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INITIAL CONDITIONS FOR NATURAL INFLATION

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ABSTRACT

We investigate the issue of initial conditions for natural inflation. Unlike many inflationary models, the pseudo-Nambu-Goldstone boson nature of the inflaton field provides a natural measure for the phase space of initial conditions. We study the effects of the inflaton kinetic term numerically and show that it shifts the range of initial field values which lead to successful inflation without altering the size of that range. The fraction of phase space in the successful range is determined by the spontaneous symmetry breaking scale, f, and is 0.7, 0.2, and 3×10^{-3} for $f = 3M_{Pl}$, M_{Pl}, and $M_{Pl}/2$ respectively. Natural inflation becomes similar to chaotic inflation for values of $f \gtrsim M_{Pl}$ and for $f \lesssim M_{Pl}$, it is more akin to new inflation. The roles of spatial curvature and spatial gradients are briefly discussed.

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The inflationary scenario[1] remains the most elegant solution to the horizon, flatness, and monopole problems. During inflation, the Universe is dominated by vacuum energy density, $\rho \simeq \rho_{vac} \simeq const.$, and the cosmic scale factor grows quasi-exponentially with time, $R(t) \propto e^{Ht}$, where $H = \dot{R}/R \simeq [8\pi G\rho_{vac}/3]^{1/2}$ is the Hubble parameter. If $R(t)$ increases by more than 60 e-foldings during inflation, a small causally connected region grows sufficiently to explain the homogeneity, isotropy, and flatness of the Universe and to dilute any overdensity of magnetic monopoles and other relics. Inflation also provides a predictive scenario for the origin of density perturbations: quantum fluctuations during inflation get stretched beyond the Hubble radius, causally generating density fluctuations on the very large scales required for galaxy formation.[2]

Inflation makes two testable predictions: that $\Omega_{Tot} = \frac{8\pi G\rho_{tot}}{3H^2} = 1$ (more precisely, that our observable universe is spatially flat) and that the spectrum of large-scale density fluctuations is approximately scale-invariant, $P(k) \propto k$. Two notable recent observations consistent with these predictions are the results of the IRAS survey[3] indicating $\Omega^{0.6}/b \simeq 1.28^{+0.75}_{-0.59}$ (where b is the biasing of galaxies relative to mass, $b \gtrsim 1$is expected) which points in the direction of a flat Universe, and the angular correlation function of the microwave background anisotropy as measured by COBE[4] which agrees well with the scale-invariant spectrum of density fluctuations ($P(k) \propto k^n, n \simeq 1 \pm 0.5$).

The simultaneous requirements of sufficient inflation and cosmic microwave background radiation (CMBR) anisotropy limits constrain the self-coupling of the inflaton field to be extremely weak.[5] Natural inflation[6], in which the inflaton is a pseudo-Nambu-Goldstone boson with a potential of the form $V(\phi) = \Lambda^4(1 + cos(\phi/f))$, was proposed as a model in which the inflaton can have a small coupling which is natural from the particle physics standpoint. In this model, the requisite weak self-coupling arises in a theory with spontaneous symmetry breaking at a scale f and explicit symmetry breaking at a lower energy scale Λ: the scalar self-coupling $\lambda \sim (\Lambda/f)^4$. Successful inflation occurs for $f \sim M_{Pl}$ and $\Lambda \sim M_{GUT}$, mass scales that arise in particle physics models with a gauge group that becomes strongly interacting at the GUT scale.[6,7]

An added bonus of natural inflation is the possible departure from scale-invariance of the power spectrum for density perturbations. Adams et al.[7] studied the implications of power-law spectra inspired by natural inflation, $P(k) \propto k^n, n \simeq 1 - (M_{Pl}^2/8\pi f^2)$, in the cold dark matter scenario of galaxy formation together with recent data from microwave background anisotropies and large-scale structure observations. In particular they showed that natural inflation can help account for much of the excess power at large scales while fitting the recent COBE data if $f \gtrsim 0.4M_{Pl}$.

In Adams et al., as in most treatments of inflation, the evolution of the inflaton was started from a spatially homogeneous initial state with zero kinetic energy, $\phi(x,t) = \phi(t), \dot{\phi}(t_i) = 0$. In this report, we address the problem of initial conditions for natural inflation. We consider in detail the effects of a non-zero initial scalar field time derivative, $\dot{\phi}$, and discuss the effects of non-zero scalar field gradients ($\vec{\nabla}\phi$) and spatial curvature ($k = \pm 1$).

