may be written as

$$G + B = \text{diag} \left( \Sigma_1, \dots, \Sigma_{d/2} \right), \quad \Sigma_j = \begin{pmatrix} e^{\psi_j} & \sigma_j \\ -\sigma_j & e^{\psi_j} \end{pmatrix}, \quad (2.10)$$

where  $G$  and  $B$  are space-time dependent and we assume  $d$  is even. The action (2.3) then simplifies to

$$S = \int d^{D+1}x \sqrt{g} \left( -R + \frac{1}{2} \sum_{j=1}^{d/2} \left[ (\nabla\psi_j)^2 + e^{-2\psi_j} (\nabla\sigma_j)^2 \right] \right). \quad (2.11)$$

One special example of an  $O(d, d)$  invariance is the *duality* symmetry associated with an interchange of  $M$  and  $M^{-1}$  [10]. The duality transformed fields are

$$\begin{aligned} e^{-\tilde{\psi}_j} &= e^{\psi_j} + e^{-\psi_j} \sigma_j^2 \\ \tilde{\sigma}_j &= - \left( e^{\psi_j} + e^{-\psi_j} \sigma_j^2 \right)^{-1} e^{-\psi_j} \sigma_j \end{aligned} \quad (2.12)$$

and a new class of solution parametrized by  $\{\tilde{\psi}_j, \tilde{\sigma}_j\}$  may be generated from a solution given in terms of  $\{\psi_j, \sigma_j\}$ .

The WDW equation for the theory (2.11) may now be derived using the standard techniques [12].

The classical Hamiltonian constraint is

$$a^D \mathcal{H} \propto -a^2 \Pi_a^2 + 2D(D-1) \sum_{j=1}^{d/2} \left[ \Pi_{\psi_j}^2 + e^{2\psi_j} \Pi_{\sigma_j}^2 \right] - \alpha^2 a^{2(D-1)} = 0, \quad (2.13)$$

where  $\mathcal{H}$  is the Hamiltonian density,  $\Pi_{z_i} \equiv \partial\mathcal{L}/\partial\dot{z}_i$  are the momenta conjugate to  $z_i$  and a dot denotes differentiation with respect to  $t$ . The constant  $\alpha = 2D(D-1)\Omega_D$  where  $\Omega_D$  is the volume of the unit  $D$ -sphere. The system is quantized by identifying the conjugate momenta with the operators

$$\Pi_a^2 = -a^{-p} \frac{\partial}{\partial a} a^p \frac{\partial}{\partial a}, \quad \Pi_{\psi_j}^2 = -\psi_j^{-q_j} \frac{\partial}{\partial \psi_j} \psi_j^{q_j} \frac{\partial}{\partial \psi_j}, \quad \Pi_{\sigma_j}^2 = -\sigma_j^{-r_j} \frac{\partial}{\partial \sigma_j} \sigma_j^{r_j} \frac{\partial}{\partial \sigma_j}, \quad (2.14)$$

where the arbitrary constants  $\{p, q_j, r_j\}$  account for ambiguities in the operator ordering. The WDW equation follows by viewing  $\mathcal{H}$  as an operator acting on the wavefunction  $\Psi[a, \psi_j, \sigma_j]$ :

$$\begin{aligned} \left[ a^{2-p} \frac{\partial}{\partial a} a^p \frac{\partial}{\partial a} - 2D(D-1) \sum_{j=1}^{d/2} \left( \psi_j^{-q_j} \frac{\partial}{\partial \psi_j} \psi_j^{q_j} \frac{\partial}{\partial \psi_j} + e^{2\psi_j} \sigma_j^{-r_j} \frac{\partial}{\partial \sigma_j} \sigma_j^{r_j} \frac{\partial}{\partial \sigma_j} \right) \right. \\ \left. - \alpha^2 a^{2(D-1)} \right] \Psi = 0. \end{aligned} \quad (2.15)$$

We proceed in the following sections to solve this equation.

### 3 Exact Solutions

Exact solutions to Eq. (2.15) may be found for  $q_j = 0$ . We first search for separable solutions of the form

$$\Psi[a, \psi_j, \sigma_j] = \Phi[a, \Psi_j] \prod_{j=1}^{d/2} C_j(\sigma_j). \quad (3.1)$$

