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Testing for Gaussianity Through the Three Point Temperature Correlation Function *

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ABSTRACT

One of the crucial aspects of density perturbations that are produced by the standard inflation scenario is that they are Gaussian. The three point correlation function of the temperature anisotropy of the cosmic microwave background radiation (CBR) provides a sensitive test of this aspect of the primordial density field. In this paper, this function is calculated in the general context of allowed non-Gaussian models. The amplitude of the correlation is found to depend logarithmically upon the beam width used in the experiment; the angular dependence of the function is found to be sensitive to the power law index n of the power spectrum. Thus an analysis of the correlation function would lead to a new constraint on n. These predictions can be tested by COBE and the forthcoming South Pole and Balloon CBR anisotropy data.

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Testing for the Gaussianity of the primordial fluctuation spectrum is of critical importance to many cosmological models. In particuliar, cosmic inflation [1-3] specifically predicts a Gaussian density fluctuation spectrum. The quantum fluctuations generated during the inflationary epoch are expected to serve as the primordial density perturbations which develop into the large scale structures we observe today [4-7]. Alternative scenarios, such as topological defects (strings, textures, domain walls, etc.), tend to produce non-Gaussian seeds on scales that were within the horizon at recombination. The anisotropy of the cosmic microwave background radiation (CBR) provides a new test of Gaussianity as we will discuss here.

After two years of operation, COBE has already provided an invaluable testing ground for cosmic structure formation theories. Two key, interesting pieces of information have been obtained from COBE anisotropy measurements [8-9]. One is the detection of rms temperature anisotropy $(\Delta T)_{rms} \approx 30\mu K$. That is a temperature fluctuation $\Delta T/T \sim 1.1 \times 10^{-5}$ at $\theta \sim 10°$ angular scale. The second is the related measurement of a two-point temperature correlation function for primordial fluctuations. Temperature fluctuations can be related to density perturbations through the Sachs-Wolfe effect [10]:

$$\frac{\delta T}{T} = \frac{\phi}{3}, \tag{1}$$

where ϕ is the gravitational potential at the last scattering surface. The two-point temperature correlation function is:

$$C_T(\hat{m}, \hat{n}) = < \frac{\delta T}{T}(\hat{m}) \frac{\delta T}{T}(\hat{n}) > = \sum_l C_l (2l+1) P_l(\hat{m} \cdot \hat{n}), \tag{2}$$

where the multipole coefficient c_l is related to the power spectrum of density perturbation $P(k)$ through [11]:

$$C_l = V^{-1} \frac{H_0^4}{2\pi} \int_0^\infty \frac{dk}{k^2} P(k) j_l^2(k\tau_0). \tag{3}$$

COBE measurements show that the observed power spectrum of the primordial density perturbations is consistent with the scale invariant spectrum predicted by inflation, $P(k) \sim k$. More specifically, COBE found for a power spectrum, $P(k) = Ak^n$, that $n = 1.2^{+0.5}_{-0.6}$.

Competing models for structure formation, including cosmological topological defects [12,13] and non-standard inflation models [14,15], will also generate a scale invariant (or nearly scale invariant) power spectrum for density perturbations. Thus, the two-point temperature correlation function is not sufficient to discriminate among these models. However, as we noted before [16], the three point temperature correlation provides a sensitive test of the Gaussian nature of density perturbation. We will use the standard inflation paradigm as our Gaussian archetype. It may be possible that a three point temperature correlation is generated via a cubic self-interaction of the inflating field [17]. But generally, the coupling constant for the cubic interaction is negligible in inflationary cosmology. From a theoretical point of view, the Lagrangian with cubic self-interaction is not bounded below so that the ground state is not well defined [18]. Based on these considerations, we adopt a vanishing three point temperature correlation function for the simple one field inflation models considered by [4]. Furthermore, if the observed CBR anisotropy is produced mainly by the gravity wave [19], the three point temperature correlation function will also vanish.