One of the alleged virtues of inflation is its insensitivity to special initial conditions (after all, this is precisely what inflation is meant to provide for the Universe), but it is often difficult to avoid imposing special initial conditions for the inflaton field when implementing a specific model of inflation. It would be preferable that inflation be a generic phenomenon. A model with a very "small" initial condition space for the onset of successful inflation is less attractive than one with a "large" initial condition space. Of course, we need a *measure* on the phase space of initial conditions before we can compare different regions of phase space. We will see that, unlike some other models of inflation, there is a well-defined measure for the $\dot{\phi} \times \phi$ phase space in natural inflation.

For definiteness and simplicity, we consider a complex scalar field, Φ, with a global U(1) symmetry to describe the natural inflation scenario. The symmetry is spontaneously broken at a temperature $T \sim f$ through the vacuum expectation value of the field, $\langle\Phi\rangle = fe^{i\phi/f}$. At temperatures below the scale f, the only relevant degree of freedom is the massless field ϕ, the angular Nambu-Goldstone mode around the bottom of the Φ potential. When the temperature subsequently reaches $T \sim \Lambda$, instanton or other effects explicitly break the angular symmetry, giving rise to the potential $V(\phi) = \Lambda^4(1 + cos(\phi/f))$. If $\dot{\phi}$ is zero at this time $t_1 = t(T \simeq \Lambda)$, inflation occurs in those regions of the Universe with values of $\phi_1 = \phi(t_1)$ that are sufficiently displaced from the minimum of the potential, $\phi_1 \leq \phi_1^{max}$, where[7]

$$sin(\phi_1^{max}/2f) \simeq [1 + \frac{M_{Pl}^2}{48\pi f^2}]^{-1/2} exp(-\frac{15M_{Pl}^2}{4\pi f^2}), \tag{1}$$

(assuming $0 < \phi_1/f < \pi$). That is, if $\phi_1 < \phi_1^{max}$, then the scale factor expands by at least 60 e-folds before ϕ reaches the end of its slow-rolling phase and the Universe exits from inflation.

To study the effect of an initial $\dot{\phi}$ on inflation, we numerically solve the homogeneous Einstein equations minimally coupled to the scalar field:

$$H^2 = \frac{8\pi}{3M_{Pl}^2}[\frac{\dot{\phi}^2}{2} + V(\phi)] - \frac{k}{R^2} \tag{2}$$

and the equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{3}$$

where a dot denotes a derivative with respect to time, a prime denotes the derivative with respect to ϕ, and k = -1, 0, 1 for open, flat, or closed universes. We define $\Pi \equiv \dot{\phi}$ and rewrite eq. (3) as two first order differential equations: $\dot{\phi} = \Pi$, and $\dot{\Pi} = -3H\Pi - V'(\phi)$, which we solve by using fourth order Runge-Kutta with a variable step-size.

There are many requirements for successful inflation[8]. The only one that depends on the initial field configuration is that the number of e-folds of growth in the scale factor during inflation, N_e, must be greater than about 60. We use the numerically integrated trajectories to calculate N_e:

$$N_e = \int_{\ddot{R}>0} H dt = \int_{\phi_1}^{\phi_2} \frac{H}{\dot{\phi}} d\phi \ , \tag{4}$$

where ϕ_2 is the value of the field at the end of inflation (which is reached when $\ddot{R} = 0$).

In figure 1, we show the boundary between "successful" and "unsuccessful" initial conditions (i.e., $N_e = 60$) in the phase-space of initial conditions for $f = 3M_{Pl}$, M_{Pl} and $M_{Pl}/2$. The results for $f = M_{Pl}$ are also shown in figure 2 over a much larger range of Π. Due to the symmetry of the potential, we only need to plot one quadrant of the phase-space plane. As f drops significantly below M_{Pl}, the size of the phase-space region leading to inflation becomes exponentially small and, as f increases beyond M_{Pl}, the size continues to increase.

To interpret these results, we must define a measure on the phase space. With no clearly defined measure, the size of any region of phase space depends on, for example, whether we use logarithmic or linear axes. For inflationary periods starting at the Planck epoch, e.g., Linde has used uncertainty principle arguments to try to get a rough estimate of the measure.[9] Since in a typical "universe creation time", the energy density can only be defined to within $O(M_{Pl}^4)$, every field configuration with a given density up to M_{Pl}^4 is given equal weight. For models in which inflation starts substantially later than the Planck time, it is usually even less clear what the measure should be. The evolution of the "Planck-time measure" to a later time is highly non-trivial and is sensitive to physics at energies well above the scale of inflation.