The WDW equation decouples to  $(d+2)/2$  differential equations:

$$\left[ a^{2-p} \frac{\partial}{\partial a} a^p \frac{\partial}{\partial a} - \alpha^2 a^{2(D-1)} - 2D(D-1) \sum_{j=1}^{d/2} \left( \frac{\partial^2}{\partial \psi_j^2} - \omega_j^2 e^{2\psi_j} \right) \right] \Phi = 0 \quad (3.2)$$

$$\sigma_l^{-r_l} \frac{d}{d\sigma_l} \sigma_l^{r_l} \frac{dC_l}{d\sigma_l} + \omega_l^2 C_l = 0, \quad l = 1, \dots, d/2, \quad (3.3)$$

where  $\omega_l$  are arbitrary separation constants. Eq. (3.3) admits the exact solutions

$$C_l = \sigma_l^{(1-r_l)/2} J_{\frac{1-r_l}{2}}(\omega_l \sigma_l), \quad \omega_l \neq 0 \quad (3.4)$$

$$C_l = \text{constant}, \quad \omega_l = 0, \quad (3.5)$$

where  $J_p$  is a Bessel function of order  $p$ . Classically Eq. (3.5) corresponds to  $\dot{\sigma}_l = 0$ , which is a particular solution to the Einstein field equations. In this case each of the  $\sigma_l$  may be set to zero without loss of generality by means of a linear translation. Thus we shall first solve Eq. (3.2) for  $\omega_l = 0$ . In this paper we are concerned only with the functional forms of the solutions to Eq. (2.15) and we therefore ignore all constants of proportionality in the wavefunction.

#### 3.1 Case A: $\omega_l = 0$

It proves convenient to make a change of variables to  $\{u, v, s_1, \dots, s_{(d-2)/2}\}$  defined in such a way that  $\Phi$  is independent of the  $s_j$ . This necessarily results in some loss of generality but allows analytical solutions to be found. It is shown in the appendix that with a factor ordering  $p = 1$ , Eq. (3.2) reduces to the simplified canonical form

$$\left[ \frac{\partial}{\partial u} \frac{\partial}{\partial v} - \frac{\alpha^2}{4} (uv)^{D-2} \right] \Phi = 0, \quad (3.6)$$

where

$$u \equiv a \exp \left[ -\frac{1}{\sqrt{2D(D-1)}} \sum_{j=1}^{d/2} \psi_j \cos \theta_j \right] \quad (3.7)$$

$$v \equiv a \exp \left[ \frac{1}{\sqrt{2D(D-1)}} \sum_{j=1}^{d/2} \psi_j \cos \theta_j \right] \quad (3.8)$$

and the set of constants  $\{\theta_j\}$  are solutions to the constraint equation

$$\sum_{j=1}^{d/2} \cos^2 \theta_j = 1. \quad (3.9)$$

The  $\{u, v\}$  variables are useful because they eliminate the direct dependence of the WDW equation on the number of axion fields present. We shall refer to them as the *canonical* coordinates. These variables are the null coordinates over minisuperspace when only one axion field is present [13]. The form of Eq. (3.6) remains invariant if one adds or removes extra massless scalar fields and the alterations are accounted for by changing the number of fields included in the summations of Eqs. (3.7) - (3.9). We note also that Eq. (3.6) is invariant under an interchange of these variables.

The variables  $u$  and  $v$  may be redefined as

$$u = \beta(y - x)^{2/(D-1)}, \quad v = \beta(y + x)^{2/(D-1)}, \quad \beta = \left( \frac{D-1}{2\alpha} \right)^{1/(D-1)}. \quad (3.10)$$

In this case Eq. (3.6) transforms to

$$\left[ \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} - y^2 + x^2 \right] \Phi = 0, \quad (3.11)$$

which is the equation for two harmonic oscillators with equal and opposite energy [7].

A singular solution to Eq. (3.2) is

$$\Phi_\gamma = K_{\pm i\epsilon} \left( \alpha a^{D-1} / (D-1) \right) \exp \left[ i\gamma \sum_{j=1}^{d/2} \psi_j \cos \Theta_j \right], \quad (3.12)$$

where

$$\sum_{j=1}^{d/2} \cos^2 \Theta_j = 1, \quad (3.13)$$

$K$  is a modified Bessel function.  $\gamma$  is an arbitrary constant and

$$\frac{\gamma}{\epsilon} \equiv \sqrt{\frac{D-1}{2D}}. \quad (3.14)$$

On the other hand, non-singular solutions to Eqs. (3.6) and (3.11) are

$$\Phi_c = \exp \left[ -\frac{\alpha}{2(D-1)} (c^{-1}u^{D-1} + cv^{D-1}) \right] \quad (3.15)$$

and

$$\Phi_n = \frac{1}{2^n n!} H_n(x) H_n(y) \exp \left[ -(x^2 + y^2) / 2 \right], \quad (3.16)$$

respectively, where  $c$  is an arbitrary integration constant and  $H_n$  are Hermite polynomials [7]. Solution (3.15) may be written directly in terms of the scale factor and axion fields:

$$\Phi_\lambda = \exp \left\{ -\frac{\alpha}{D-1} a^{D-1} \cosh \left[ \frac{\gamma}{\epsilon} \left( \sum_{j=1}^{d/2} \psi_j \cos \theta_j \right) + \lambda \right] \right\}, \quad (3.17)$$

where  $\lambda \equiv \ln c$ . Solutions equivalent to (3.16) and (3.17) have been found previously in a different context by Zhuk [14]. They do not oscillate as the spatial metric degenerates and they are exponentially damped at infinity. Consequently they may be viewed as quantum wormholes with asymptotic topology  $\mathfrak{R} \times S^D$ .