Once the primordial density perturbation is generated by inflation, the temperature fluctuations of the CBR are related to the density perturbations through the Sachs-Wolfe formula in Eq.(1). The three-point temperature correlation function is given by [16]

$$\xi_T^{SW} = -\frac{H_0^6}{(4\pi)^6}\int d^3k_1 d^3k_2 \frac{< \delta_{k_1}\delta_{k_2}\delta_{-k_1-k_2} >}{k_1^2 k_2^2 |k_1+k_2|^2} e^{i\vec{k_1}\cdot(\vec{m_1}-\vec{m_3})\eta_0} \cdot e^{i\vec{k_2}\cdot(\vec{m_2}-\vec{m_3})\eta_0}. \quad (4)$$

It is also possible that the density fluctuations are generated through a cosmic vacuum phase transition [20]. In this case, the temperature fluctuations are related to the density perturbation [21,22] through

$$(\frac{\delta T}{T})_{LT} = 2\int \frac{\partial \Phi}{\partial \eta} d\eta. \quad (5)$$

By making the following anastz [23]

$$\delta(k,\eta) = \delta(k) f_k(\eta), \quad (6)$$

the three point temperature correlation is given by

$$\xi_T^{LT} = -\frac{(3H_0^2)^3}{(2\pi)^6}\int d^3k_1 d^3k_2 \frac{< \delta_{k_1}\delta_{k_2}\delta_{-k_1-k_2} >}{k_1^2 k_2^2 |k_1+k_2|^2}$$

$$\times \int_{\eta_i}^{\eta_0} e^{i\vec{k_1}\cdot\vec{m_1}(\eta-\eta_0)} \dot{f}_{k_1}(\eta) d\eta \int_{\eta_i}^{\eta_0} e^{i\vec{k_2}\cdot\vec{m_2}(\eta-\eta_0)} \dot{f}_{k_2}(\eta) d\eta \int_{\eta_i}^{\eta_0} e^{-i(\vec{k_1}+\vec{k_2})\cdot\vec{m_3}(\eta-\eta_0)} \dot{f}_{k_3}(\eta) d\eta \quad (7)$$

In the realistic model where the density perturbations are generated by a phase transition, $f_k(\eta)$ is a step function, and the time derivative is a δ-function, $\dot{f}_k(\eta) = \delta(\eta - \eta_z)$, where

$\eta_z = \eta_0/\sqrt{1+z}$ is the conformal time when the phase transition occurs. The three point temperature correlation function reduces to:

$$\xi_T^{LT} = -\frac{(3H_0^2)^3}{(2\pi)^6} \int d^3k_1 d^3k_2 \frac{<\delta_{k_1}\delta_{k_2}\delta_{-k_1-k_2}>}{k_1^2 k_2^2 |k_1+k_2|^2} e^{i\vec{k_1}\cdot(\vec{m_1}-\vec{m_3})\eta'} \cdot e^{i\vec{k_2}\cdot(\vec{m_2}-\vec{m_3})\eta'}, \quad (9)$$

where $\eta' = \eta_0 - \eta_z = \eta_0(1 - \frac{1}{\sqrt{1+z}}) \approx \eta_0$ when $z \gg 1$. The expression for the three point temperature correlation function in which the density perturbations are generated at a late-time phase transition is identical to Eq. (4) except that the amplitude is 6 times larger (after the amplitude of power spectrum being properly normalized). Potentially, this result can be used to tell whether the density perturbation is generated primordially or through some physical processes at late time (after decoupling) [24].

From the discussion above, it is clear that the three point temperature correlation function is directly related to the bispectrum of the density perturbations. For a large variety of physical models that generate non-Gaussian density perturbations, the density perturbation δ is well modeled by

$$\delta = [\phi + \alpha(\phi^2 - 1)]\delta_0, \quad (10)$$

where ϕ is a random Gaussian field with a variance of unity, δ_0 is the amplitude, and α is a constant. In the limit when $\alpha \ll 1$, this model reduces to the two-field inflation model [25] which is a prototype for non-Gaussian adiabatic perturbations; when $\alpha \gg 1$, the model reduces to the O(N) σ model [26] which is a prototype for non-Gaussian isocurvature perturbations. It is worthwhile to note that the skewness K, which is defined as $K =< \delta^3 >$, has different scaling relations with respect to the variance $< \delta^2 >$ in different limiting cases [16]. In the case, when $\alpha \ll 1$, $< \delta^3 > \sim \alpha < \delta^2 >^2$, which is the same as the skewness generated by gravitational evolution [27]. On other hand, $\alpha \gg 1$, $< \delta^3 > \sim \alpha^3 < \phi^2 >^3, < \delta^2 > \sim \alpha^2 < \phi^2 >^2$. Thus, $< \delta^3 > / < \delta^2 >^{3/2} \sim const.$.

The analysis of IRAS galaxy survey [28] shows that the observed skewness scales quadratically with the variance in both the linear regimes and non-linear regime, which suggests that either the primordial density perturbations are Gaussian or if they are non-Gaussian, the coefficient α in Eq. (5) must be small. In the following discussion of non-Gaussian perturbations, we concetrate only on this latter case.