Even though natural inflation starts well after the Planck epoch ($t_1 \sim M_{Pl}/\Lambda^2 >> t_{Pl}$), it is easy to see what the measure for natural inflation should be. Due to the angular symmetry that exists until the onset of inflation each value of ϕ_1 should be given equal weight. (The idea is that ϕ is laid down randomly for example by the Kibble mechanism on scales larger than the Hubble radius at some time $t < t_1$.) At first it might not be clear how to weight different values of Π, but we find that the measure in the Π direction is irrelevant since for each value of Π, the range of successful ϕ values has the same size. Thus, using the appropriate measure we see that the fraction of phase space that leads to sufficient inflation is 0.7, 0.2 and 3×10^{-3} for $f = 3M_{Pl}$, M_{pl} and $M_{pl}/2$ respectively.[10]

Goldwirth and Piran[11] found that for both chaotic and new inflation, the effect of the kinetic term is to shift the range of ϕ_1 that leads to sufficient inflation by an amount

$$\Delta\phi = \frac{M_{Pl}}{\sqrt{12\pi}}(\ln(\frac{\Pi_1}{\sqrt{2V}}) + 1), \quad \text{for } \Pi_1^2 > 2V \tag{5a}$$

$$\Delta\phi = \frac{\Pi_1}{(24\pi V)^{1/2}} M_{Pl}, \quad \text{for } \Pi_1^2 < 2V. \tag{5b}$$

These results should also apply to natural inflation since the only assumption about the potential that went into the approximations is that it be relatively flat. Indeed, equations 5 agree well with our numerical solutions. If the lines $\phi_1^{bdrf}(\Pi_1) = \phi_1^{bdrf}(\Pi_1 = 0) + \Delta\phi$ were plotted on figure 1, they would be nearly indistinguishable from the numerical boundaries. This boundary line is plotted in figure 2 where it can be seen that the deviation is slight and qualitatively insignificant.

Goldwirth and Piran also show that the size of the phase space leading to successful inflation is 'large' for chaotic inflation and 'small' for new inflation. Their results are related to the dependence of our results on f. For $f \gtrsim M_{Pl}$, natural inflation is similar to chaotic inflation: during the final 60 e-foldings of inflation the potential is well-approximated by $V(\tilde{\phi}) = m^2\tilde{\phi}^2/2$, where $\tilde{\phi} = \phi - \pi f$ and $m^2 = \Lambda^4/f^2$. For $f \lesssim M_{Pl}$, natural inflation is similar to new inflation: the inflation occurs near the maximum at $\phi = 0$, where the potential is similar to $V(\phi) = \lambda(\phi^2 - \sigma^2)^2$ with $\sigma^2 = 6f^2$ and $\lambda = 1/24(\Lambda/f)^4$. Thus the dependence of our results on f can be understood in terms of our understanding of 'chaotic' and 'new' inflation, with the advantage that the angular symmetry of the field constrains the parameter space and gives it a natural measure.

With the inclusion of a spatial curvature term in the analysis, our conclusions cannot be as precise. We expect a typical value at the Planck time for the curvature term in the expression for H^2 to be of the order of M_{Pl}^2. Thus we can imagine a universe at the Planck time that is a patching together of Friedmann-Robertson-Walker spacetimes, each with some value of k/R^2 between $-M_{Pl}^2$ and M_{Pl}^2.[12]. Unless there is an inflationary stage between the Planck time and the onset of natural inflation, some of the closed regions of the Universe will collapse before inflating. These regions of the Universe will appear as black holes from the open space-times which will continue to expand. The curvature contribution to H^2 redshifts as R^{-2} and hence has a typical value of Λ^2 when the epoch of natural inflation is reached. Since vacuum energy does not redshift, this will only slightly delay the onset of inflation [13]. The phase space results for an open universe will not be significantly different from those for a flat universe.

Many authors have considered the influence of scalar field gradients on the onset of inflation [11,14]. Here we mention two aspects of the problem that are peculiar to natural inflation. The first is again the angular symmetry of theory which keeps the dynamics insensitive to different values of ϕ (and as we've shown II) up to the the explicit symmetry breaking scale Λ. When the universe cools to temperatures $T \sim \Lambda$, the energy density in gradients can be as high as $\rho_{grad} \sim f^2 H^2$ on the scale of the Hubble radius. For $f \sim M_{Pl}$, $\rho_{grad} \sim \Lambda^4 \sim V(\phi)$. Since ρ_{grad} redshifts while $V(\phi) \sim const$, the main effect would be a slight delay of the onset of inflation.