The ground state wavefunction of Eq. (3.15) corresponds to  $c = 1$  ( $\lambda = 0$ ) and excited states, corresponding to  $c \neq 1$ , are generated by linear translations on the  $\psi_j$  fields [8]. Thus there exists a continuous spectrum of these excited wormhole states. On the other hand Eq. (3.16) represents a discrete spectrum of excited states [7], but the two spectra have the equivalent ground state. The duality transformation (2.12) maps  $\psi_j$  to  $-\psi_j$  when the  $\sigma_j$  fields vanish and this is equivalent to an interchange of the canonical coordinates. Hence it follows from Eq. (3.15) that the ground state may be interpreted as the self-dual solution to the WDW equation. In this sense the ground state is associated with the solution of maximum symmetry. In figure 1 this ground state and an excited state of the continuous wormhole spectrum (3.15) are plotted to illustrate this feature.

**Figure 1**

Additional wormhole solutions may be found by factoring out the continuous spectrum given by Eq. (3.15). If we write

$$\Phi(u, v) \equiv \Delta(u, v)\Phi_c(u, v), \quad (3.18)$$

it follows after substitution into Eq. (3.6) that Eq. (3.18) is a solution to the WDW equation if  $\Delta$  satisfies

$$\frac{\partial^2 \Delta}{\partial u \partial v} - \frac{\alpha c v^{D-2}}{2} \frac{\partial \Delta}{\partial u} - \frac{\alpha u^{D-2}}{2c} \frac{\partial \Delta}{\partial v} = 0. \quad (3.19)$$

When  $c = 1$  a comparison with Eq. (3.16) implies that  $\Delta$  may be written as a product of two Hermite polynomials. For  $c \neq 1$  one solution to Eq. (3.19) is

$$\Delta = c v^{D-1} - c^{-1} u^{D-1} \quad (3.20)$$

$$= 2a^{D-1} \sinh \left[ \frac{\gamma}{\epsilon} \left( \sum_{j=1}^{d/2} \psi_j \cos \theta_j \right) + \lambda \right]. \quad (3.21)$$

### 3.2 Case B: $\omega_l \neq 0$

The question now arises as to whether singular and non-singular solutions to Eq. (3.2) can be found when  $\omega_l \neq 0$ . If we assume the separable ansatz

$$\Phi[a, \psi_j] = A(a) \prod_{j=1}^{d/2} B_j(\psi_j) \quad (3.22)$$

for arbitrary functions  $A(a)$  and  $B_j(\psi_j)$ , we find

$$a^{2-p} \frac{d}{da} a^p \frac{dA}{da} + \left( 2D(D-1)z^2 - \alpha^2 a^{2(D-1)} \right) A = 0 \quad (3.23)$$

and

$$\frac{1}{B_j} \frac{d^2 B_j}{d\psi_j^2} - \omega_j^2 e^{2\psi_j} = -z_j^2 = -z^2 \cos^2 \Theta_j, \quad (3.24)$$

where  $\Theta_j$  again satisfy the integrability condition (3.13) and  $z$  is an arbitrary constant.

Eq. (3.24) reduces to a Bessel equation after the change of variables  $\psi_j = \ln \xi_j$ . Hence one solution to Eq. (3.2) is

$$\Phi_z = a^{(1-p)/2} K_{\pm s} \left( \alpha a^{D-1} / (D-1) \right) \prod_{j=1}^{d/2} K_{\pm iz_j} \left( \omega_j e^{\psi_j} \right), \quad (3.25)$$

where

$$s^2 = \frac{1}{(D-1)^2} \left[ \left( \frac{1-p}{2} \right)^2 - 2D(D-1)z^2 \right] \quad (3.26)$$

and the expression for the full wavefunction  $\Psi$  is

$$\Psi[a, \psi_j, \sigma_j] = a^{(1-p)/2} K_s \left( \alpha a^{D-1} / (D-1) \right) \prod_{j=1}^{d/2} \left\{ K_{iz_j} \left( \omega_j e^{\psi_j} \right) J_{\frac{1-r_j}{2}} \left( \omega_j \sigma_j \right) \sigma_j^{(1-r_j)/2} \right\}. \quad (3.27)$$