The bispectrum $P(k_1, k_2)$ for the non-Gaussian density fields which is described by Eq.

(10) is given by:

$$P(k_1,k_2) = 2\alpha[P(k_1)P(k_2) + P(k_1)P(|k_1+k_2|) + P(k_2)P(|k_1+k_2|)]. \qquad (11)$$

Putting this expression into the Eq. (4), we obtain the formula for the three point correlation function:

$$\xi_T^{SW} = \frac{2\alpha H_0^6}{(4\pi)^6}\int d^3k_1 d^3k_2 \frac{[P(k_1)P(k_2) + P(k_1)P(|k_1+k_2|) + P(k_2)P(|k_1+k_2|)]}{k_1^2k_2^2|k_1+k_2|^2}$$

$$\times e^{i\vec{k_1}\cdot(\vec{m_1}-\vec{m_3})\eta_0}\cdot e^{i\vec{k_2}\cdot(\vec{m_2}-\vec{m_3})\eta_0}. \qquad (12)$$

As shown in the Appendix, this integral can be evaluated analytically for a power-law power spectrum: $P(k) = Ak^n$. The result is that the three point temperature correlation function can expand into a Legendre polynomial in the same manner as the two-point correlation function (as shown in Eq. (2)) except that the argument for the Legendre function and the multipole coefficients is more complicated. In the limit when the beam width is small, $\sigma \ll 1$, the observed three point temperature correlation function is given by following expression:

$$\xi_T(\hat{m}_1,\hat{m}_2,\hat{m}_3) = \sum_l (2l+1)C_l(\hat{m}_1\cdot\hat{m}_3,\hat{m}_2\cdot\hat{m}_3)\exp(-1.5(l+0.5)^2\sigma^2)$$

$$\times P_l\left(\frac{\hat{m}_1\cdot\hat{m}_2 - \hat{m}_1\cdot\hat{m}_3 - \hat{m}_2\cdot\hat{m}_3 + 1}{|\hat{m}_1-\hat{m}_3|\cdot|\hat{m}_2-\hat{m}_3|}\right), \qquad (13)$$

where

$$C_l(\hat{m}_1\cdot\hat{m}_3,\hat{m}_2\cdot\hat{m}_3) =$$

$$-\frac{2\alpha(H_0)^6A^2}{(4\pi)^4}\Big[\int dk_1dk_2 j_l(k_1\eta_0|\hat{m}_1-\hat{m}_3|)j_l(k_2\eta|\hat{m}_2-\hat{m}_3|)Q_l\left(\frac{k_1^2+k_2^2}{2k_1k_2}\right)k_1^nk_2^n$$

$$+\int dk_1dk_2(k_1^n+k_2^n)a_l(k_1,k_2)j_l(k_1\eta_0|\hat{m}_1-\hat{m}_3|)j_l(k_2\eta_0|\hat{m}_2-\hat{m}_3|)\Big], \qquad (14)$$

and

$$a_l(k_1,k_2) = \frac{1}{2}\int_{-1}^{1} dx P_l(x)(k_1^2+k_2^2-2k_1k_2x)^{(\frac{n}{2}-1)}. \qquad (15)$$

The formula above is far too complicated to be of any intuitive use. In the following discussions, we use the leading order of C_l, which reads as

$$C_l \sim -\frac{2\alpha(H_0)^{4+2n}A^2}{(4\pi)^4}\ln\left(\frac{1}{\sigma}\right)F(n,l)(|\hat{m}_1-\hat{m}_3|\cdot|\hat{m}_2-\hat{m}_3|)^{(-n-1)},$$

where

$$F(n,l) = \int_0^\infty x^{(2n-2)} j_l^2(x) dx = \frac{\pi}{8} \frac{\Gamma(3-2n)}{\Gamma(2-n)^2} \frac{\Gamma[(2l+2n-1)/2]}{\Gamma[(2l+5-2n)/2]}. \quad (16)$$

Discussion:

(1) The amplitude of the three point correlation function depends on the Gaussianity of the density perturbation. A Gaussian density field gives a vanishing three point temperature correlation function and the correlation function deviates linearly from zero with the skewness of the primordial density perturbations (non-linear effects can generate a non-vanishing three point temperature correlation function for the Guassian primordial density perturbation, but it is negligible compared with the case we are studying when skewness $\alpha > 0.01$). Furthurmore, the correlation function depends upon the beam width σ used in the experiment as well as the epoch when the density perturbations are generated (before or after photon decoupling). Thus, it is helpful to analyze data with a different angular resolution. The three point correlation function will be enhanced as the beam width decreases. In particular, by comparing the three point temperature correlation done with COBE, where the data is smeared over $\sim 10°$ and the correlation function that might be obtained with a full sky map done with an instrument that can get the resolution down to 1°, one might be able to tell whether the temperature fluctuations on different angular scales are generated by gravitational potential at the last scattering surface (Sachs-Wolfe effect, which is the dominant effect for adiabatic perturbation generated by inflation) or by the change of the gravitational potential along the photon path (modified Rees-Sciama effects, which are the dominant effect for isocurvature perturbation generated by topological defects or by a late-time phase transition). The correlation amplitude on a 1° scale will be one order of magnitude larger if the perturabtions on that scale are generated by causal processes that occurred after decoupling. This is one of the unique properties of the three point correlation function and should be testable by future CBR experiments.

(2) We consider a special case when $\hat{m}_1 - \hat{m}_3$ and $\hat{m}_2 - \hat{m}_3$ are unit vectors. This is true when $\hat{m}_1 \cdot \hat{m}_3 = \hat{m}_2 \cdot \hat{m}_3 = 0.5$. In this case, ξ_T is a function of $\hat{m}_1 \cdot \hat{m}_2$ only:

$$\xi_T = \sum_l C_l (2l+1) P_l(\hat{m} \cdot \hat{n}) e^{-1.5(l+0.5)^2\sigma^2},$$

where $C_l \sim \frac{1}{2l+1}$ for $n = 1$ H-Z power spectrum. This function is plotted in Fig. (1) as a function of θ, where $\cos(\theta) = \hat{m} \cdot \hat{n}$, with monopole, dipole and quadrupole removed. The shape is similar to the two point correlation function with proper normalization.

(3) Generally, the angular dependence of ξ_T is:

$$\xi_T \sim (|\hat{m}_1 - \hat{m}_3| \cdot |\hat{m}_2 - \hat{m}_3|)^{(-n-1)} \cdot P_l(\frac{\hat{m}_1 \cdot \hat{m}_2 - \hat{m}_1 \cdot \hat{m}_3 - \hat{m}_2 \cdot \hat{m}_3 + 1}{|\hat{m}_1 - \hat{m}_3| \cdot |\hat{m}_2 - \hat{m}_3|}).$$

The shape of the three point temperature correlation function is plotted in Fig (2) for another interesting special case [17] when the three beams have the same angle with respect to each other, e.g., $\hat{m}_1 \cdot \hat{m}_2 = \hat{m}_2 \cdot \hat{m}_3 = \hat{m}_3 \cdot \hat{m}_1 = \cos(\beta)$, as a function of β, with two different power spectral indices: n=1 (Harrison-Zeldovich spectrum) and n=0.7.

As mentioned earlier, the COBE DMR experiment has put a constraint on the power spectra index [9]. Since the two point correlation function has a weak dependence on the power spectrum, there still exists a large range of parameters that can fit the DMR data. However, as shown in Fig. (2), the three point temperature correlation function has a stronger dependence on n. Thus, it is hoped that an analysis of the three point correlation function can put a more stringent limit on n.

In conclusion, we have shown the importance of testing the predictions of inflation by a three point temperature correlation function. Valuable information, including the Gaussinality of the density perturbation and the power spectra index can be obtained from this analysis. Furthermore, the existing COBE data should be sufficient to carry out an initial significant exploration of this test.

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Appendix

In this appendix, we present the mathematical details on deriving Eq. (13).

First, we can expand the cross term in the denominator in Eq. (12) by the Legendre polynomial:

$$|k_1 + k_2|^n = \sum_l (2l+1) C_l(k_1, k_2) P_l(\hat{k}_1 \cdot \hat{k}_2),$$

then, by using the following expansion for the plane wave:

$$e^{i\vec{k}\cdot\vec{r}} = \sum_l i^l (2l+1) j_l(kr) P_l(\cos\theta),$$

and the sum rule of the spherical harmonics,

$$P_l(\hat{m}\cdot\hat{n}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\hat{m}) Y^*_{lm}(\hat{n})$$

The integration over solid angle of $\hat{k}_1$ and $\hat{k}_2$ gives the anglar dependence on the beam directions:

$$\xi \sim P_l\left(\frac{(\hat{m}_1 - \hat{m}_3)\cdot(\hat{m}_2 - \hat{m}_3)}{|\hat{m}_1 - \hat{m}_3| \cdot |\hat{m}_2 - \hat{m}_3|}\right) j_l(k_1 \eta_0 |\hat{m}_1 - \hat{m}_3|) j_l(k_2 \eta_0 |\hat{m}_2 - \hat{m}_3|).$$

The integration over the k_1 and k_2 gives the multipole coefficients and the angular dependence.