Finally, another interesting feature of natural inflation is the likelihood of an earlier inflationary period when the complex scalar field Φ relaxed to its vaccum expectation value. This earlier inflation does not leave any observable traces if the following inflation in the ϕ direction occurs for more than 60 e-folds and thus does not require a small self-coupling. The advantage it brings is that even if it only lasted for a few e-folds it would help the onset of natural inflation by smoothing out inhomogeneities. This earlier inflation and its likelihood are discussed more thoroughly in Ref. 7.

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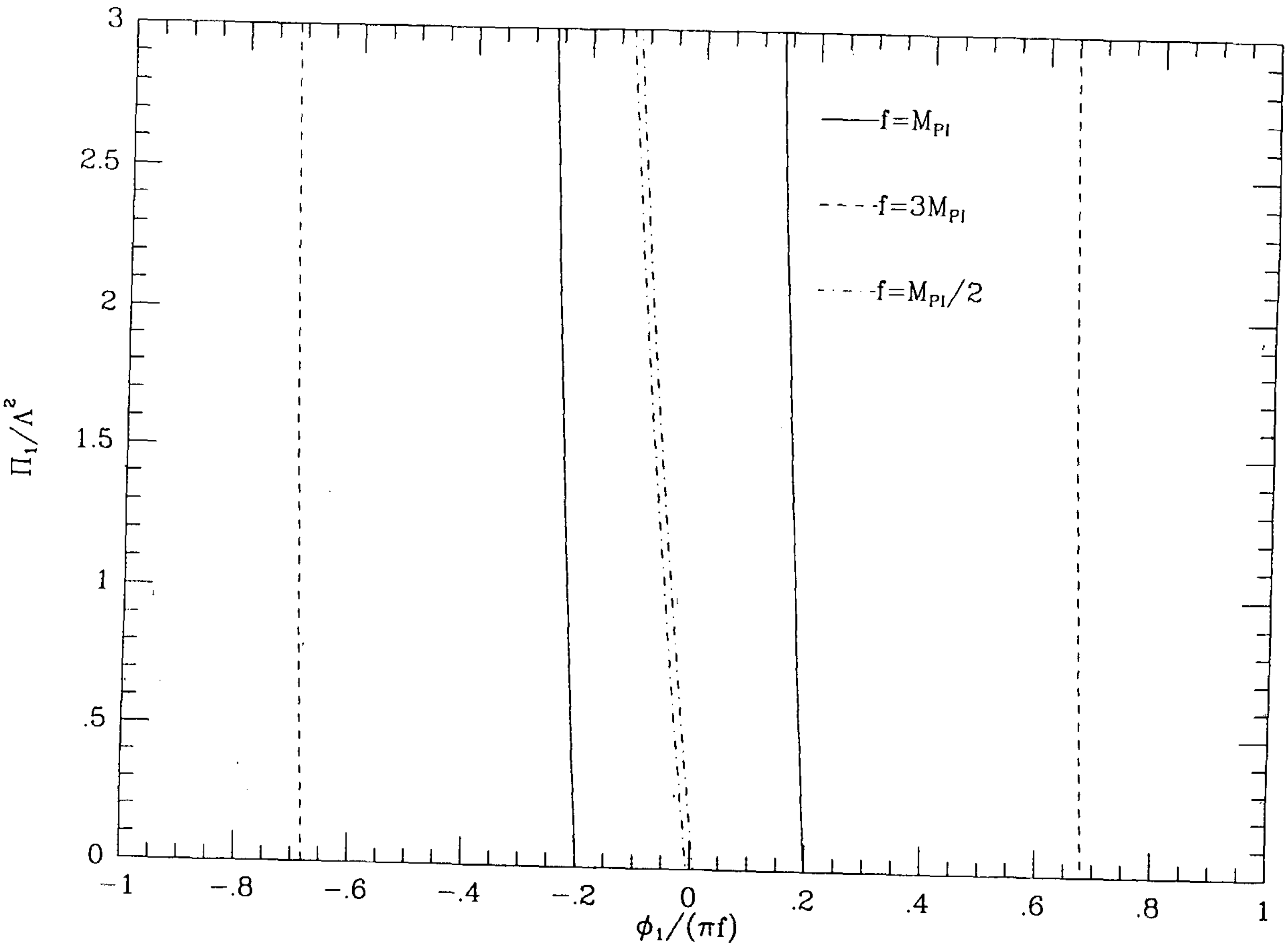
Figure Captions

Figure 1: The three pairs of lines plotted are the numerically calculated boundaries, for three different values of f, between regions of the initial condition phase space that lead to sufficient inflation and regions that lead to insufficient inflation.