For simplicity let us consider the case  $p = 1$  and  $r_j = 0$ . This leaves the WDW equation invariant under a change of minisuperspace coordinates [13] and the  $C_j$  functions represent plane wave solutions. The modified Bessel function  $K_q(x)$  is exponentially damped at infinity, but diverges as  $K_q \propto x^{-q}$  in the limit  $x \rightarrow 0$ . It follows from Eq. (3.25) that

$$\Phi_z \propto \exp \left[ -iz \left( \sqrt{2D(D-1)} \ln a + \sum_{j=1}^{d/2} \psi_j \cos \Theta_j \right) \right] \quad (3.28)$$

as  $\{a, e^{\psi_j}\} \rightarrow 0$  and the wavefunction oscillates an infinite number of times. This corresponds to an initial singularity, but may be removed by integrating over the separation constant  $z$ . This is analogous to taking the Fourier transform [8]. The Riemann-Lebesgue lemma [15] states that

$$\lim_{p \rightarrow +\infty} \int_{-\infty}^{+\infty} dx e^{ipx} = 0 \quad (3.29)$$

and it follows that the transformed wavefunction

$$\bar{\Phi} = \int_{-\infty}^{+\infty} dz \Phi_z \quad (3.30)$$

is damped both at the origin and at infinity. It is therefore Euclidean for all values of  $a$ .

Finally new solutions may now be generated from Eq. (3.27) in terms of the duality transformed fields defined in Eq. (2.12). After a simple rearrangement we find

$$e^{\psi_j} = e^{-\tilde{\psi}_j} \left( 1 + \tilde{\sigma}_j^2 e^{-2\tilde{\psi}_j} \right)^{-1} \quad (3.31)$$

and

$$\sigma_j = -\bar{\sigma}_j e^{-2\bar{\psi}_j} \left(1 + \bar{\sigma}_j^2 e^{-2\bar{\psi}_j}\right)^{-1}. \quad (3.32)$$

Substitution of these expressions into Eq. (3.27) generates a more complicated class of singular solution  $\Psi[a, \bar{\psi}_j, \bar{\sigma}_j]$ .

## 4 Power Series Solutions via a Method of Successive Approximations

Additional solutions to Eq. (3.6) may be derived by employing a method of successive approximations.

We assume the wavefunction may be expanded as the infinite sum

$$\Phi(u, v) \equiv \sum_{m=0}^{\infty} \Phi_m(u, v). \quad (4.1)$$

This is a solution to Eq. (3.6) if the functions  $\Phi_0$  and  $\Phi_m$  ( $m \geq 1$ ) satisfy

$$\frac{\partial^2 \Phi_0}{\partial u \partial v} = 0 \quad (4.2)$$

and

$$\frac{4}{\Phi_{m-1}} \frac{\partial^2 \Phi_m}{\partial u \partial v} = \alpha^2 (uv)^{D-2}, \quad m \geq 1, \quad (4.3)$$

respectively. Eq. (4.2) is the canonical form of the one-dimensional wave equation and admits the *general* solution

$$\Phi_0 = f(u) + g(v), \quad (4.4)$$

where  $f(u)$  and  $g(v)$  are arbitrary twice continuously differentiable functions.

An iteration procedure may now be established by substituting the solution for  $\Phi_0$  into Eq. (4.3), solving for  $\Phi_1$ , and then repeating the process. The symmetry of Eqs. (4.3) and (4.4) allows this to be done analytically by expressing the solutions in terms of quadratures with respect to the canonical

coordinates. For simplicity we assume for the moment that  $g(v) = 0$ . It follows that for  $m \geq 1$   $\Phi_m$  may be written in terms of quadratures with respect to  $u$ :

$$\Phi_m = \left(\frac{\alpha}{2}\right)^{2m} \frac{v^{m(D-1)}}{m!(D-1)^m} \int^u du_m u_m^{D-2} \int^{u_m} du_{m-1} u_{m-1}^{D-2} \dots \int^{u_2} du_1 u_1^{D-2} f(u_1). \quad (4.5)$$

By symmetry however, if  $\Phi_0 = g(v)$ , the form of  $\Phi_m$  is identical to Eq. (4.5) but with  $u$  replaced by  $v$  and  $f(u_1)$  replaced by  $g(v_1)$ . A more general expression for the wavefunction is therefore

$$\Phi = \Phi_0 + \sum_{m=1}^{\infty} \left(\frac{\alpha}{2}\right)^{2m} \frac{(uv)^{m(D-1)}}{m!(D-1)^m} [F_m(u) \pm G_m(v)] \quad (4.6)$$

$$F_m(u) \equiv \frac{1}{u^{m(D-1)}} \int^u du_m u_m^{D-2} \int^{u_m} du_{m-1} u_{m-1}^{D-2} \dots \int^{u_2} du_1 u_1^{D-2} f(u_1) \quad (4.7)$$

$$G_m(v) \equiv \frac{1}{v^{m(D-1)}} \int^v dv_m v_m^{D-2} \int^{v_m} dv_{m-1} v_{m-1}^{D-2} \dots \int^{v_2} dv_1 v_1^{D-2} g(v_1). \quad (4.8)$$