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FIGUERE CAPTIONS

Fig.1: The three point correlation as a function of the angle between two beams. The dotted line is the two point temperature correlation function with monopole, dipole and quadrupole terms removed. The power spectrum is assumed to be H-Z (n=1).

Fig.2: The dependence of the three point correlation function on the power law index n. The dotted line is for $n = 0.7$, while the solid line is for $n = 1$.

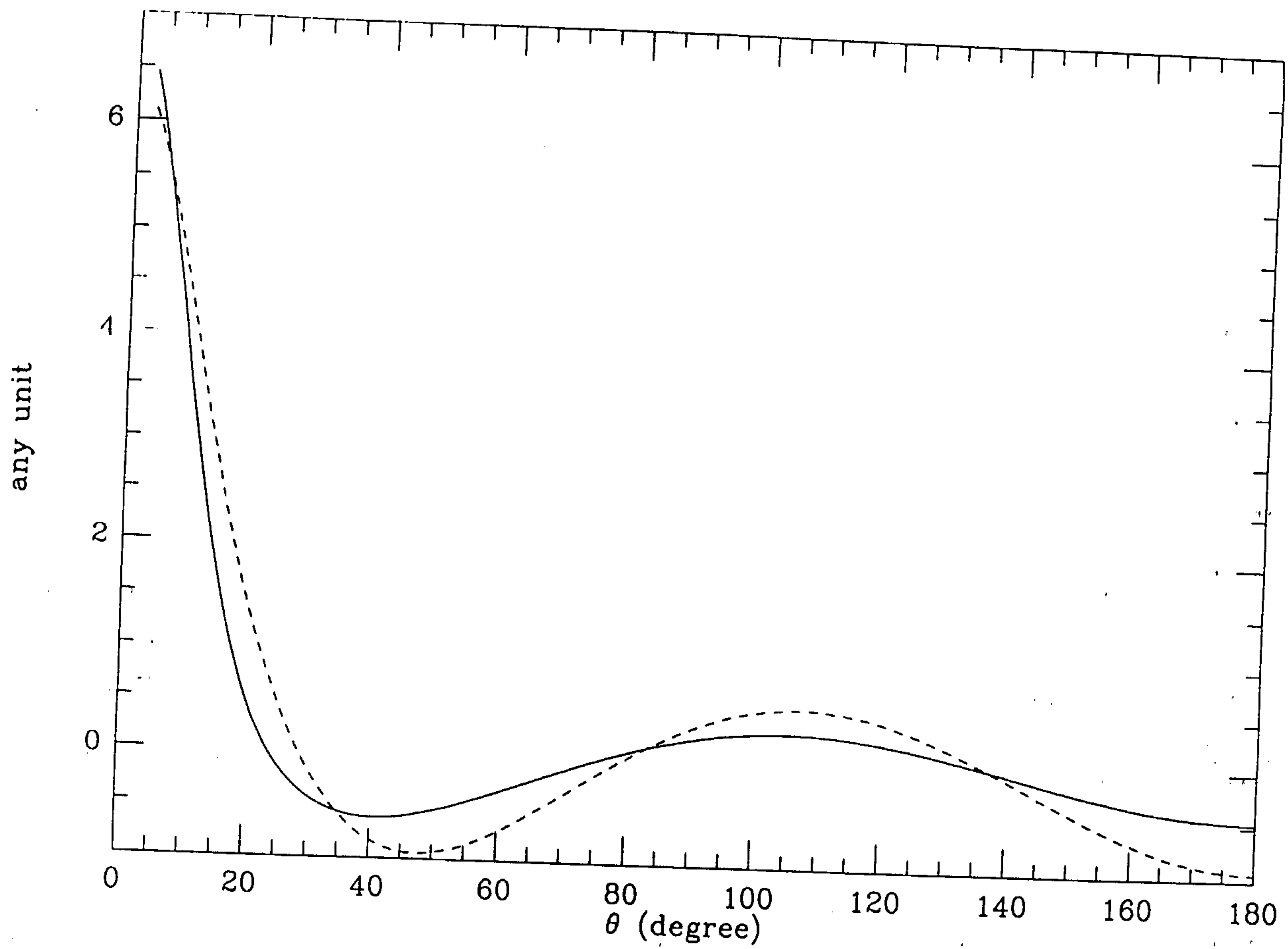

Fig. 1

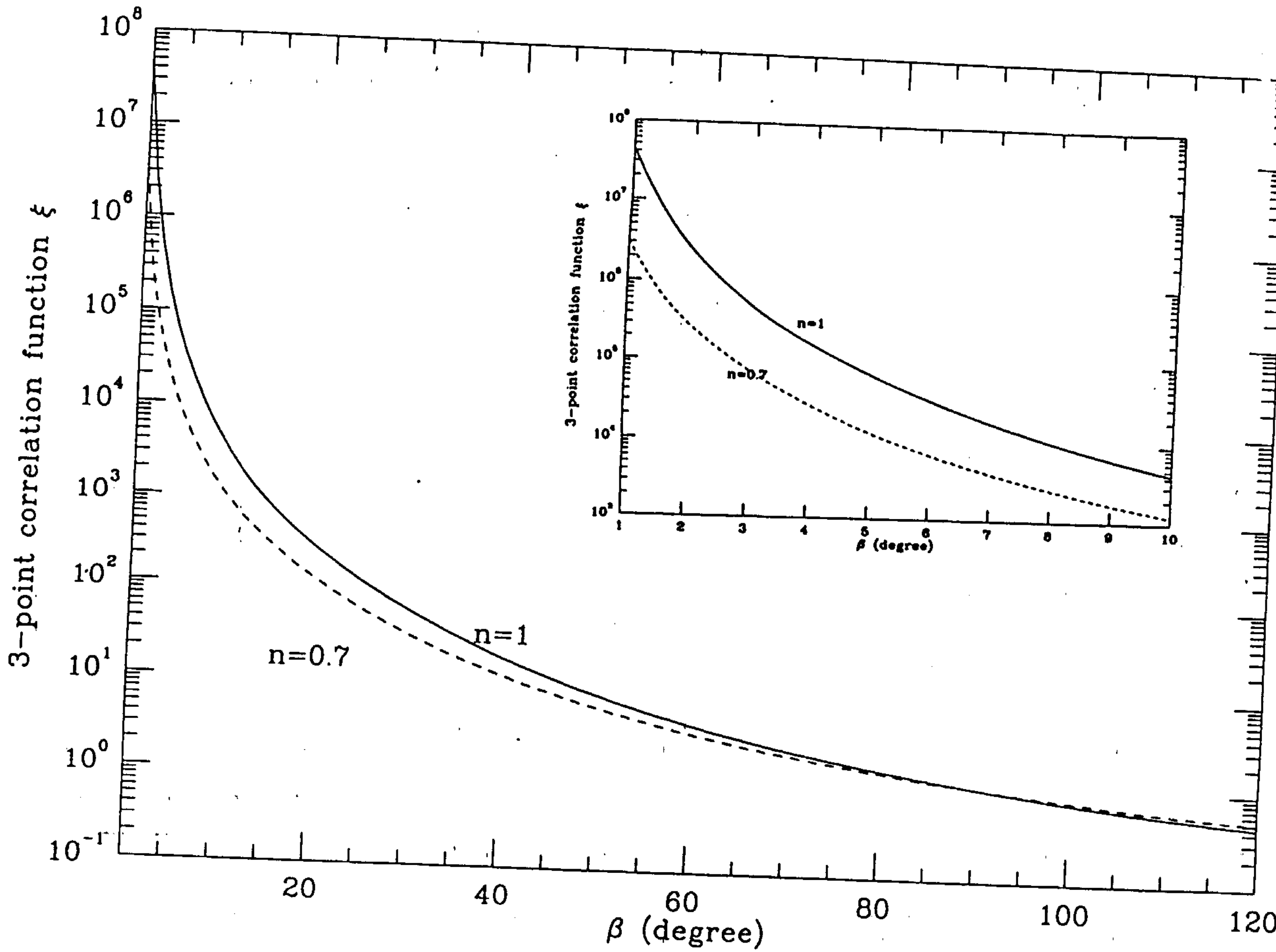
3-point correlation function ξ
β (degree)
n=1
n=0.7
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10^1
10^0
10^-1
20
40
60
80
100
120

Fig. 2