Figure 2: The boundaries for the case $f = M_{Pl}$ are plotted over a much larger range of Π, much larger than we expect to occur at the onset of inflation. The solid line is from the numerical solution and the dotted line is from equations 5a and 5b. Here we take $\ln(M_{Pl}^2/\Lambda^2) = 14.3$.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f=$3M_{Pl}$
f=$M_{Pl}/2$
Π_1/Λ^2
$\phi_1/(\pi f)$
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2.5
2
1.5
1
.5
0
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−.8
−.6
−.4
−.2
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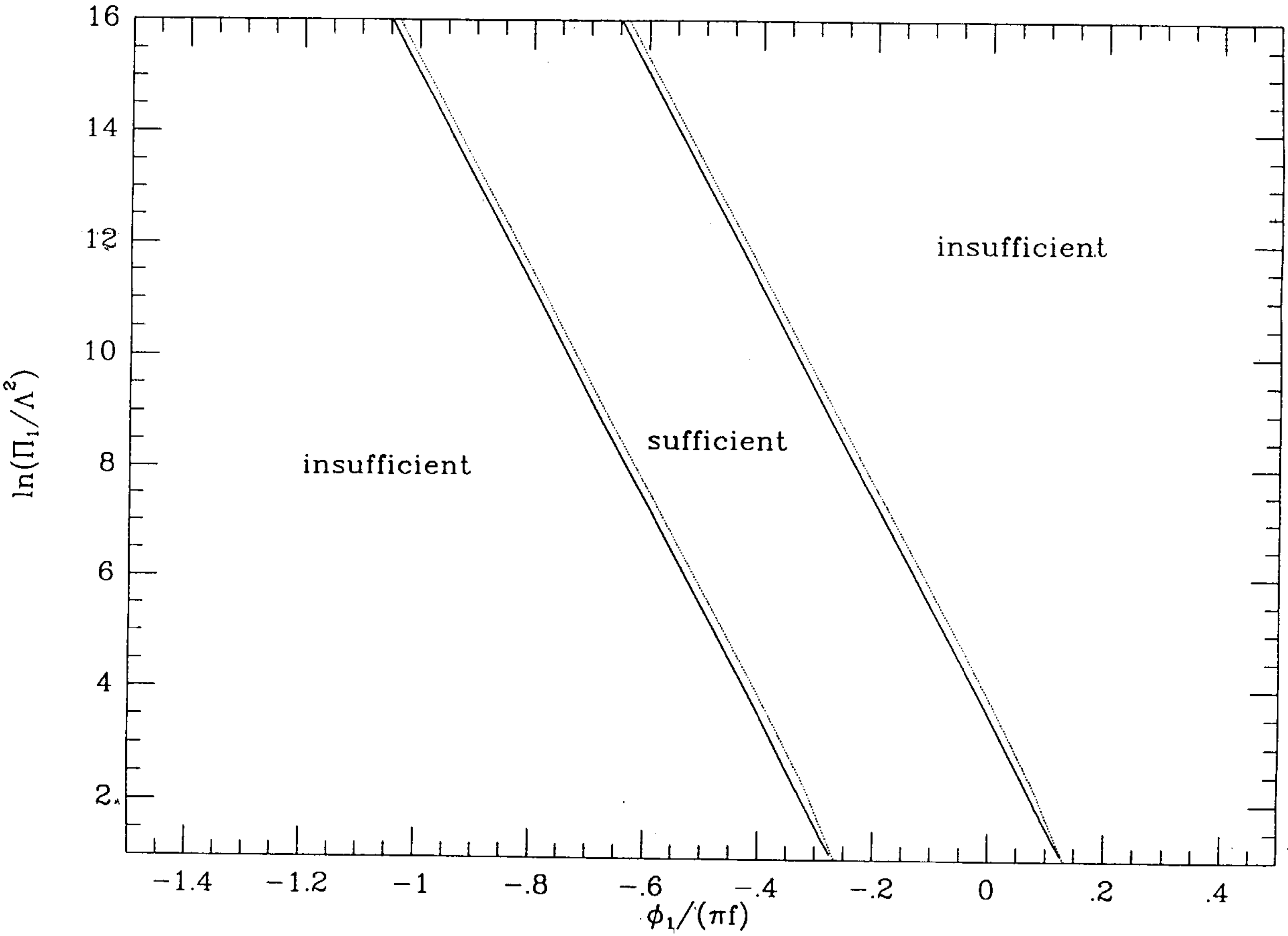
.4
.6
.8
1



insufficient
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$\ln(\Pi_1/\Lambda^2)$
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