The integrals in Eqs. (4.7) and (4.8) may be evaluated analytically when

$$\Phi_0 = v^b \pm u^b, \quad b = \text{constant} \quad (4.9)$$

and it is straightforward to show that

$$F_m + G_m = \frac{\Phi_0}{[D-1+b][2(D-1)+b] \dots [m(D-1)+b]}. \quad (4.10)$$

It follows that the wavefunction reduces to the series solution

$$\Phi = \Phi_0 \left[ 1 + \sum_{m=1}^{\infty} \left(\frac{\alpha}{2}\right)^{2m} \frac{(uv)^{m(D-1)}}{m!(D-1)^m} \frac{1}{[D-1+b] \dots [m(D-1)+b]} \right]. \quad (4.11)$$

Closed expressions for  $\Phi$  exist when  $b = 0$  and  $b = \pm(D-1)/2$  and these solutions are shown in Table 1. They are Euclidean for all values of the scale factor in the sense that they do not oscillate. However they diverge exponentially fast at infinity and therefore do not represent quantum wormholes in the Hawking-Page sense. Solution I is identified as the Hartle-Hawking ground state [16]. On the other hand a linear combination of solutions II and III is

$$\begin{aligned} \Phi_{\text{III}} - \Phi_{\text{II}} &= \left(v^{-(D-1)/2} + u^{-(D-1)/2}\right) \exp\left(-\frac{\alpha(uv)^{(D-1)/2}}{D-1}\right) \\ &= \left(\frac{8\alpha}{\pi(D-1)}\right)^{1/2} K_{1/2}\left(\frac{\alpha}{D-1} a^{D-1}\right) \cosh\left[\left(\frac{D-1}{8D}\right)^{1/2} \sum_{j=1}^{d/2} \psi_j \cos \theta_j\right]. \end{aligned} \quad (4.12)$$

Solution	$b$	Wavefunction
I	0	$I_0 \left( \frac{\alpha(uv)^{(D-1)/2}}{D-1} \right)$
II	$\frac{D-1}{2}$	$(u^{-b} \pm v^{-b}) \sinh \left( \frac{\alpha(uv)^{(D-1)/2}}{D-1} \right)$
III	$\frac{1-D}{2}$	$(v^b \pm u^b) \cosh \left( \frac{\alpha(uv)^{(D-1)/2}}{D-1} \right)$

Table 1: Closed solutions to the canonical form of the WDW equation when  $\Phi_0 = v^b \pm u^b$ .  $I_0$  is the modified Bessel function of the first kind of order zero. All solutions diverge at infinity, but linear combinations may be found which are exponentially damped at infinity. It is straightforward to verify by differentiation that these are solutions to Eq. (3.6).

This solution has the correct asymptotic behaviour at infinity to be interpreted as a quantum wormhole. Although it diverges as  $a^{(1-D)/2}$  when the spatial metric degenerates, this may be acceptable behaviour for a quantum wormhole [9]. Moreover, Eq. (3.6) was derived from the field theoretic limit of the string effective action, and one might expect the analysis of Sect. 2 to be invalid for scales below the Planck length. One could therefore argue that this divergence will not arise in a more accurate analysis.

## 5 An Exact Solution in the Bianchi IX Cosmology

The dimensionally reduced action (2.11) is essentially  $(D+1)$ -dimensional Einstein gravity minimally coupled to a set of massless scalar fields with non-standard kinetic terms. It is therefore possible when  $D=3$  to extend the results from the previous sections to the class of anisotropic and homogeneous Bianchi cosmologies. The line element for the class A Bianchi spaces is [17]

$$ds^2 = -dt^2 + e^{2\bar{a}(t)} \left( e^{2\beta(t)} \right)_{ij} \epsilon^i \epsilon^j, \quad (5.1)$$

where  $e^{6\bar{a}}$  is the determinant of the metric on the three-surface,  $\beta_{ij}$  is a tracefree, symmetric matrix and  $\epsilon^j$  are the one-forms determining the isometry of the three-surface. Without loss of generality  $\beta_{ij}$

may be diagonalized:

$$\beta_{ij}(t) = \text{diag} \left[ \beta_+(t) + \sqrt{3}\beta_-(t), \beta_+(t) - \sqrt{3}\beta_-(t), -2\beta_+(t) \right]. \quad (5.2)$$

Following an identical procedure to that presented at the end of Sect. 2, the WDW equation derived from the action (2.11) is found to be

$$\left[ e^{-p\bar{a}} \frac{\partial}{\partial \bar{a}} e^{p\bar{a}} \frac{\partial}{\partial \bar{a}} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} + U(\bar{a}, \beta_{\pm}) - 12 \sum_{j=1}^{d/2} \left( \frac{\partial^2}{\partial \psi_j^2} + e^{2\psi_j} \frac{\partial^2}{\partial \sigma_j^2} \right) \right] \Psi = 0, \quad (5.3)$$

where the potential  $U(\bar{a}, \beta_{\pm})$  depends on the spatial curvature in the model. For simplicity we choose a factor ordering  $q_j = r_j = 0$  for the scalar field momentum operators and rescale such that  $\alpha = 1$ . The WDW equation (2.15) for the positively curved, isotropic FRW model is recovered by setting  $\beta_+ = \beta_- = 0$  in the potential and removing the  $\partial^2 \Psi / \partial \beta_{\pm}^2$  terms in Eq. (5.3). When  $\beta_- = 0$  the Bianchi IX universe simplifies to the Taub cosmology [18].

To proceed we assume the separable ansatz:

$$\Psi[\bar{a}, \beta_+, \beta_-, \psi_j, \sigma_j] = \Phi[\bar{a}, \beta_+, \beta_-] S[\psi_j, \sigma_j]. \quad (5.4)$$

The WDW equation separates into two differential equations:

$$\left[ \frac{\partial^2}{\partial \bar{a}^2} + p \frac{\partial}{\partial \bar{a}} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} + U - m^2 \right] \Phi = 0 \quad (5.5)$$

$$\left[ 12 \sum_{j=1}^{d/2} \left( \frac{\partial^2}{\partial \psi_j^2} + e^{2\psi_j} \frac{\partial^2}{\partial \sigma_j^2} \right) - m^2 \right] S = 0, \quad (5.6)$$

where  $m$  is an arbitrary separation constant.

The curvature potential for the type IX may be written in the form [19]

$$3U = e^{4\bar{a}} \left[ e^{-8\beta_+} - 4e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + 2e^{4\beta_+} (\cosh 4\sqrt{3}\beta_- - 1) \right]. \quad (5.7)$$

It is well known that Eq. (5.7) exhibits a triangular symmetry in the  $(\beta_+, \beta_-)$  plane, but recently it was shown [20] that there exists an additional symmetry

$$\left( \frac{\partial \chi}{\partial \bar{a}} \right)^2 - \left( \frac{\partial \chi}{\partial \beta_+} \right)^2 - \left( \frac{\partial \chi}{\partial \beta_-} \right)^2 + U = 0, \quad (5.8)$$

where

$$\chi = \frac{1}{6}e^{2\tilde{a}}\text{Tr}\left(e^{2\beta}\right) = \frac{1}{6}e^{2\tilde{a}}\left[e^{-4\beta_+} + 2e^{2\beta_+}\cosh 2\sqrt{3}\beta_-\right]. \quad (5.9)$$

This additional symmetry may be employed to find new exact solutions. If we substitute the secondary ansatz

$$\Phi = W(\tilde{a})e^{-\chi} \quad (5.10)$$

into Eq. (5.5), and employ the identities

$$\frac{\partial^2\chi}{\partial\tilde{a}^2} - \frac{\partial^2\chi}{\partial\beta_+^2} - \frac{\partial^2\chi}{\partial\beta_-^2} + 12\chi = \frac{\partial\chi}{\partial\tilde{a}} - 2\chi = 0, \quad (5.11)$$

it follows that Eq. (5.5) simplifies to

$$\frac{\partial^2W}{\partial\tilde{a}^2} + 12\chi W + p\frac{\partial W}{\partial\tilde{a}} - 2p\chi W - 4\chi\frac{\partial W}{\partial\tilde{a}} - m^2W = 0. \quad (5.12)$$

For specific choices of the factor ordering parameter  $p$ , Eq. (5.12) admits the exponential solution  $W = \exp(-c\tilde{a})$ , where  $c$  is a decay constant. Eq. (5.12) reduces to the constraint equation

$$c^2 - pc - m^2 + 2(p - 2c - 6)\chi = 0, \quad (5.13)$$

which is solved for

$$p = 2(c + 3), \quad c = -3 \pm \sqrt{(3 + m)(3 - m)}. \quad (5.14)$$

Requiring the factor ordering to be real implies  $|m| \leq 3$ . This solution is a generalization of the vacuum solution corresponding to  $m = 0$  found in [21]. We find a new vacuum solution for  $c = p = -6$ . It is worth remarking that the  $e^{-\chi}$  factor in Eq. (5.10) appears to be independent of the choice of factor ordering.

Finally we may combine these results with the solution to Eq. (5.6). This equation is solved by separating it into Eqs. (3.3) and (3.24) as in Sect. 3. Hence, one solution to Eq. (5.3) for the Bianchi IX cosmology is

$$\psi = e^{-c\tilde{a} - \chi} \prod_{j=1}^{d/2} \left\{ K \frac{m}{\sqrt{12}} \cos\theta_j \left( \omega_j e^{\psi_j} \right) e^{\pm i\omega_j \sigma_j} \right\}, \quad (5.15)$$

where  $\omega_j$  are arbitrary constants that may be imaginary and the  $\theta_j$  must satisfy  $\sum_{j=1}^{d/2} \cos \theta_j^2 = 1$ .

A new class of solution may now be generated from Eq. (5.15) as a consequence of the duality symmetry of the action (2.11), as summarized in Eqs. (3.31) and (3.32). However, the real-phase part of the solution,  $\Phi$ , remains invariant after such a transformation.

When  $c = 0$  it was noticed in [21] that as  $\tilde{a}$  increases the wavefunction  $\Phi$  becomes peaked in the  $(\beta_+, \beta_-)$  plane around the isotropic FRW solution  $\beta_+ = \beta_- = 0$ . Our solutions also exhibit this feature for values of  $c$  given by Eq. (5.13). In principle, a solution of the form  $\Phi = e^{-c\tilde{a} - \chi}$  exists for any WDW equation that contains a separable component of the form (5.5). For example such a separation is possible with any matter sector that behaves as a stiff perfect fluid at the classical level. Dynamically this is equivalent to a massless scalar field and at the quantum-mechanical level the matter is an eigenstate of some operator with eigenvalue  $m$ . The constant  $2\pi^2 m$  represents a conserved scalar flux.

## 6 Discussion

In this paper the WDW equation was derived from the dimensionally reduced string effective action assuming a toroidal compactification. The  $(D+1)$ -dimensional FRW and the four-dimensional Bianchi IX (mixmaster) cosmologies were investigated. In the former example singular solutions were found and these may be interpreted as spatially closed Lorentzian universes. Euclidean solutions were also found which may be viewed as quantum wormholes if appropriate boundary conditions are satisfied. The correct quantum theory of gravity should explain, either from symmetry or probabilistic considerations, why compactification to a four-dimensional space-time is observed. For this reason the dimensionality of the external space was left unspecified in the analysis.

In the Bianchi type IX case exact solutions were obtained for specific choices of factor ordering. The wavefunction is strongly peaked around the FRW universe at large three-geometries. Recently a symmetry of the form (5.8) has been found for the Bianchi type II metric [22] and it appears that this

symmetry exists for all class A spaces. In principle solutions of the form Eq. (5.15) can therefore be found for these spaces.

In both the FRW and Bianchi IX examples the  $O(d, d)$  symmetry of the string effective action may be employed to generate new classes of solutions which may otherwise be difficult to derive using standard techniques. This was illustrated using the duality symmetry of the action, which is a subgroup of  $O(d, d)$ . The action (2.11) is also symmetric under the group  $SL(2, R)$  and new solutions may also be generated from this symmetry [6].

A method of successive approximations was employed for the FRW cosmology to derive further solutions. This method should be useful for finding perturbative solutions when  $uv = a^2$  is small (but not so small that the analysis of Sect. 2 breaks down). In this region one may view the power series as a truncated Taylor series up to some appropriate order in the scale factor.

Finally, in the regime where the curvature term of Eq. (3.6) becomes negligible, the Wheeler-DeWitt equation (3.6) reduces to the one-dimensional wave equation:

$$\frac{\partial^2 \Phi}{\partial u \partial v} = 0 \tag{6.1}$$

with general solution  $\Phi = f(u) + g(v)$ . Eq. (6.1) may also be treated as the exact WDW equation for a spatially flat Friedmann universe with nontrivial spatial topology such as the 3-torus  $T^3 = S^1 \times S^1 \times S^1$  [23]. This illustrates a further advantage of employing the canonical coordinates. In this case one may easily find forms of  $f(u)$  and  $g(v)$  which satisfy the Hawking-Page boundary conditions and this suggests that a positive spatial curvature is not a necessary condition for the existence of quantum wormholes [24].

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## Appendix

The purpose of this appendix is to derive the expressions for the canonical coordinates when  $\omega_j = 0$ . We make an arbitrary change of variables in Eq. (3.2) and search for solutions that depend on only two of these new variables  $u$  and  $v$ . We choose  $p = 1$  and define  $\bar{a} = \ln a$  for convenience. The differential momentum operators of Eq. (3.2) take the form

$$\frac{\partial^2 \Phi}{\partial w^2} = \frac{\partial^2 u}{\partial w^2} \frac{\partial \Phi}{\partial u} + \left( \frac{\partial u}{\partial w} \right)^2 \frac{\partial^2 \Phi}{\partial u^2} + 2 \frac{\partial u}{\partial w} \frac{\partial v}{\partial w} \frac{\partial^2 \Phi}{\partial u \partial v} + \frac{\partial^2 v}{\partial w^2} \frac{\partial \Phi}{\partial v} + \left( \frac{\partial v}{\partial w} \right)^2 \frac{\partial^2 \Phi}{\partial v^2}, \quad (\text{A.1})$$

where the variable  $w$  is to be identified with  $\ln a$  and  $\psi_j$ .

One may arrange for the majority of these terms to cancel after substitution into Eq. (3.2). Only the contributions containing  $\partial^2 \Phi / \partial u \partial v$  remain if  $u$  is a solution of the  $d/2$ -dimensional wave equation:

$$\frac{\partial^2 u}{\partial \bar{a}^2} - 2D(D-1) \sum_{j=1}^{d/2} \frac{\partial^2 u}{\partial \psi_j^2} = 0, \quad (\text{A.2})$$

subject to the integrability condition

$$\left( \frac{\partial u}{\partial \bar{a}} \right)^2 - 2D(D-1) \sum_{j=1}^{d/2} \left( \frac{\partial u}{\partial \psi_j} \right)^2 = 0. \quad (\text{A.3})$$

Equivalent equations also apply for  $v$ . The first integral of Eq. (A.2) is

$$\frac{\partial u}{\partial \psi_j} = \pm \frac{\cos \theta_j}{\sqrt{2D(D-1)}} \frac{\partial u}{\partial \bar{a}}, \quad (\text{A.4})$$

where the constants of integration  $\theta_j$  satisfy the constraint equation  $\sum_{j=1}^{d/2} \cos^2 \theta_j = 1$ . It follows that

$$u = f \left( \ln a \pm [2D(D-1)]^{-1/2} \sum_{j=1}^{d/2} \psi_j \cos \theta_j \right), \quad (\text{A.5})$$

where  $f$  is some arbitrary function. To ensure that the  $\partial^2 \Psi / \partial u \partial v$  terms do not cancel in the WDW equation, the argument in  $v$  for the scalar fields must take the opposite sign to that in  $u$ , i.e.

$$v = g \left( \ln a \mp [2D(D-1)]^{-1/2} \sum_{j=1}^{d/2} \psi_j \cos \theta_j \right), \quad (\text{A.6})$$

where  $g$  is a second arbitrary function.

If we assume the separable ansatz

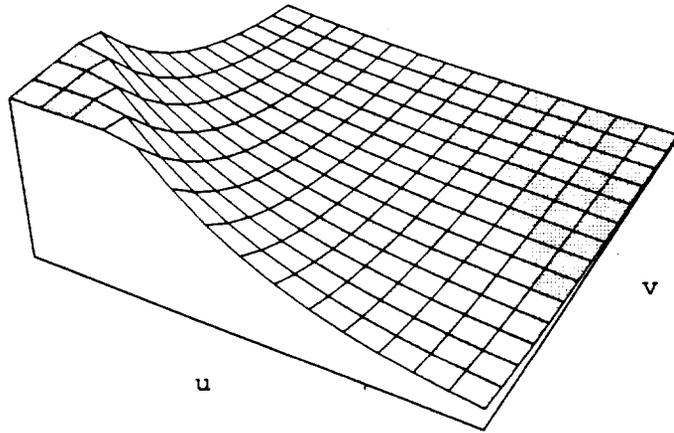
$$u = u(\bar{a}) \prod_{j=1}^{d/2} u_j(\psi_j), \quad v = v(\bar{a}) \prod_{j=1}^{d/2} v_j(\psi_j), \quad (\text{A.7})$$

it is easy to verify by differentiation that one solution to these equations is given by Eqs. (3.7), (3.8) and (3.9).

## Figure Caption

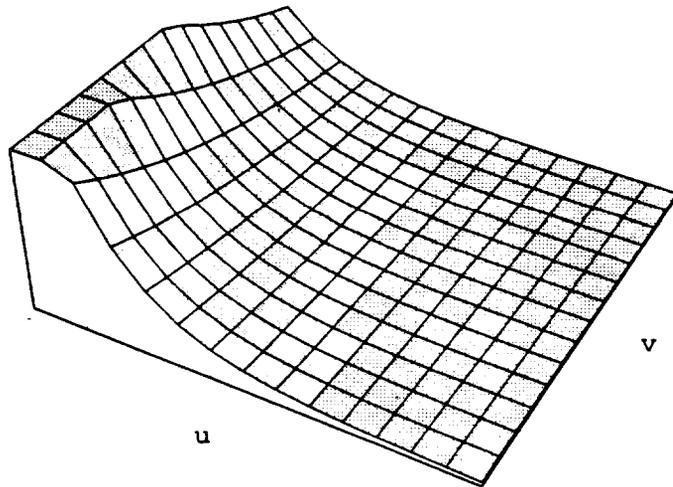
*Figure 1:* In Figure (1a) a schematic plot of the self-dual ground state of both the continuous and discrete wormhole spectra (3.15) and (3.16) is shown as a function of the canonical coordinates. The symmetric nature of the solution is clearly illustrated. In Figure (1b) an excited state of (3.15) is plotted for  $c = 1/2$ . As noted in the text the ground state may be viewed as the solution of maximum symmetry.

Self-Dual Ground State



(a)

An Excited State



(b)